Optimal control of a fourth-order parabolic equation modeling epitaxial thin film growth

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Abstract

In this paper, for the fourth-order parabolic equation modeling epitaxial thin film growth, the optimal control under boundary condition is given, and the existence of optimal solution to the equation is proved.

1 Introduction

In this paper, we study the optimal control problem of the following fourth-order parabolic equation

$$u_t + D^4 u - D\varphi(Du) = 0, \ (x,t) \in (0,1) \times (0,T), \tag{1.1}$$

where $D = \frac{\partial}{\partial x}$, $\varphi(s) = s^3 - s$. On the basis of physical considerations, Eq.(1.1) is supplemented by the following boundary conditions and initial condition

$$Du(x,t) = D^3 u(x,t) = 0, \ x = 0, 1, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in (0,1). \tag{1.3}$$

Eq.(1.1) arises in epitaxial growth of nanoscale thin films [2, 11], where $u(x,t)$ denotes the height from the surface of the film in epitaxial growth. The term $D^4 u$ denotes the capillarity-driven surface diffusion, and $D\varphi(Du)$ correspond to the upward hopping of atoms.

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In 2003, King, Stein and Winkler [2] studied the following equation

$$u_t + \Delta^2 u - \operatorname{div}\big(f(\nabla u)\big) = h(x).$$

In their paper, they proved the existence, uniqueness and regularity of solutions in an appropriate function space for the initial boundary value problem. Kohn and Yan [5] considered the following equation

$$u_t + \Delta^2 u + \nabla \cdot \big(2(1-|\nabla u|^2)\nabla u\big) = 0,$$

in two space dimensions. Numerical simulations and heuristic arguments indicate that the standard deviation of u grows like $t^{-\frac{1}{3}}$, and the energy per unit area decays like $t^{-\frac{1}{3}}$. There is much literature concerned with the fourth-order parabolic equation modeling epitaxial thin film growth. For more recent results we refer the reader to [1, 4] and the references therein.

The optimal control plays an important role in modern control theories, and has a wider application in modern engineering. Many papers have already been published to study the control problems of nonlinear parabolic equations. In 1991, Yong and Zheng [10] studied the feedback stabilization and optimal control of the Cahn-Hilliard equation in a bounded domain with smooth boundary. Tian et al [8, 9] considered the optimal control problems for parabolic equations, such as viscous Camassa-Holm equation, viscous Degasperis-Procesi equation and so on. There are also many papers were denoted to the optimal control problem, for example [6, 7, 12] and so on.

In this paper, suppose that $Q_0 \subseteq Q = (0,1)\times(0,T)$, $C \in \mathcal{L}(W(0,T;V), L^2(Q_0))$ is the observer, we are concerned with distributed optimal control problem

$$\min J(u,\bar{\omega}) = \frac{1}{2}\int_{Q_0}(Cu - z_d)^2 dxdt + \frac{\delta}{2}\int_{Q_0}|\bar{\omega}|^2 dxdt, \tag{1.4}$$

subject to

$$\begin{cases} u_t + D^4 u - D\varphi(Du) = B\bar{\omega}, \quad (x,t) \in \Omega\times(0,T), \\ Du(0,t) = Du(1,t) = D^3u(0,t) = D^3u(1,t) = 0, \\ u(0) = u_0. \end{cases} \tag{1.5}$$

The control target is to match the given desired state z_d in L^2-sense by adjusting the body force $\bar{\omega}$ in a control volume Q_0 in the L^2-sense.

Assume that $H = L^2(0,1)$, $U = \{u \in H^1(0,1); \int_0^1 u dx = 0\}$ and

$$V = \{u \in H^2(0,1) \big| Du(0,t) = Du(1,t) = 0, \int_0^1 u dx = 0\}.$$

Assume further V', U' and H' are dual spaces of V, U and H. Then,

$$V \hookrightarrow U \hookrightarrow H = H' \hookrightarrow U' \hookrightarrow V',$$

each embedding being dense. The extension operator $B \in \mathcal{L}(L^2(Q_0), L^2(0,T;H))$ which is called the controller is introduced as

$$Bq = \begin{cases} q, & q \in Q_0, \\ 0, & q \in Q \setminus Q_0. \end{cases}$$

We supply H with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, and define a space $W(0,T;V)$ as

$$W(0,T;V)=\Big\{y: y\in L^2(0,T;V),\ y_t\in L^2(0,T;V')\Big\},$$

which is a Hilbert space endowed with common inner product.

This paper is organized as follows. In the next section, we prove the existence and uniqueness of weak solution to the equation in a special space. We also discuss the relation among the norms of weak solution, initial value and control item. In section 3, we consider the optimal control problem and prove the existence of optimal solution. Finally in Section 4, conclusions are obtained.

2 Existence and uniqueness of weak solution

In this section, we prove the existence and uniqueness of weak solution for the equation

$$\begin{cases} u_t+D^4u-D\varphi(Du)=B\bar{\omega}, \\ Du(0,t)=Du(1,t)=D^3u(0,t)=D^3u(1,t)=0, \\ u(x,0)=u_0(x), \end{cases} \tag{2.1}$$

where $x\in(0,1)$, $t\in(0,T)$, $B\bar{\omega}\in L^2(0,T;H)$ and a control $\bar{\omega}\in L^2(Q_0)$.

Now, we give the definition of the weak solution in the space $W(0,T;V)$.

Definition 2.1. For all $\eta\in V,\ t\in(0,T)$, a function $u(x,t)\in W(0,T;V)$ is called a weak solution to problem (2.1), if

$$(\frac{d}{dt}u,\eta)+(D^2u,D^2\eta)+\Big((Du)^3-Du,D\eta\Big)=(B\bar{\omega},\eta). \tag{2.2}$$

We shall give Theorem 2.2 on the existence and uniqueness of weak solution to problem (2.1).

Theorem 2.2. *Suppose $u_0\in V$, $B\bar{\omega}\in L^2(0,T;H)$, then problem (2.1) admits a unique weak solution $u(x,t)\in W(0,T;V)$.*

Proof. Galerkin method is applied to the proof.

Denote $\mathbb{A}=(-\partial_x^2)^2$ as a differential operator, let $\{\psi_i\}_{i=1}^{\infty}$ denote the eigenfunctions of the operator $\mathbb{A}=(-\partial_x^2)^2$. For $n\in N$, define the discrete ansatz space by

$$V_n=\operatorname{span}\{\psi_1,\psi_2,\cdots,\psi_n\}\subset V.$$

Let $u_n=\sum_{i=1}^n u_i^n(t)\psi_i(x)$ require $u_n(0,\cdot)\to u_0$ in H holds true.

By analyzing the limiting behavior of sequences of smooth function $\{u_n\}$, we can prove the existence of a weak solution to problem (2.1).

Performing the Galerkin procedure for (2.1), we obtain

$$u_{n,t}+D^4u_n-D\big((Du_n)^3-Du_n\big)=B\bar{\omega}, \tag{2.3}$$

with

$$Du_n(x,t) = D^3u_n(x,t) = 0, \ x = 0,1, \tag{2.4}$$

$$u_n(x,0) = u_{n,0}(x). \tag{2.5}$$

Eq.(2.3) is an ordinary differential equation and according to ODE theory, there exists a unique solution to Eq.(2.3) in the interval $[0,t_n)$. What we should do is to show that the solution is uniformly bounded when $t_n \to T$. We need also to show that the times t_n there are not decaying to 0 as $n \to \infty$.

Then, we shall prove the existence of solution in the following steps.

Step 1, multiplying Eq.(2.3) by u_n, integrating with respect to x on $(0,1)$, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2 + \|D^2u_n\|^2 + \|Du_n\|_4^4 = (B\bar{\omega}, u_n) + \|Du_n\|^2. \tag{2.6}$$

Noticing that

$$\|Du_n\|^2 = -\int_0^1 u_n D^2u_n dx \le \frac{1}{2}\|D^2u_n\|^2 + \frac{1}{2}\|u_n\|^2.$$

Therefore

$$\frac{d}{dt}\|u_n\|^2 + \|D^2u_n\|^2 \le 2(B\bar{\omega}, u_n) + \|u_n\|^2 \le 2\|u_n\|^2 + \|B\bar{\omega}\|^2.$$

Since $B\bar{\omega} \in L^2(0,T;H)$ is the control item, we can assume $\|B\bar{\omega}\| \le M$, where M is a positive constant. Then,

$$\frac{d}{dt}\|u_n\|^2 + \|D^2u_n\|^2 \le 2\|u_n\|^2 + M^2. \tag{2.7}$$

Using Gronwall's inequality, we obtain

$$\|u_n\|^2 \le e^{2t}\|u_{n,0}\|^2 + \frac{M^2}{2} \le e^{2T}\|u_{n,0}\|^2 + \frac{M^2}{2} = c_1, \ t \in [0,T]. \tag{2.8}$$

Multiplying Eq.(2.3) by D^2u_n, integrating with respect to x on $(0,1)$, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|Du_n\|^2 + \|D^3u_n\|^2 + (D((Du_n)^3 - Du_n), D^2u_n) = -(B\bar{\omega}, D^2u_n),$$

which means

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|Du_n\|^2 + \|D^3u_n\|^2 + (D((Du_n)^3), D^2u_n)\\ =&\|D^2u_n\|^2 - (B\bar{\omega}, D^2u_n).\end{aligned} \tag{2.9}$$

On the other hand, we have

$$\Big(D[(Du_n)^3], D^2u_n\Big) = 3\int_0^1 |Du_n|^2|D^2u_n|^2 dx \ge 0.$$

Therefore, by (2.9), we derive that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|Du_n\|^2+\|D^3u_n\|^2\\ \le&\|B\bar{\omega}\|\|D^2u_n\|+\|D^2u_n\|^2\le\frac{1}{4}M^2+2\|D^2u_n\|^2\\ \le&\frac{1}{4}M^2+2\|Du_n\|^2+\frac{1}{2}\|D^3u_n\|^2.\end{aligned}\tag{2.10}$$

That is

$$\frac{d}{dt}\|Du_n\|^2+\|D^3u_n\|^2\le\frac{1}{2}M^2+4\|Du_n\|^2.$$

Using Gronwall's inequality, we obtain

$$\begin{aligned}\|Du_n\|^2\le&e^{4t}\|Du_{n,0}\|^2+\frac{M^2}{8}\le e^{4T}\|Du_{n,0}\|^2+\frac{M^2}{8}\\ =&c'\|Du_{n,0}\|^2+c_3,\ t\in[0,T].\end{aligned}\tag{2.11}$$

Multiplying Eq.(2.3) by D^4u_n, integrating with respect to x on $(0,1)$, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|D^2u_n\|^2+\|D^4u_n\|^2+(D(Du_n-(Du_n)^3),D^4u_n)dx=(B\bar{\omega},D^4u_n),$$

that is

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|D^2u_n\|^2+\|D^4u_n\|^2\\ =&\|D^3u_n\|^2+(D((Du_n)^3),D^4u_n)+(B\bar{\omega},D^4u_n).\end{aligned}\tag{2.12}$$

On the other hand, by Nirenberg's inequality, we obtain

$$\|Du_n\|_8^8\le\|Du_n\|^7(\|D^4u_n\|^{\frac{1}{8}}+\|Du_n\|^{\frac{1}{8}})^8\le\varepsilon\|D^4u_n\|^2+c$$

and

$$\|D^2u_n\|_4^4\le\|Du_n\|^{\frac{7}{3}}(\|D^4u_n\|^{\frac{5}{12}}+\|Du\|^{\frac{5}{12}})^4\le\varepsilon\|D^4u_n\|^2+c.$$

Hence

$$\begin{aligned}(D((Du_n)^3),D^4u_n)=&3\int_0^1|Du_n|^2D^2u_nD^4u_ndx\\ \le&\varepsilon\|D^4u_n\|^2+c_1'\|Du_n\|_8^8+c_2'\|D^2u_n\|_4^4\\ \le&(1+c_1'+c_2')\varepsilon\|D^4u_n\|^2+c.\end{aligned}\tag{2.13}$$

We also have

$$\|D^3u_n\|^2=-\int_0^1D^2u_nD^4u_ndx\le\frac{1}{4}\|D^4u_n\|^2+\|D^2u_n\|^2$$

and

$$(B\bar{\omega},D^4u_n)\le\|B\bar{\omega}\|\|D^4u_n\|\le\frac{1}{4}\|D^4u_n\|^2+M^2.$$

Summing up, we obtain

$$\frac{d}{dt}\|D^2u_n\|^2 + (1 - 2(1 + c_1' + c_2')\varepsilon)\|D^4u_n\|^2 \leq 2\|D^2u_n\|^2 + 2c + 2M^2,$$

where ε is small enough, it satisfies $1 - 2(1 + c_1' + c_2'\varepsilon) > 0$. Using Gronwall's inequality, we deduce that

$$\begin{aligned} \|D^2u_n\|^2 \leq& e^{2t}\|D^2u_{n,0}\|^2 + c + M^2 \\ \leq& e^{2T}\|D^2u_{n,0}\|^2 + c + M^2 = c_4,\ t \in [0, T]. \end{aligned} \tag{2.14}$$

Using Sobolev's embedding theorem, we immediately obtain

$$\|Du_n\|_{L^\infty} \leq c_5. \tag{2.15}$$

Then, by (2.8), (2.11) and (2.14), we obtain

$$\|u_n(x,t)\|_{H^2}^2 \leq c. \tag{2.16}$$

Step 2, we prove a uniform $L^2(0, T; V')$ bound on a sequence $\{u_{n,t}\}$. Noticing that

$$\begin{aligned} &(D((Du_n)^3 - Du_n), \eta) = (Du_n, D\eta) - ((Du_n)^3, D\eta) \\ &\leq \|Du_n\|\|D\eta\| + \sup_{x\in[0,1]} |Du_n|^2 \cdot \|Du_n\|\|D\eta\| \leq c\|Du_n\|\|\eta\|_V, \\ &(B\bar{\omega}, \eta) \leq \|B\bar{\omega}\|\|\eta\| \leq \|B\bar{\omega}\|\|\eta\|_V. \end{aligned}$$

Therefore, by (2.15), we have

$$\|u_{n,t}\|_{V'} + \|D^4u_n\|_{V'} \leq \|B\bar{\omega}\| + c\|Du_n\| \leq M + c.$$

Hence, we get

$$\|u_{n,t}\|_{L^2(0,T;V)} \leq (M + c)T = c_6. \tag{2.17}$$

Adding (2.16) and (2.17) together gives

$$\|u_n(x,t)\|_{W(0,T;V)} \leq c.$$

Collecting the previous we get:

(1) For every $t \in [0, T]$, the sequence $\{u_n\}_{n\in N}$ is bounded in $L^2(0, T; V)$ by a constant which is independent of the dimension of ansatz space n.

(2) For every $t \in [0, T]$, the sequence $\{u_{n,t}\}_{n\in N}$ is bounded in $L^2(0, T; V')$ by a constant which is independent of the dimension of ansatz space n.

By the above discussion, we obtain $u(x,t) \in W(0, T; V)$. It's easy to check that $W(0, T; V)$ is continuously embedded into $C(0, T; U)$ which denote the space of continuous functions. We concludes convergence of a subsequences, again denoted by $\{u_n\}$ weak into $W(0, T; V)$, weak-star in $L^\infty(0, T; U)$ and strong in $L^2(0, T; U)$ to functions $u(x,t) \in W(0, T; V)$. Since the proof of uniqueness is easy, we omit it.

Then, Theorem 2.2 is proved. ∎

Now, we shall discuss the relation among the norm of weak solution and initial value and control item.

Theorem 2.3. *Suppose $B\bar{\omega} \in L^2(0,T;H)$, $u_0 \in V$, then there exists positive constants C' and C'' such that*

$$\|u\|^2_{W(0,T;V)} \le C'(\|u_0\|^2_V + \|\bar{\omega}\|^2_{L^2(Q_0)}) + C''. \tag{2.18}$$

Proof. Clearly, (2.18) means

$$\|u\|^2_{L^2(0,T;V)} + \|u_t\|^2_{L^2(0,T;V')} \le C'(\|u_0\|^2_V + \|B\bar{\omega}\|^2_{L^2(0,T;H)}) + C''. \tag{2.19}$$

Passing to the limit in (2.6), (2.9), (2.12), we have

$$\|u\|^2 \le c(\|u_0\|^2 + \|B\bar{\omega}\|^2),\ \|Du\|^2 \le c(\|Du_0\|^2 + \|B\bar{\omega}\|^2),$$

and

$$\|Du\|_{L^\infty} \le c_5,\ \|D^2u\|_{L^2(H)} \le c\|D^2u_0\|^2 + \|B\bar{\omega}\|^2 + c. \tag{2.20}$$

Hence

$$\|u\|^2_{L^2(H)} \le cT\|u_0\|^2 + c\|B\bar{\omega}\|^2_{L^2(H)}, \tag{2.21}$$

$$\|Du\|^2_{L^2(H)} \le cT\|Du_0\|^2 + c\|B\bar{\omega}\|^2_{L^2(H)}, \tag{2.22}$$

and

$$\|D^2u\|^2_{L^2(H)} \le cT\|D^2u_0\|^2 + \|B\bar{\omega}\|^2_{L^2(H)} + cT. \tag{2.23}$$

Adding (2.21), (2.22) and (2.23) together gives

$$\|u\|^2_{L^2(0,T;V)} \le c_7(\|B\bar{\omega}\|^2_{L^2(0,T;H)} + \|u_0\|^2_V) + c_8. \tag{2.24}$$

On the other hand, by (2.20), we have

$$\begin{aligned}&(D((Du)^3 - Du),\eta) = -(Du,D\eta) + ((Du)^3,D\eta)\\ &\le \|Du\|\|D\eta\| + \sup_{x\in[0,1]} |Du|^2 \cdot \|Du\|\|D\eta\| \le c\|Du\|\|\eta\|_V,\\ &(B\bar{\omega},\eta) \le \|B\bar{\omega}\|\|\eta\| \le \|B\bar{\omega}\|\|\eta\|_V.\end{aligned}$$

Therefore, by (2.1), we have

$$\|u_t\|_{V'} + \|D^4u\|_{V'} \le \|B\bar{\omega}\| + c\|Du\| \le c(\|B\bar{\omega}\| + (\|Du_0\|^2 + \|B\bar{\omega}\|^2)^{\frac{1}{2}}).$$

Hence, we get

$$\|u_t\|^2_{L^2(0,T;V')} \le c_9(\|B\bar{\omega}\|^2_{L^2(0,T;H)} + \|u_0\|^2_V). \tag{2.25}$$

By (2.24), (2.25) and the definition of extension operator B, we obtain (2.19). Then, Theorem 2.3 is proved. ∎

3 Optimal control problem

In this section, we consider the optimal control problem associated with the fourth-order parabolic equation and prove the existence of optimal solution basing on Lions' theory (see [3]).

In the following, we suppose $L^2(Q_0)$ is a Hilbert space of control variables and we also suppose $B \in \mathcal{L}(L^2(Q_0), L^2(0,T;H))$ is the controller and a control $\bar{\omega} \in L^2(Q_0)$, consider the following control system

$$\begin{cases} u_t + D^4u - D((Du)^3 - Du) = B\bar{\omega}, \\ Du(0,t) = Du(1,t) = D^3u(0,t) = D^3u(1,t) = 0, \\ u(0) = u_0, \quad x \in (0,1). \end{cases} \tag{3.1}$$

Here in (3.1), it is assume that $u_0 \in V$. By virtue of Theorem 2.2, we can define the solution map $\bar{\omega} \to u(\bar{\omega})$ of $L^2(Q_0)$ into $W(0,T;V)$. The solution u is called the state of the control system (3.1). The observation of the state is assumed to be given by Cu. Here $C \in \mathcal{L}(W(0,T;V), L^2(Q_0))$ is an operator, which is called the observer. The cost functional associated with the control system (3.1) is given by

$$J(u,\bar{\omega}) = \frac{1}{2}\int_{Q_0}(Cu - z_d)^2 dxdt + \frac{\delta}{2}\int_{Q_0}|\bar{\omega}|^2 dxdt, \tag{3.2}$$

where $z_d \in L^2(Q_0)$ is a desired state and $\delta > 0$ is fixed. An optimal control problem about problem (3.1) is

$$\min J(u,\bar{\omega}). \tag{3.3}$$

Let $X = W(0,T;V) \times L^2(Q_0)$ and $Y = L^2(0,T;V) \times H$. We define an operator $e = e(e_1, e_2) : X \to Y$, where

$$\begin{cases} e_1 = G = (\Delta^2)^{-1}(u_t + D^4u - D((Du)^3 - Du) - B\bar{\omega}), \\ e_2 = u(x,0) - u_0. \end{cases}$$

Here Δ^2 is an operator from V to V'. Then, we write (3.3) in following form

$$\min J(u,\bar{\omega}) \ \text{ subject to } \ e(u,\bar{\omega}) = 0.$$

Theorem 3.1. *Suppose $B\bar{\omega} \in L^2(0,T;H)$, $u_0 \in V$, then there exists an optimal control solution $(u^*, \bar{\omega}^*)$ to the problem (3.1).*

Proof. Suppose $(u,\bar{\omega})$ satisfy $e(u,\bar{\omega}) = 0$. In view of (3.2), we deduce that

$$J(u,\bar{\omega}) \geq \frac{\delta}{2}\|\bar{\omega}\|^2_{L^2(Q_0)}.$$

By Theorem 2.3, we obtain

$$\|u\|_{W(0,T;V)} \to \infty \ \text{ yields } \ \|\bar{\omega}\|_{L^2(Q_0)} \to \infty.$$

Therefore,

$$J(u,\bar{\omega}) \to \infty, \ \text{ when } \ \|(u,\bar{\omega})\|_X \to \infty. \tag{3.4}$$

As the norm is weakly lower semi-continuous, we achieve that J is weakly lower semi-continuous. Since for all $(u,\bar{\omega}) \in X$, $J(u,\bar{\omega}) \geq 0$, there exists $\lambda \geq 0$ defined by

$$\lambda = \inf\{J(u,\bar{\omega})|(u,\bar{\omega}) \in X,\ e(u,\bar{\omega}) = 0\},$$

which means the existence of a minimizing sequence $\{(u^n,\bar{\omega}^n)\}_{n\in N}$ in X such that

$$\lambda = \lim_{n\to\infty} J(u^n,\bar{\omega}^n) \text{ and } e(u^n,\bar{\omega}^n) = 0, \quad \forall n \in N.$$

From (3.4), there exists an element $(u^*,\bar{\omega}^*) \in X$ such that when $n \to \infty$,

$$u^n \to u^*, \text{ weakly}, \ u \in W(0,T;V), \tag{3.5}$$

$$\bar{\omega}^n \to \bar{\omega}^*, \text{ weakly}, \ \bar{\omega} \in L^2(Q_0). \tag{3.6}$$

Using (3.5), we get

$$\lim_{n\to\infty} \int_0^T (u_t^n(x,t) - u_t^*, \psi(t))_{V',V} dt = 0, \quad \forall \psi \in L^2(0,T;V). \tag{3.7}$$

Based on the definition of $W(0,T;V)$, we can derive that $Du^n \to Du^*$ strongly in $L^2(0,T;L^\infty)$ as $n \to \infty$. We can also deduce that $Du^n \to Du^*$ strongly in $C(0,T;H)$ when $n \to \infty$.

Since sequence $\{Du^n\}_{n\in N}$ converge weakly and $\{u^n\}$ is bounded in $W(0,T;V)$, based on the embedding theorem, we can obtain $\{Du^n\}_{L^2(0,T;L^\infty)}$ is also bounded. Because $Du^n \to Du^*$ strongly in $L^2(0,T;L^\infty)$ as $n \to \infty$, we know that $\|Du^*\|_{L^2(L^\infty)}$ is bounded too.

Using (3.6) again, we derive that

$$\left| \int_0^T \int_0^1 (B\bar{\omega} - B\bar{\omega}^*)\eta dx dt \right| \to 0, \quad n \to \infty, \ \forall \eta \in L^2(0,T;H). \tag{3.8}$$

By (3.5), we deduce that

$$\begin{aligned}
&\left| \int_0^T \int_0^1 (D((Du^n)^3 - (Du^*)^3)\eta dx dt \right| \\
=&\left| \int_0^T \int_0^1 ((Du^n)^3 - (Du^*)^3) D\eta dx dt \right| \\
=&\left| \int_0^T \int_0^1 (Du^n - Du^*)(|Du^n|^2 + Du^n Du^* + |Du^*|^2) D\eta dx dt \right| \\
\leq& \int_0^T \|(Du^n)^2 + Du^n Du^* + (Du^*)^2\|_{L^\infty} \|Du^n - Du^*\|_H \|D\eta\|_H dt \\
\leq& \|(Du^n)^2 + Du^n Du^* + (Du^*)^2\|_{L^2(L^\infty)} \|Du^n - Du^*\|_{C(H)} \|D\eta\|_{L^2(H)} \\
\to& 0, \quad n \to \infty, \ \forall \eta \in L^2(0,T;U),
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\left| \int_0^T \int_0^1 (D^2u^n - D^2u^*)\eta dx dt \right| = \left| \int_0^T \int_0^1 (u^n - u^*) D^2\eta dx dt \right| \\
\leq& \|u^n - u^*\|_{C(H)} \|\eta\|_{L^2(V)} \to 0, \quad n \to \infty, \ \forall \eta \in L^2(0,T;V).
\end{aligned} \tag{3.10}$$

Hence we have $u = u(\bar{\omega})$ and therefore

$$J(u,\bar{\omega}) \leq \lim_{n\to\infty} J(u^n,\bar{\omega}^n) = \lambda.$$

In view of the above discussions, we get

$$e_1(u^*,\bar{\omega}^*) = 0, \quad \forall n \in N.$$

Noticing that $u^* \in W(0,T;V)$, we derive that $u^*(0) \in H$. Since $u^n \to u^*$ weakly in $W(0,T;V)$, we can infer that $u^n(0) \to u^*(0)$ weakly when $n \to \infty$. Thus, we obtain

$$(u^n(0) - u^*(0),\eta) \to 0, \;\; n \to \infty, \;\; \forall \eta \in H,$$

which means $e_2(u^*,\bar{\omega}^*) = 0$. Therefore, we obtain

$$e(u^*,\bar{\omega}^*) = 0, \quad \text{in } Y.$$

So, there exists an optimal solution $(u^*,\bar{\omega}^*)$ to problem (3.1).

Then, Theorem 3.1 is proved. ∎

4 Conclusions

The fourth-order parabolic equation in epitaxial growth of nanoscale equation is an important mathematical physical equation that has many practical meanings. Because of the complexity of nonlinear parts of the equation, there has been no research on the optimal control and boundary control of this equation. In this paper, we study the distributed optimal control problem for problem (1.1)-(1.3) using a series of mathematical estimates. Our research is motivated by the study of the optimal control problem for the viscous Degasperis-Procesi equation, viscous Camassa-Holm equation [8, 9], and the existence theory of optimal control of distributed parameter systems. We also prove the existence of an optimal solution to problem (1.1)-(1.3). In order to realize optimal solutions of optimal control problems in practice one must be able to recompute the optimal solutions in the presence of disturbances in real time unless one gives up optimality. We will use mathematical theory and related numerical methods to solve that problem numerically, which is our intention in the future.

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