

# Some results on Best Proximity Points for Cyclic Mappings\*

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## Abstract

In this paper we consider a cyclic  $\varphi_A$ -contraction mapping defined on a partially ordered orbitally complete metric space and prove some fixed point and best proximity point theorems. We also discuss some relationship between points of coincidence and common best proximal points. It is shown that, under certain condition, a point of coincidence, a common best proximal point and a common fixed point coincide.

## 1 Introduction and preliminaries

Let  $(X, \leq)$  be a partially ordered set. A self mapping  $T : X \rightarrow X$  is said to be monotone nondecreasing if  $Tx \leq Ty$  whenever  $x \leq y, x, y \in X$ . In 2005, Nieto and Rodriguez-Lopez [7] studied fixed point theory in partially ordered metric spaces and established the following results.

**Theorem 1.1.** [7] Let  $(X, \leq)$  be a partially ordered set and let there exist a metric  $d$  in  $X$  which makes  $(X, d)$  into a complete metric space. Let  $T$  be a continuous and nondecreasing self mapping on  $X$  for which there exists  $k \in [0, 1)$  such that  $d(Tx, Ty) \leq kd(x, y)$  for each  $y \leq x$ . If there exists  $x_0 \in X$  with  $x_0 \leq T(x_0)$ , then  $T$  has a fixed point.

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**Theorem 1.2.** [7] Let  $(X, \leq)$  be a partially ordered set and let there exist a metric  $d$  in  $X$  which makes  $(X, d)$  into a complete metric space. Assume that  $X$  satisfies the condition

$$\text{if a nondecreasing sequence } x_n \rightarrow x \in X, \text{ then } x_n \leq x, \forall n. \quad (1)$$

Let  $T : X \rightarrow X$  be a monotone and nondecreasing mapping for which there exists  $k \in [0, 1)$  such that  $d(Tx, Ty) \leq kd(x, y)$  for each  $y \leq x$ . If there exists  $x_0 \in X$  with  $x_0 \leq T(x_0)$ , then  $T$  has a fixed point.

Since then, there has been a lot of activity in this area and several interesting results have appeared.

Let  $A$  and  $B$  be two nonempty subsets of a metric space  $X := (X, d)$ ,  $T : A \cup B \rightarrow A \cup B$ . The mapping  $T$  is said to be

- cyclic if  $T(A) \subseteq B$  and  $T(B) \subseteq A$ ;
- cyclic contraction [6], if it is cyclic and

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)dist(A, B)$$

for some  $k \in (0, 1)$  and for all  $x \in A$  and  $y \in B$ , where

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\};$$

- cyclic  $\varphi$ -contraction [3, 4], if it is cyclic,  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is a strictly increasing function and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(dist(A, B))$$

for all  $x \in A$  and  $y \in B$ ;

- $x \in A \cup B$  is a best proximity point for  $T$  if  $d(x, Tx) = dist(A, B)$ .

Notice that a cyclic contraction mapping is cyclic  $\varphi$ -contraction with  $\varphi(t) = (1 - k)t$  for  $t \geq 0$  and  $0 \leq k < 1$ .

Recently, Abkar and Gabeleh [1] proved best proximity points results for cyclic mappings in partially ordered complete metric spaces.

- $T : A \cup B \rightarrow A \cup B$  is said to be cyclic contraction mapping in the sense of Abkar and Gabeleh [1], if it is cyclic and

$$d(Tx, T^2x') \leq kd(x', Tx) + (1 - k)dist(A, B) \quad (1.2)$$

for some  $k \in (0, 1)$  and for all  $(x, x') \in A \times A$ .

- $T : A \cup B \rightarrow A \cup B$  is said to be cyclic  $\varphi_A$ - contraction mapping, if it is cyclic,  $\varphi_A : [0, +\infty) \rightarrow [0, +\infty)$  is a strictly increasing function and

$$d(Tx, T^2x') \leq d(x', Tx) - \varphi_A(d(x', Tx)) + \varphi_A(\text{dist}(A, B)) \tag{1.3}$$

for some  $k \in (0, 1)$  and for all  $(x, x') \in A \times A$ .

Notice that a cyclic contraction mapping in the sense of Abkar and Gabeleh [1] is cyclic  $\varphi_A$ - contraction with  $\varphi_A(t) = (1 - k)t$  for  $t \geq 0$  and  $0 \leq k < 1$ . For more details on this subject, we refer to [5, 8, 9, 11].

Let  $(X, d)$  be a metric space and let  $T : A \cup B \rightarrow A \cup B$  be a mapping. The orbit of  $T^2$  at the point  $x_0 \in A$  is the set

$$\mathcal{O}(x_0, T^2) = \{x_0, T^2x_0, T^4x_0, \dots, T^{2n}x_0, \dots\}.$$

**Definition 1.1.** The set  $\mathcal{O}(x_0, T^2)$  is said to be  $T^2$ -orbitally complete if any Cauchy subsequence  $\{T^{2n_i}x_0\}$  in orbit  $\mathcal{O}(x_0, T^2)$ ,  $x_0 \in A$ , converges in  $A$ .

**Definition 1.2.** An operator  $T : X \rightarrow X$  is said to be orbitally continuous if  $T^{n_i}x \rightarrow p \Rightarrow T(T^{n_i}x) \rightarrow Tp$  as  $i \rightarrow \infty$ .

Notice that a complete metric space  $(X, d)$  is orbitally complete with respect to any self-mapping of  $X$ , and that a continuous mapping is always orbitally continuous.

In this paper we consider a cyclic  $\varphi_A$ -contraction mapping defined on a partially ordered orbitally complete metric space. We prove some fixed point and best proximity point theorems and discuss some relationship between points of coincidence and common best proximal points. It is also shown that, under certain condition, a point of coincidence, a common best proximal point and a common fixed point coincide.

## 2. Fixed Point Theorems

Throughout this section, let  $\varphi_A : [0, +\infty) \rightarrow [0, +\infty)$  be a strictly increasing function. Denote by  $\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t), \dots$  to  $\varphi_A(t), \varphi_A(t - \varphi_A(t)), \varphi_A(t - \varphi_A(t) - \varphi_A(t - \varphi_A(t))), \varphi_A(t - \varphi_A(t) - \varphi_A(t - \varphi_A(t)) - \varphi_A(t - \varphi_A(t) - \varphi_A(t - \varphi_A(t))))), \dots$ , respectively. Let  $\Phi$  be the class of functions  $\varphi_A$  satisfying the following conditions:

- (a<sub>1</sub>)  $t - \varphi_A(t)$  is a nondecreasing function of  $t$ ,  $\forall t > 0$ ,
- (a<sub>2</sub>)  $t - \varphi_A(t) \rightarrow 0$  as  $t \rightarrow 0$ .
- (a<sub>3</sub>)  $t - \sum_{r=1}^{2^n-1} \phi_r(t) \rightarrow 0$  as  $n \rightarrow \infty$ ,  $\forall t > 0$ .

**Remark 2.1.** Since  $\varphi_A : [0, +\infty) \rightarrow [0, +\infty)$  is a strictly increasing function, it is injective and  $\varphi_A(0) = 0$ .

The following lemma is obvious.

**Lemma 2.1.** Let  $A, B$  be two nonempty subsets of a metric space  $(X, d)$  and  $T : A \cup B \rightarrow A \cup B$  be a cyclic  $\varphi_A$ -contraction mapping. For  $x_0 \in A$ , define  $x_{n+1} = Tx_n$  for each  $n \geq 0$ . Then

(a)  $-\varphi_A(d(x', Tx)) + \varphi_A(\text{dist}(A, B)) \leq 0$  for all  $(x, x') \in A \times A$ .

(b)  $d(Tx', T^2x) \leq d(x', Tx)$  for all  $(x, x') \in A \times A$ .

(c) if  $\text{dist}(A, B) = 0$ , then  $d(x_{2n-1}, x_{2n}) = d(Tx_{2n-2}, T^2x_{2n-2}) \leq d(x_0, Tx_0) - \sum_{r=1}^{2n-1} \phi_r(d(x_0, Tx_0))$  for all  $n \geq 1$ .

**Lemma 2.2.** Let  $A, B$  be two nonempty subsets of a metric space  $(X, d)$  and  $T : A \cup B \rightarrow A \cup B$  be a cyclic  $\varphi_A$ -contraction mapping. Let there exist  $x_0 \in A$  such that  $\mathcal{O}(x_0, T^2)$  is  $T^2$ -orbitally complete and  $T^2$  is nondecreasing on  $\mathcal{O}(x_0, T^2)$ . Let " $\leq$ " be a partial order relation on  $\mathcal{O}(x_0, T^2)$  such that

$$d(Tx', T^2x) \leq d(x', Tx) - \varphi_A(d(x', Tx)) + \varphi_A(\text{dist}(A, B)) \quad (2.1)$$

for all  $(x, x') \in \mathcal{O}(x_0, T^2) \times \mathcal{O}(x_0, T^2)$  with  $x \leq x'$ . If  $x_0 \leq T^2x_0$ , then  $d(x_n, x_{n+1}) \rightarrow \text{dist}(A, B)$  as  $n \rightarrow \infty$  where  $x_{n+1} = Tx_n$ .

*Proof.* Assume that  $x_0 \in A$  with  $x_0 \leq T^2(x_0)$  and  $T^2$  is nondecreasing on  $\mathcal{O}(x_0, T^2)$ , then it follows that

$$x_0 \leq T^2(x_0) \leq \dots \leq T^{2n}x_0 \leq \dots$$

Put  $r_n = d(x_n, x_{n+1})$  for all  $n \geq 1$ . By Lemma 2.1(b), it is clear that  $\{r_n\}$  is decreasing and bounded. Thus  $\lim_{n \rightarrow \infty} r_n = \zeta$  for some  $\zeta \geq d(A, B)$ . It follows that  $r_{2n} \rightarrow \zeta$  and also  $r_{2n+1} \rightarrow \zeta$ . If  $r_{2n_0} = 0$  for some  $n_0 \geq 1$ , there is nothing to prove. So we assume that  $r_{2n} > 0$  for each  $n \geq 1$ . By (2.1) we have

$$r_{2n+1} = d(Tx_{2n}, T^2x_{2n}) \leq d(x_{2n}, Tx_{2n}) - \varphi_A(d(x_{2n}, Tx_{2n})) + \varphi_A(\text{dist}(A, B)).$$

This yields

$$\varphi_A(r_{2n}) \leq r_{2n} - r_{2n+1} + \varphi_A(\text{dist}(A, B)) \quad (2.2)$$

for each  $n \geq 1$ . Since  $\varphi_A$  is strictly increasing and  $r_{2n} \geq \zeta \geq d(A, B)$  for each  $n \geq 1$ , it follows from (2.2) that

$$\lim_{n \rightarrow \infty} \varphi_A(r_{2n}) = \varphi_A(\zeta) = \varphi_A(d(A, B)).$$

Since  $\varphi_A$  is strictly increasing, we have  $\zeta = d(A, B)$  i.e.,  $r_{2n} \rightarrow \text{dist}(A, B)$ . We can similarly show that  $r_{2n+1} \rightarrow \text{dist}(A, B)$ . It follows that  $r_n \rightarrow \text{dist}(A, B)$  as  $n \rightarrow \infty$ .

**Theorem 2.3.** Let  $A, B$  be two nonempty subsets of a metric space  $(X, d)$ . Let there exist  $x_0 \in A$  such that  $\mathcal{O}(x_0, T^2)$  is  $T^2$ -orbitally complete, and let " $\leq$ " be a partially ordered relation on  $\mathcal{O}(x_0, T^2)$ ,  $T$  be orbitally continuous,  $T^2$  be nondecreasing on  $\mathcal{O}(x_0, T^2)$  and  $T : A \cup B \rightarrow A \cup B$  be a cyclic mapping such that

$$d(Tx', T^2x) \leq d(x', Tx) - \varphi_A(d(x', Tx)) \tag{2.1a}$$

for all  $(x, x') \in \mathcal{O}(x_0, T^2) \times \mathcal{O}(x_0, T^2)$  with  $x \leq x'$  and  $\varphi_A \in \Phi$ . If  $x_0 \leq T^2x_0$ , then  $A \cap B \neq \emptyset$ , hence  $T$  has a fixed point  $p \in A \cap B$ . Moreover, if  $x_{n+1} = Tx_n$ , then  $x_{2n} \rightarrow p$ .

*Proof.* Suppose  $T^2x_0 = x_0$ , then  $x_0$  is a fixed point of  $T$ . Indeed, by (2.1a), we have

$$d(Tx_0, x_0) = d(Tx_0, T^2x_0) \leq d(x_0, Tx_0) - \varphi_A(d(x_0, Tx_0))$$

which implies  $\varphi_A(d(x_0, Tx_0)) = 0$ . By Remark 2.1,  $d(x_0, Tx_0) = 0$ . It follows that  $x_0$  is a fixed point of  $T$ . Suppose that  $T^2x_0 \neq x_0$ . By Lemma 2.1(c), it follows that  $\lim_{n \rightarrow \infty} r_{2n}$  exists. Suppose  $\lim_{n \rightarrow \infty} r_{2n} = \delta \geq 0$ . We show that  $\delta = 0$ . Suppose  $\delta > 0$ . Let  $N > 0$  be an integer such that  $r_{2n} \geq \frac{\delta}{2}$  for all  $n \geq N$ . Then

$$\liminf_{n \rightarrow \infty} \varphi_A(r_{2n}) \geq \varphi_A\left(\frac{\delta}{2}\right) > 0. \tag{2.3}$$

Now(2.1a) yields

$$\varphi_A(r_{2n}) \leq r_{2n} - r_{2n+1}. \tag{2.1b}$$

Similarly, we see that

$$\varphi_A(r_{2n+1}) \leq r_{2n+1} - r_{2n+2}. \tag{2.1c}$$

Thus, from (2.1b) and (2.1c) we obtain

$$\sum_{n=1}^{\infty} \varphi_A(r_n) < \infty,$$

from which it follows that  $\{\varphi_A(r_n)\}$  is a Cauchy sequence in  $[0, +\infty)$  and

$$\liminf_{n \rightarrow \infty} \varphi_A(r_{2n}) = 0,$$

contradicting (2.3). Thus,  $\delta = 0$ . It follows that  $\{x_{2n}\}$  is a Cauchy sequence in  $\mathcal{O}(x_0, T^2)$ . Therefore, there exists  $p \in \mathcal{O}(x_0, T^2)$  such that  $x_{2n} \rightarrow p$ . As  $T$  is orbitally continuous,  $Tx_{2n} \rightarrow Tp$ . Using (2.1a) again we have

$$d(p, Tp) = \lim_{n \rightarrow \infty} r_{2n} \leq \lim_{n \rightarrow \infty} r_{2n-1} - \liminf_{n \rightarrow \infty} \varphi_A(r_{2n-1}) = 0.$$

**Theorem 2.4.** Let  $A, B$  be two nonempty subsets of a metric space  $(X, d)$ . Let there exist  $x_0 \in A$  such that  $\mathcal{O}(x_0, T^2)$  is  $T^2$ -orbitally complete and  $\mathcal{O}(x_0, T^2)$  satisfies the condition (1), and let " $\leq$ " be a partially ordered relation on  $\mathcal{O}(x_0, T^2)$ ,  $T$  be orbitally continuous,  $T^2$  be nondecreasing on  $\mathcal{O}(x_0, T^2)$  and  $T : A \cup B \rightarrow A \cup B$



$\{(\frac{2}{3^{2n}}, 0) : n = 0, 1, 2, \dots\}$ . Clearly  $\mathcal{O}(x_0, T^2)$  is  $T^2$ -orbitally complete. We observe that, for  $x = (\frac{2}{3^{2n}}, 0)$  and  $x' = (\frac{2}{3^{2m}}, 0)$  with  $n \geq m$  i.e.,  $x \leq x'$ ,

$$\begin{aligned} d(Tx', T^2x) &= d\left(\left(0, \frac{2}{3^{2m+1}}\right), \left(\frac{2}{3^{2n+2}}, 0\right)\right) = \max\left\{\left|0 - \frac{2}{3^{2n+2}}\right|, \left|\frac{2}{3^{2m+1}} - 0\right|\right\} \\ &= \frac{2}{3^{2m+1}} = \frac{1}{3}\left(\frac{2}{3^{2m}}\right) = \frac{2}{3^{2m}} - \frac{2}{3}\left(\frac{2}{3^{2m}}\right) \\ &= d\left(\left(\frac{2}{3^{2m}}, 0\right), \left(0, \frac{2}{3^{2n+1}}\right)\right) - \varphi_A\left(d\left(\left(\frac{2}{3^{2m}}, 0\right), \left(0, \frac{2}{3^{2n+1}}\right)\right)\right) \\ &= d(x', Tx) - \varphi_A(d(x', Tx)) \end{aligned}$$

Thus we see that  $T$  satisfies all the conditions of Theorem 2.3, hence  $T$  has a fixed point. Indeed,  $p = (0, 0) \in A \cap B$  is a fixed point of  $T$ .

It may be remarked that in Example 2.1 above neither  $A$  nor  $B$  is complete. Hence our Theorem 2.3 is a proper generalization of the corresponding result i.e., Theorem 2.2 of Abkar and Gabeleh [1].

### 3. Best Proximity Points

In this section we obtain some results on the existence and convergence of best proximity points for  $\varphi_A$ -cyclic mappings.

**Theorem 3.1.** Let  $A, B$  be two nonempty closed subsets of a metric space  $(X, d)$ . Let there exist  $x_0 \in A$  such that  $\mathcal{O}(x_0, T^2)$  is  $T^2$ -orbitally complete and  $\mathcal{O}(x_0, T^2)$  satisfies the condition (1), and let " $\leq$ " be a partially ordered relation on  $\mathcal{O}(x_0, T^2)$ ,  $T$  be orbitally continuous,  $T^2$  be nondecreasing on  $\mathcal{O}(x_0, T^2)$  and  $T : A \cup B \rightarrow A \cup B$  be a cyclic mapping such that

$$d(Tx', T^2x) \leq d(x', Tx) - \varphi_A(d(x', Tx)) + \varphi_A(\text{dist}(A, B))$$

for all  $(x, x') \in \mathcal{O}(x_0, T^2) \times \mathcal{O}(x_0, T^2)$  with  $x \leq x'$  and  $\varphi_A \in \Phi$ . If  $x_0 \leq T^2x_0$ , define  $x_{n+1} = Tx_n$ , and  $\{x_{2n}\}$  has a convergent subsequence in  $A$ , then  $T$  has a best proximity point  $p \in A$ .

*Proof.* Let  $\{x_{2n_k}\}$  be a subsequence of  $\{x_{2n}\}$  converging to some  $p \in A$ . Then

$$\text{dist}(A, B) \leq d(p, x_{2n_k-1}) \leq d(p, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}).$$

If  $k \rightarrow \infty$ , then by Lemma 2.2 we have  $d(x_{2n_k}, x_{2n_k-1}) \rightarrow \text{dist}(A, B)$ . Thus, it follows from the above inequality that  $d(p, x_{2n_k}) \rightarrow \text{dist}(A, B)$ . Since  $T^2$  is nondecreasing and condition (1) holds, it follows from Lemma 2.1(b) that

$$\begin{aligned} \text{dist}(A, B) &\leq d(x_{2n_k}, Tp) = d(Tp, T^2(x_{2n_k-2})) \\ &\leq d(p, T(x_{2n_k-2})) = d(p, x_{2n_k-1}). \end{aligned}$$

Again if  $k \rightarrow \infty$ , we obtain

$$d(p, Tp) = \lim_{k \rightarrow \infty} d(x_{2n_k}, Tp) = \text{dist}(A, B).$$

**Theorem 3.2.** Let  $A, B$  be two nonempty, closed and convex subsets of a Banach space  $(X, \|\cdot\|)$ . Let there exist  $x_0 \in A$  such that  $\mathcal{O}(x_0, T^2)$  is  $T^2$ -orbitally complete and bounded and satisfies the condition (1), and let " $\leq$ " be a partially ordered relation on  $\mathcal{O}(x_0, T^2)$ ,  $T$  be orbitally continuous,  $T^2$  be nondecreasing on  $\mathcal{O}(x_0, T^2)$  and  $T : A \cup B \rightarrow A \cup B$  be a cyclic mapping such that

$$d(Tx', T^2x) \leq d(x', Tx) - \varphi_A(d(x', Tx)) + \varphi_A(\text{dist}(A, B))$$

for all  $(x, x') \in \mathcal{O}(x_0, T^2) \times \mathcal{O}(x_0, T^2)$  with  $x \leq x'$  and  $\varphi_A \in \Phi$ . If  $x_0 \leq T^2x_0$ , define  $x_{n+1} = Tx_n$ , and  $T$  is weakly continuous on  $\mathcal{O}(x_0, T^2)$ , then  $T$  has a best proximity point  $p \in A$ .

*Proof.* It follows by Lemma 2.2 that  $\|x_{2n} - Tx_{2n}\| \rightarrow \text{dist}(A, B)$ . Now applying similar argument as in Theorem 10 of [3] we obtain the desired conclusion. So we omit the details.

**Remark 3.1.** If in Theorem 3.2 we assume that  $X$  is a uniformly convex Banach space, then the best proximity point of  $T$  is unique.

In a recent paper, Suzuki, Kikkawa and Vetro [10] introduced the geometric notion of property UC as follows:

**Definition 3.1.** Let  $A, B$  be two nonempty subsets of a metric space  $(X, d)$ . A pair  $(A, B)$  is said to satisfy property UC iff the following holds: If  $\{x_n\}$  and  $\{z_n\}$  are sequences in  $A$ , and  $\{y_n\}$  is a sequence in  $B$  such that

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = \lim_{n \rightarrow \infty} d(z_n, y_n) = \text{dist}(A, B),$$

then  $\lim_{n \rightarrow \infty} d(x_n, z_n) = 0$ .

In the following we prove the existence and convergence of best proximity points in ordered metric spaces with this geometric property.

**Lemma 3.3.** [10] Let  $A, B$  be two nonempty subsets of a metric space  $(X, d)$ . Let the pair  $(A, B)$  satisfy the property UC. If  $\{x_n\}$  and  $\{y_n\}$  are sequences in  $A$  and  $B$ , respectively, such that either of the following condition holds:

$$\lim_{m \rightarrow \infty} \sup_{n \geq m} d(x_m, y_n) = \text{dist}(A, B) \quad \text{or} \quad \lim_{n \rightarrow \infty} \sup_{m \geq n} d(x_m, y_n) = \text{dist}(A, B).$$

Then  $\{x_n\}$  is a Cauchy sequence.

**Theorem 3.4.** Let  $(X, \leq)$  be a partially ordered set and  $d$  be a metric on  $X$ . Let  $A, B$  be two nonempty subsets of  $X$  such that the pair  $(A, B)$  satisfies the property UC. Let there exist  $x_0 \in A$  and  $y_0 \in B$  such that  $\mathcal{O}(x_0, T^2)$  is  $T^2$ -orbitally complete and satisfies the condition (1), and let " $\leq$ " be a partially ordered relation on  $\mathcal{O}(x_0, T^2)$ ,  $T$  and  $T^2$  be nondecreasing on  $\mathcal{O}(x_0, T^2)$  and  $T : A \cup B \rightarrow A \cup B$  be a cyclic mapping such that

$$d(Tx', T^2x) \leq d(x', Tx) - \varphi_A(d(x', Tx)) + \varphi_A(\text{dist}(A, B))$$



and

$$d(Ty', T^2y) \leq d(y', Ty) - \varphi_A(d(y', Ty)) + \varphi_A(\text{dist}(A, B))$$

for all  $(x, x') \in \mathcal{O}(x_0, T^2) \times \mathcal{O}(x_0, T^2)$ ,  $(y, y') \in \mathcal{O}(y_0, T^2) \times \mathcal{O}(y_0, T^2)$  with  $x \leq x'$ ,  $y \leq y'$ , and  $\varphi_A \in \Phi$ . If  $x_0 \leq T^2x_0$ , define  $x_{n+1} = Tx_n$ , then  $T$  has a best proximity point  $p \in A$  and  $x_{2n} \rightarrow p$ .

*Proof.* Since  $T^2$  and  $T$  are nondecreasing on  $\mathcal{O}(x_0, T^2)$  and  $x_0 \leq T^2x_0$ , it follows that  $\{T^{2n}x_0\}$  and  $\{T^{2n-1}x_0\}$  are nondecreasing. By Lemma 2.2, we have  $d(T^{2n}x_0, T^{2n+1}x_0) \rightarrow \text{dist}(A, B)$  and  $d(T^{2n+2}x_0, T^{2n+1}x_0) \rightarrow \text{dist}(A, B)$ . Since the pair  $(A, B)$  satisfies the property UC,  $d(T^{2n}x_0, T^{2n+2}x_0) \rightarrow 0$ . Fix  $\epsilon > 0$  and choose  $k \in \mathbb{N}$  such that for each  $m \geq k$  we have

$$d^*(T^{2m}x_0, T^{2m+1}x_0) < \epsilon, \quad d^*(T^{2m+2}x_0, T^{2m+1}x_0) < \epsilon,$$

and

$$d(T^{2m}x_0, T^{2m+2}x_0) < [1 - \{\epsilon - \varphi_A(\epsilon) + \varphi_A(\text{dist}(A, B)) - \varphi_A(\epsilon - \varphi_A(\epsilon) + \varphi_A(\text{dist}(A, B)))\}] \epsilon,$$

where  $d^*(a, b) := d(a, b) - \varphi_A(\text{dist}(A, B))$  for  $(a, b) \in A \times B$ . The above inequality can be justified for sufficiently small  $\epsilon$  because  $t - \varphi_A(t) \rightarrow 0$  as  $t \rightarrow 0$ . Now, fix  $m \in \mathbb{N}$  with  $m \geq k$ . We shall show by induction

$$d^*(T^{2m}x_0, T^{2n+1}x_0) < \epsilon \tag{3.1}$$

for all  $n \in \mathbb{N}$  with  $n \geq m$ . It is obvious that (3.1) holds when  $n = m$ . We assume that (3.1) holds for some  $n \geq m$ . Then we have

$$\begin{aligned} & d^*(T^{2m}x_0, T^{2n+3}x_0) \\ & \leq d(T^{2m}x_0, T^{2m+2}x_0) + d^*(T^{2m+2}x_0, T^{2n+3}x_0) \\ & < [1 - \{\epsilon - \varphi_A(\epsilon) + \varphi_A(\text{dist}(A, B)) - \varphi_A(\epsilon - \varphi_A(\epsilon) + \varphi_A(\text{dist}(A, B)))\}] \epsilon \\ & \quad + d(T^{2n+2}x_0, T^{2m+1}x_0) - \varphi_A(d(T^{2n+2}x_0, T^{2m+1}x_0)) \\ & \leq [1 - \{\epsilon - \varphi_A(\epsilon) + \varphi_A(\text{dist}(A, B)) - \varphi_A(\epsilon - \varphi_A(\epsilon) + \varphi_A(\text{dist}(A, B)))\}] \epsilon \\ & \quad + d(T^{2m}x_0, T^{2n+1}x_0) - \varphi_A(d(T^{2m}x_0, T^{2n+1}x_0)) + \varphi_A(\text{dist}(A, B)) \\ & \quad - \varphi_A(d(T^{2m}x_0, T^{2n+1}x_0) - \varphi_A(d(T^{2m}x_0, T^{2n+1}x_0)) + \varphi_A(\text{dist}(A, B))) \\ & \leq [1 - \{\epsilon - \varphi_A(\epsilon) + \varphi_A(\text{dist}(A, B)) - \varphi_A(\epsilon - \varphi_A(\epsilon) + \varphi_A(\text{dist}(A, B)))\}] \epsilon \\ & \quad + \epsilon - \varphi_A(\epsilon) + \varphi_A(\text{dist}(A, B)) - \varphi_A(\epsilon - \varphi_A(\epsilon) + \varphi_A(\text{dist}(A, B))) \\ & \leq \epsilon \end{aligned}$$

and so (3.1) holds when  $n := n + 1$ . Hence by induction, (3.1) holds for all  $n \in \mathbb{N}$  with  $n \geq k$ . This, in turn, implies that

$$\limsup_{m \rightarrow \infty} \sup_{n \geq m} d^*(T^{2m}x_0, T^{2n+3}x_0) = 0.$$

Since  $(A, B)$  satisfies the property UC, it follows from Lemma 3.3 that  $\{x_{2n}\}$  is a Cauchy sequence, and since  $\mathcal{O}(x_0, T^2)$  is  $T^2$ -orbitally complete, there exists

$p \in \mathcal{O}(x_0, T^2)$  such that  $x_{2n} \rightarrow p$ . Further, since  $T^2$  is nondecreasing on  $\mathcal{O}(x_0, T^2)$  and that the condition (1) holds, we conclude that  $x_{2n} \leq p$ . Therefore

$$\begin{aligned} d(p, Tp) &= \lim_{n \rightarrow \infty} d(T^{2n}x_0, Tp) = \lim_{n \rightarrow \infty} d(Tp, T^2(T^{2n-2}x_0)) \\ &= \lim_{n \rightarrow \infty} [d(p, T^{2n-1}x_0) - \varphi_A(d(p, T^{2n-1}x_0))] + \varphi_A(\text{dist}(A, B)) = \text{dist}(A, B). \end{aligned}$$

Hence  $p$  is a best proximity point of  $T$  in  $A$ , and  $T^{2n}x_0 \rightarrow p$ .

**Example 3.1.** Let  $X = \mathbb{R}^2$  be endowed with metric  $d$  as defined in Example 2.1. Let

$$\begin{aligned} A &= \{(1, -a) \in \mathbb{R}^2 : 0 \leq a < 1\} \cup \{(1, -2)\}, \\ B &= \{(-1, b) \in \mathbb{R}^2 : 0 \leq b < 1\} \cup \{(-1, 2)\}. \end{aligned}$$

Then  $\text{dist}(A, B) = 2$ . Define  $T : A \cup B \rightarrow A \cup B$  by  $T(1, -a) = (-1, \frac{a}{3})$  for  $0 \leq a < 1$ ,  $T(1, -2) = (-1, \frac{2}{3})$ ,  $T(-1, b) = (1, -\frac{b}{3})$  for  $0 \leq b < 1$ ,  $T(-1, 2) = (1, -\frac{2}{3})$ , and  $\varphi_A : [0, +\infty) \rightarrow [0, +\infty)$  by  $\varphi_A(t) = \frac{2t}{3}$  for all  $t \in [0, +\infty)$ . Then  $T$  is a cyclic  $\varphi_A$ -contraction mapping. We define the partial order relation " $\leq$ " on  $\mathcal{O}(x_0, T^2)$  in the following way:

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq x_2, y_1 \leq y_2,$$

for all  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ . Pick  $x_0 = (1, -2) \in A$ , then  $T^2$ -orbitally complete such that  $\mathcal{O}(x_0, T^2)$  satisfies the condition (1) and  $\mathcal{O}(x_0, T^2) = \{(1, -\frac{2}{3^{2n}}) : n = 0, 1, 2, \dots\}$ . We observe that, for  $x = (1, -\frac{2}{3^{2n}})$  and  $x' = (1, -\frac{2}{3^{2m}})$  with  $n \geq m$  i.e.,  $x \leq x'$ ,

$$\begin{aligned} d(Tx', T^2x) &= d\left(-1, \frac{2}{3^{2m+1}}, 1, -\frac{2}{3^{2n+2}}\right) = \max\{|-1-1|, |\frac{2}{3^{2m+1}} + \frac{2}{3^{2n+2}}|\} \\ &= 2 = 2 - \frac{4}{3} + \frac{4}{3} \\ &= d\left(1, -\frac{2}{3^{2m}}, -1, \frac{2}{3^{2n+1}}\right) - \varphi_A\left(d\left(1, -\frac{2}{3^{2m}}, -1, \frac{2}{3^{2n+1}}\right)\right) \\ &\quad + \varphi_A(\text{dist}(A, B)) \\ &= d(x', Tx) - \varphi_A(d(x', Tx)) + \varphi_A(\text{dist}(A, B)) \end{aligned}$$

Thus we see that  $T$  satisfies all the conditions of Theorem 3.4. Now, if  $x_{n+1} = T(x_n)$ , then  $x_0 \leq T^2x_0$  and  $\{x_{2n}\}, \{x_{2n-1}\}$  are nondecreasing on  $\mathcal{O}(x_0, T^2)$ . Therefore,  $T$  has a best proximity point. Clearly this point is  $p = (1, 0)$ .

**Example 3.2.** Let  $X = \mathbb{R}^2$  be endowed with Euclidean metric  $d$ . Let

$$A = \{(1, a) \in \mathbb{R}^2 : -1 \leq a < 1\}, \quad B = \{(-1, b) \in \mathbb{R}^2 : -1 < b \leq 1\}.$$

Then  $\text{dist}(A, B) = 2$ . Define  $T : A \cup B \rightarrow A \cup B$  by  $T(1, a) = (-1, -\frac{a}{2})$  for all  $-1 \leq a < 1$ ,  $T(-1, b) = (1, -\frac{b}{2})$  for all  $-1 < b \leq 1$ , and  $\varphi_A : [0, +\infty) \rightarrow [0, +\infty)$

by  $\varphi_A(t) = \frac{t}{t+1}$  for all  $t \in [0, +\infty)$ . Then  $T$  is a cyclic  $\varphi_A$ -contraction mapping. We define the partial order relation “ $\leq$ ” on  $\mathcal{O}(x_0, T^2)$  in the following way:

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq x_2, y_1 \leq y_2,$$

for all  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ . Pick  $x_0 = (1, 0) \in A$ , then  $T^2$ -orbitally complete such that  $\mathcal{O}(x_0, T^2)$  satisfies the condition (1) and  $\mathcal{O}(x_0, T^2) = \{(1, 0)\}$ . We observe that, for any  $x, x' \in \mathcal{O}(x_0, T^2)$  and  $x \leq x'$ ,

$$\begin{aligned} d(Tx', T^2x) &= d\left((-1, 0), (1, 0)\right) = 2 = 2 - \frac{2}{2+1} + \frac{2}{2+1} \\ &= d\left((1, 0), (-1, 0)\right) - \varphi_A\left(d\left((1, 0), (-1, 0)\right)\right) + \varphi_A(\text{dist}(A, B)) \\ &= d(x', Tx) - \varphi_A(d(x', Tx)) + \varphi_A(\text{dist}(A, B)) \end{aligned}$$

Thus we see that  $T$  satisfies conditions of Theorem 3.4. Now, if  $x_{n+1} = T(x_n)$ , then  $x_0 \leq T^2x_0$  and  $\{x_{2n}\}, \{x_{2n-1}\}$  are nondecreasing on  $\mathcal{O}(x_0, T^2)$ . Therefore,  $T$  has a best proximity point. Clearly this point is  $p = (1, 0)$ .

The following concept [2] is a proper generalization of nontrivial weakly compatible mappings which do have a coincidence point.

**Definition 3.2.** Two self-mappings  $S$  and  $T$  of a set  $X$  are occasionally weakly compatible (owc) iff there is a point  $u$  in  $X$  which is a coincidence point of  $S$  and  $T$  at which  $S$  and  $T$  commute.

Let  $A, B$  be two nonempty subsets of a metric space  $(X, d)$ ,  $S, T : A \cup B \rightarrow A \cup B$  be mappings and let  $C_A(S, T), PC_B(S, T)$  denote the set of coincidence points and the set of points of coincidence of mappings  $S$  and  $T$  in  $A$  and in  $B$ , respectively.

**Definition 3.3.** A point  $x \in A \cup B$  is called a common best proximity point for  $S$  and  $T$  if  $d(x, Sx) = d(x, Tx) = \text{dist}(A, B)$ .

**Theorem 3.5.** Let  $S, T : A \cup B \rightarrow A \cup B$  be cyclic mappings and occasionally weakly compatible satisfying the condition

$$\varphi_A(d^*(Sx, Ty)) \leq d(x, Ty) - \varphi_A(d(x, Ty)) \tag{3.2}$$

for all  $x \in B, y \in A$  and  $\varphi_A \in \Phi$ . Then  $\exists y' \in PC_B(S, T)$  such that  $d(y', Sy') = d(y', Ty') = \text{dist}(A, B)$ . Further, if  $\text{dist}(A, B) = 0$ , then  $y'$  is a unique common fixed point of  $S$  and  $T$ .

*Proof.* Since  $S, T$  are occasionally weakly compatible, it follows that  $\exists u \in A$  such that  $Su = Tu$ . Now (3.2) yields

$$\varphi_A(d^*(S^2u, Tu)) \leq d(Su, Tu) - \varphi_A(d(Su, Tu)) = 0.$$

Thus  $\varphi_A(d^*(S^2u, Su)) = 0$ . Since  $\varphi_A$  is strictly increasing, it follows that  $d^*(S^2u, Su) = 0$  i.e.,  $\exists y' (= Su) \in PC_B(S, T)$  such that  $d(y', Sy') = \text{dist}(A, B)$ .

Again, by owc of the pair, we have  $STu = TSu$  i.e.,  $Sy' = Ty'$ . It follows that  $d(y', Sy') = d(y', Ty') = \text{dist}(A, B)$ . Hence  $y'$  a common best proximity point for  $S$  and  $T$ .

Assume that  $\text{dist}(A, B) = 0$ , then it is obvious that  $y'$  is a common fixed point for  $S$  and  $T$ . This completes the proof.

Notice that in Theorem 3.5, without affecting the conclusion, one can replace the condition (3.2) with the following condition:

$$d^*(Sx, Ty) \leq d(x, Ty) - \varphi_A(d(x, Ty)). \quad (3.2')$$

Our next example validates all the conditions of Theorem 3.5.

**Example 3.3.** Let  $X = \mathbb{R}^2$  be endowed with metric  $d$  as defined in Example 2.1. Let

$$A = \{(1, a) \in \mathbb{R}^2 : -1 \leq a \leq 1\}, \quad B = \{(-1, b) \in \mathbb{R}^2 : -1 \leq b \leq 1\}.$$

Then  $\text{dist}(A, B) = 2$ . Define  $S, T : A \cup B \rightarrow A \cup B$  by  $S(1, a) = (-1, -\frac{a}{4})$  for all  $-1 \leq a \leq 1$ ,  $S(-1, b) = (1, -\frac{b}{4})$  for all  $-1 \leq b \leq 1$ ,  $T(1, a) = (-1, -\frac{a}{2})$  for all  $-1 \leq a \leq 1$ ,  $T(-1, b) = (1, -\frac{b}{2})$  for all  $-1 \leq b \leq 1$ , and  $\varphi_A : [0, +\infty) \rightarrow [0, +\infty)$  by  $\varphi_A(t) = \frac{t}{2}$  for all  $t \in [0, +\infty)$ . Then  $S, T$  are cyclic mappings. Clearly,  $C_A(S, T) = \{(1, 0)\}$ . Now, for any  $(-1, b) \in B$  and  $(1, a) \in A$ , we have

$$\begin{aligned} \varphi_A(d^*(S(-1, b), T(1, a))) &= \varphi_A(d(S(-1, b), T(1, a)) - \text{dist}(A, B)) \\ &= \varphi_A(0) = 0 \leq \frac{1}{2}d((-1, b), T(1, a)) \\ &= d((-1, b), T(1, a)) - \varphi_A(d((-1, b), T(1, a))). \end{aligned}$$

Therefore, all the conditions of Theorem 3.5 are satisfied, and hence  $\exists y' = (-1, 0)$  ( $= S(1, 0)$ )  $\in PC_B(S, T)$  such that  $d(y', Sy') = d(y', Ty') = \text{dist}(A, B)$ .

**Remark 3.2.** It may be remarked that

(i) Nieto and Rodriguez-Lopez [7] proved that every continuous and nondecreasing contraction on a complete partially ordered metric space has a fixed point (see Theorem 1.1). This result was generalized by Abkar and Gabeleh [1] to cyclic contractions which requires completeness of  $A$  only. We have further generalized this theorem to cyclic  $\varphi_A$ -contractions under much weaker condition. Indeed, we have used  $T^2$ -orbital completeness on  $\mathcal{O}(x_0, T^2)$  for some  $x_0$  in  $A$  rather than completeness of  $A$  as used by Abkar and Gabeleh [1].

(ii) Theorem 1.2 of Nieto and Rodriguez-Lopez [7] states that every monotone and nondecreasing contraction has a fixed point. This result was extended to cyclic contractions by Abkar and Gabeleh [1]. We further extended the corresponding result of Abkar and Gabeleh [1] to  $\varphi_A$ -contractions under much weaker condition on the space.

(iii) In section 3, we have established several theorems on the existence and convergence of best proximity points for cyclic  $\varphi$ -contractions under much weaker condition on the space. These results improve and extend corresponding results of Abkar and Gabeleh [1] from cyclic mappings to cyclic  $\varphi_A$ -contraction mappings.

(iv) In Examples 3.1 and 3.2 above neither  $A$  nor  $B$  is complete. Hence our Theorem 3.4 is a proper generalization of the corresponding result i.e., Theorem 3.5 of Abkar and Gabeleh [1].

(v) Our Examples 3.3 appreciably exhibits a close relationship between points of coincidence and common best proximal points.

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