Approximation in compact balls by convolution operators of quaternion and paravector variable

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Abstract

Attaching to a compact ball $\overline{\mathbb{B}_r}$ in the quaternion field $\mathbb{H}$ and to analytic functions in Weierstrass sense (slice regular functions on $\overline{\mathbb{B}_r}$) some convolution operators, the exact orders of approximation in $\overline{\mathbb{B}_r}$ for these operators are obtained. The results in this paper extend to quaternionic variables those in the case of approximation of analytic functions of a complex variable in disks by convolution operators of a complex variable, extensively studied in the Chapter 3 of the book [5]. More in general, the results extend also to the setting of analytic functions of paravector variable with coefficients in a Clifford algebra.

1 Introduction and preliminaries

The noncommutative field $\mathbb{H}$ of quaternions consists of elements of the form $q = x_1 + x_2 i + x_3 j + x_4 k$, $x_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, where the imaginary units $i, j, k \notin \mathbb{R}$ satisfy

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

Since, obviously, $\mathbb{C} \subset \mathbb{H}$, it extends the class of complex numbers. On $\mathbb{H}$ we consider the norm $\|q\| = \sqrt{x_1^2 + x_2^2 + x_3^3 + x_4^2}$.

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Let us denote by $\mathbb{S}$ the unit sphere of purely imaginary quaternion, i.e.

$$\mathbb{S} = \{q = ix_1 + jx_2 + kx_3, \text{ such that } x_1^2 + x_2^2 + x_3^3 = 1\}.$$

Note that if $I \in \mathbb{S}$, then $I^2 = -1$. For this reason the elements of $\mathbb{S}$ are also called imaginary units. For any fixed $I \in \mathbb{S}$ we define $\mathbb{C}_I := \{x + Iy; \mid x, y \in \mathbb{R}\}$. It is immediate that $\mathbb{C}_I$ can be identified with a complex plane, moreover $\mathbb{H} = \bigcup_{I\in\mathbb{S}} \mathbb{C}_I$. The real axis belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$ and thus a real quaternion can be associated to any imaginary unit I. Any non real quaternion q is uniquely associated to the element $I_q \in \mathbb{S}$ defined by $I_q := (ix_1 + jx_2 + kx_3)/\|ix_1 + jx_2 + kx_3\|$ and q belongs to the complex plane $\mathbb{C}_{I_q}$.

For our purposes we need some suitable concepts of analyticity of functions of a quaternion variable.

Definition 1.1. ([10, Definition 1.1]) Let U be an open set in $\mathbb{H}$ and let $f : U \to \mathbb{H}$ be real differentiable. f is called *left slice regular* if for every $I \in \mathbb{S}$, its restriction f_I to the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ satisfies

$$\overline{\partial}_I f(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0, \quad \text{on } U \cap \mathbb{C}_I.$$

In this case, the so called left (slice) I-derivative of f at a point $q = x + Iy$ is given by $\partial_I f_I(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) - I\frac{\partial}{\partial y} f_I(x + Iy)\right)$.

Analogously, one can give the notion of function right slice regular and its right I-derivative, see [3]. Let us now introduce a suitable notion of derivative:

Definition 1.2. Let U be an open set in $\mathbb{H}$, and let $f : U \to \mathbb{H}$ be a slice regular function. The slice derivative $\partial_s f$ of f is defined by:

$$\partial_s(f)(q) = \begin{cases} \partial_I(f)(q) & \text{if } q = x + Iy,\ y \neq 0, \\ \dfrac{\partial f}{\partial x}(x) & \text{if } q = x \in \mathbb{R}. \end{cases}$$

The definition of slice derivative is well posed because it is applied only to slice regular functions and thus $\frac{\partial}{\partial x} f(x + Iy) = -I\frac{\partial}{\partial y} f(x + Iy)$, for all $I \in \mathbb{S}$, and therefore, analogously to what happens in the complex case, $\partial_s(f)(x + Iy) = \partial_I(f)(x + Iy) = \partial_x(f)(x + Iy)$. If f is a slice regular function, then also its slice derivative is slice regular, in fact $\overline{\partial}_I(\partial_s f(x + Iy)) = \partial_s(\overline{\partial}_I f(x + Iy)) = 0$, and therefore

$$\partial_s^n f(x + Iy) = \frac{\partial^n f}{\partial x^n}(x + Iy).$$

We have the following result:

Theorem 1.3. *([10, Theorem 2.7]) Let $\mathbb{B}_R = \{q \in \mathbb{H}; \|q\| < R\}$. A function $f : \mathbb{B}_R \to \mathbb{H}$ is left slice regular on $\mathbb{B}_R$ if and only if it has a series representation of the form*

$$f(q) = \sum_{n=0}^{\infty} q^n \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(0),$$

uniformly convergent on $\mathbb{B}_R$.

Analogously, f is right slice regular if and only if it has a series representation with the coefficients written on the left.
The equivalence established in Theorem 1.3, i.e. the fact that slice regular functions can be expanded into power series, holds just when the domain of a slice regular function is a ball with center at a real point. Indeed, the function $f(q) = (q - q_0)^n$ is not slice regular when $q_0 \in \mathbb{H} \setminus \mathbb{R}$. However, on balls with center at the origin, as an alternative to Definition 1.1, we can present the following.

Definition 1.4. (see [6], [7]) One says that $f : \mathbb{B}_R \to \mathbb{H}$ is left analytic in the Weierstrass sense in $\mathbb{B}_R$ (shortly *left W-analytic*) if $f(q) = \sum_{k=0}^{\infty} c_k q^k$, for all $q \in \mathbb{B}_R$, where $c_k \in \mathbb{H}$ for all $k = 0, 1, 2, ...,$. Also, f is called *right W-analytic* in $\mathbb{B}_R$ if $f(q) = \sum_{k=0}^{\infty} q^k c_k$, for all $q \in \mathbb{B}_R$.

Here the convergence of the partial sums $\sum_{k=0}^{n} c_k q^k$ and $\sum_{k=0}^{n} q^k c_k$ to f is understood uniformly in any closed ball $\overline{\mathbb{B}_r} = \{q \in \mathbb{H}; \|q\| \leq r\}$, $0 < r < R$, with respect to the metric $d(x, y) = \|x - y\|$.

If $f(q) = \sum_{k=0}^{n} c_k q^k$ ($f(q) = \sum_{k=0}^{n} q^k c_k$) then f is called left (respectively right) polynomial of degree $\leq n$.

Remark 1.5. As we have already discussed, on balls with center at real points the concept of right W-analytic function f coincides with that of left slice regular function, and the concept of left W-analytic function coincides with that of right slice regular function. Although there is this equivalence, for the purpose of this paper, the terminology of W-analytic function is more suitable than that of slice regular function. However, in our proofs we will use some general results on slice regular functions. In the sequel, in order to avoid redundances, we will always write slice regular, W-analytic and polynomial, instead of left slice regular, right W-analytic, right polynomial, respectively.

Among the useful tools from the general theory on slice regular functions, we recall the Cauchy theorem and the formula to compute the derivatives, see [3, Theorems 4.5.3, 4.5.4] and the definition of axially symmetric s-domain, see [3, Definitions 4.1.4, 4.3.1]. For our purposes, it is enough to know that balls in $\mathbb{H}$ are examples of axially symmetric s-domains.

In Chapter 3 of the recent book [5], are introduced and studied the approximation properties of complex convolution operators acting on analytic functions in compact disks with center at the origin. The operators considered are of the form

$$T_n(f)(z) = \alpha_n \int_a^b f(ze^{iu}) K_n(u) du, \quad \text{(here } i^2 = -1\text{)}, \tag{1.1}$$

where usually $a = -\pi$, $b = \pi$ (or equivalently $a = 0$, $b = 2\pi$) and $K_n(u)$ is a positive, even, trigonometric kernel (i.e. a trigonometric polynomial), or $a = -\infty$, $b = +\infty$ and $K_n(u)$ is a positive, continuous kernel. Here $\alpha_n > 0$ is a constant that may depend on n, but it is independent of f and it is chosen such that $T_n(e_0)(z) = 1$, for all z, where $e_0(z) = 1$, for all z.

The main purpose of this paper is to make a similar study for the approximation of W-analytic functions by convolution operators of a quaternion variable. We will show how to generalize some definitions and results that hold in the complex case to the non commutative setting of quaternions and Clifford algebras.

The plan of the paper is the following: in section 2 we will introduce some preliminaries and the convolutions operators of quaternion variable. In sections 3 and 4 we will discuss the case of approximation properties of the quaternionic convolution based on the de La Vallée Poussin kernel and of the Gauss-Weierstrass kernel, respectively. Finally, in section 5 we discuss how to further generalize these results to the setting of W-analytic functions of a paravector variable and with values in a Clifford algebra.

2 Convolution operators of a quaternion variable

To introduce convolution operators of a quaternion variable, we need a suitable exponential function of quaternion variable. For any $I \in \mathbb{S}$, we choose the following well-known definition for the exponential: $e^{It} = \cos(t) + I\sin(t),\ t \in \mathbb{R}$, see [11]. The Euler's kind formula holds : $(\cos(t) + I\sin(t))^k = \cos(kt) + I\sin(kt)$, and therefore we can write $[e^{It}]^k = e^{Ikt}$.

For any $q \in \mathbb{H} \setminus \mathbb{R}$, let $r := \|q\|$; then, see [11], there exists a unique $a \in (0, \pi)$ such that $\cos(a) := \frac{x_1}{r}$ and a unique $I_q \in \mathbb{S}$, such that

$$q = re^{I_q a},\ \text{ with } I_q = iy + jv + ks,\ y = \frac{x_2}{r\sin(a)},\ v = \frac{x_3}{r\sin(a)},\ s = \frac{x_4}{r\sin(a)}.$$

Now, if $q \in \mathbb{R}$, then we choose $a = 0$, if $q > 0$ and $a = \pi$ if $q < 0$, and as I_q we choose an arbitrary fixed $I \in \mathbb{S}$. So that if $q \in \mathbb{R} \setminus \{0\}$, then again we can write $q = \|q\|(\cos(a) + I\sin(a))$ (but with a non unique I). The above is called the trigonometric form of the quaternion number $q \neq 0$. For $q = 0$ we do not have a trigonometric form for q (exactly as in the complex case). Analogously to the case of complex variable in the formula (1.1), we can introduce the following.

Definition 2.1. Let $K_n(u)$ and α_n be under the hypothesis in the formula (1.1).

If $f : \mathbb{B}_R \to \mathbb{H}$ is W-analytic on $\mathbb{B}_R$, then we can define the right convolution operator of quaternion variable

$$T_{n,r}(f)(q) = \alpha_n \int_a^b f(qe^{I_q u})K_n(u)du, \qquad q \in \mathbb{H} \setminus \mathbb{R},\ q = re^{I_q t} \in \mathbb{B}_R,$$

$$T_{n,r}(f)(q) = \alpha_n \int_a^b f(qe^{Iu})K_n(u)du, \qquad q \in \mathbb{R} \setminus \{0\}, q = \|q\|e^{It} \in \mathbb{B}_R, t = 0 \text{ or } \pi, \tag{2.1}$$

where $I \in \mathbb{S}$ is fixed (arbitrary), and $T_{n,r}(f)(0) = \alpha_n f(0) \int_a^b K_n(u)du$.

If $f : \mathbb{B}_R \to \mathbb{H}$ is left W-analytic on $\mathbb{B}_R$, then we can define in an analogous way the left convolution operator of quaternion variable by taking $f(e^{I_q u}q)$ instead of $f(qe^{I_q u})$ in the integrals (2.1).

The integral in (2.1), for example, is understood to be in Riemann sense and it is of the form

$$T_{n,r}(f)(q) = \int_a^b P_n du + i\int_a^b Q_n du + j\int_a^b R_n du + k\int_a^b S_n du,$$

where $P_n, Q_n R_n, S_n$ are real valued functions of the variables x_1, x_2, x_3, x_4, u, at least continuous and

$$\alpha_n f(qe^{I_q u})K_n(u) := P_n(x_1, x_2, x_3, x_4, u) + iQ_n(x_1, x_2, x_3, x_4, u) \\ + jR_n(x_1, x_2, x_3, x_4, u) + kS_n(x_1, x_2, x_3, x_4, u).$$

We have :

Theorem 2.2. *Let $K_n(u)$ be a positive and even trigonometric kernel, $a = -\pi$, $b = +\pi$ and $f : \mathbb{B}_R \to \mathbb{H}$. If f is W-analytic, that is $f(q) = \sum_{k=0}^{\infty} q^k c_k$, then $T_{n,r}(f)(q)$ is a W-analytic function given by the formula*

$$T_{n,r}(f)(q) = \sum_{k=0}^{\infty} q^k c_k A_{k,n},\ q \in \mathbb{B}_R,$$

where $A_{k,n} = \alpha_n \int_{-\pi}^{\pi} \cos(ku)K_n(u)du \in \mathbb{R}$, $k = 0, 1, ...,$. An analogous statement holds when f is left W-analytic.

Proof. (i) Suppose first that q is not real. Since the kernel $K_n(u)$ is real valued, by the trigonometric form $q = r(\cos(\alpha) + I_q \sin(\alpha))$, we easily get

$$\alpha_n f(qe^{I_q u})K_n(u) = \alpha_n \sum_{k=0}^{\infty} [r(\cos(\alpha + u) + I_q \sin(\alpha + u))]^k c_k K_n(u) \\ = \alpha_n \sum_{k=0}^{\infty} r^k \cos(k(\alpha + u))K_n(u)c_k + I_q \alpha_n \sum_{k=0}^{\infty} r^k \sin(k(\alpha + u))K_n(u)c_k.$$

Since both of the last two series evidently are uniformly and absolutely convergent with respect to the real variable u, they can be integrated term by term, so that finally it easily follows

$$T_{n,r}(f)(q) = \sum_{k=0}^{\infty} r^k \left[\alpha_n \int_{-\pi}^{\pi} \cos(k(\alpha + u))K_n(u)du\right] c_k \\ + I_q \sum_{k=0}^{\infty} r^k \left[\alpha_n \int_{-\pi}^{\pi} \sin(k(\alpha + u))K_n(u)du\right] c_k.$$

Since $\int_{-\pi}^{\pi} \sin(ku)K_n(u)du = 0$, we get

$$T_{n,r}(f)(q) = \sum_{k=0}^{\infty} r^k \cos(k\alpha)A_{k,n}c_k + I_q \sum_{k=0}^{\infty} r^k \sin(k\alpha)A_{n,k}c_k = \sum_{k=0}^{\infty} q^k c_k A_{k,n},$$

which proves the formula in (i) for q not real.

If $q = 0$, then $f(0) = c_0$ and $T_{n,r}(f)(0) = \alpha_n c_0 A_{0,n} = \alpha_n c_0 \int_a^b K_n(u)du$.

Now, suppose that $q \in \mathbb{R} \setminus \{0\}$. Then the proof follows as in the complex case. Summarizing all the above cases, we get that the representation for $T_{n,r}(f)(q)$ is valid for all $q \in \mathbb{B}_R$. The proof in the case of left W-analytic functions is similar. ∎

Remark 2.3. Similar formulas hold for the convolution operators of quaternion variable, in the case when $a = -\infty$, $b = +\infty$ and $K_n(u)$ is a continuous, positive, even and bounded kernel on $(-\infty, +\infty)$.

3 Approximation by quaternion polynomial convolutions

It is well-known that the classical de la Vallée Poussin kernel

$$K_n(u) = \frac{(n!)^2}{(2n)!}\left(2\cos\frac{u}{2}\right)^{2n} = 1 + 2\sum_{j=1}^{n} \frac{(n!)^2}{(n-j)!(n+j)!}\cos(ju), u \in \mathbb{R}$$

and the de la Vallée Poussin convolution trigonometric polynomials of real variable attached to a 2π-periodic real function g, defined by

$$V_n(g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x-u)K_n(u)dt, x \in \mathbb{R}, n \in \mathbb{N},$$

were introduced in [13] in order to give a constructive solution to the second approximation theorem of Weierstrass, by proving there that $\lim_{n\to\infty} V_n(f)(x) = f(x)$, uniformly on $\mathbb{R}$. A quantitative upper estimate of $|V_n(f)(x) - f(x)|$ in terms of the second order modulus of smoothness $\omega_2(f; 1/\sqrt{n})$ was obtained in [2].

Replacing in the above integral the translation $x - u \in \mathbb{R}$ by the rotation ze^{-iu} or $ze^{iu} \in \mathbb{C}$, for an analytic function f in a disk D_R, the complex convolution polynomials defined by

$$V_n(f)(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(ze^{it})K_n(t)dt, z \in D_R, n \in \mathbb{N},$$

were firstly introduced and studied in [12], by proving that all $V_n(f)(z)$, $n \in \mathbb{N}$ preserve the spirallikeness and convexity of f in the unit disk. These nice shape preserving properties do not hold for the partial sums of the Taylor's expansion of f. We will come back to this issue in the following Remark 3.7. On the other hand, it was also natural to study the approximation properties of the complex de la Vallée Poussin polynomials, see [5], pp. 182-187 for the details. At this point, we should point out that the approximation by the partial sums of the Taylor expansion provides a better upper estimate (of geometrical order) than the approximation given by the de la Vallée Poussin complex polynomials. However, for the latter one, the exact order of approximation and a Voronovskaja-type result can be obtained

The first goal of this section is to extend the approximation properties of the de la Vallée Poussin polynomials of complex variable, to the case of quaternion variable. Thus, taking in Definition 2.1 $\alpha_n = \frac{1}{2\pi}$ and $K_n(u)$ the above kernel and taking into account Theorem 2.2, the de la Vallée-Poussin convolution operator of a quaternion variable for a W-analytic function $f : \mathbb{B}_R \to \mathbb{H}$, $f(q) = \sum_{k=0}^{\infty} q^k c_k$, will be

$$P_{n,r}(f)(q) = \sum_{k=0}^{n} q^k c_k \frac{(n!)^2}{(n-k)!(n+k)!}, \ q \in \mathbb{B}_R. \tag{3.1}$$

Here we used in Theorem 2.2, the formula $A_{k,n} = \frac{(n!)^2}{(n-k)!(n+k)!}$ for $0 \le k \le n$, and $A_{k,n} = 0$ for $k > n$, formulas obtained in the complex case in [5], p. 182.

Defining $|||f|||_d = \sup\{\|f(q)\|;\ \|q\| \le d\}$, firstly upper estimates in approximation of f by $P_{n,r}(f)$ with explicit constants are presented.

Theorem 3.1. *Let $R > 1$, $\mathbb{B}_R = \{q \in \mathbb{H}; \|q\| < R\}$ and let us suppose that $f : \mathbb{B}_R \to \mathbb{H}$ is W-analytic in $\mathbb{B}_R$, that is we can write $f(q) = \sum_{k=0}^{\infty} q^k c_k$, for all $q \in \mathbb{B}_R$.*

(i) Denoting $M_d(f) = \sum_{k=1}^{\infty} \|c_k\| k^2 d^k < \infty$, for any $d \in [1, R)$ we have

$$\|P_{n,r}(f) - f\|_d \leq \frac{M_d(f)}{n}, n \in \mathbb{N}.$$

(ii) If $1 \leq d < r_1 < R$ and $p \in \mathbb{N}$ then we have

$$\|\partial_s^p P_{n,r}(f) - \partial_s^p f\|_d \leq \frac{r_1 p! M_{r_1}(f)}{(r_1 - d)^{p+1} n}, n \in \mathbb{N},$$

where ∂_s^p denotes the slice derivative of order p.

Proof. (i) Denote $e_k(q) = q^k$. Since we can write $e_k(q)c_k = \sum_{j=0}^{\infty} q^j c_k a_j$ with $a_k = 1$ and $a_j = 0$ for all $j \neq k$, by (3.1) it is immediate that

$$P_{n,r}(e_k c_k)(q) = e_k(q) c_k \frac{(n!)^2}{(n-k)!(n+k)!} = P_{n,r}(e_k)(q) c_k, \text{ for all } 0 \leq k \leq n,$$

and that $P_{n,r}(e_k) = 0$ for $k > n$. This implies $P_{n,r}(f)(q) = \sum_{k=0}^{\infty} P_{n,r}(e_k)(q) c_k$ and for all $\|q\| \leq d$ we get

$$\|P_{n,r}(f)(q) - f(q)\| \leq \sum_{k=1}^{\infty} \|c_k\| \, \|P_{n,r}(e_k)(q) - e_k(q)\|$$
$$\leq \sum_{k=1}^{n} \|c_k\| \, \|P_{n,r}(e_k)(q) - e_k(q)\| + \sum_{k=n+1}^{\infty} \|c_k\| \frac{k^2}{n} d^k.$$

But by

$$P_{n,r}(e_k)(q) = e_k(q) \frac{(n!)^2}{(n-k)!(n+k)!} = e_k(q) \Pi_{j=1}^{k} \left(1 - \frac{k}{n+j}\right), \text{ if } k \leq n,$$

for all $0 \leq k \leq n$ and $\|q\| \leq d$ we get

$$\|P_{n,r}(e_k)(q) - e_k(q)\| \leq \left|1 - \Pi_{j=1}^{k} \left(1 - \frac{k}{n+j}\right)\right| d^k \leq k d^k \sum_{j=1}^{k} \frac{1}{n+j} \leq \frac{k^2 d^k}{n}.$$

Here we used the inequality $1 - \Pi_{i=1}^{k} x_i \leq \sum_{i=1}^{k} (1 - x_i)$, for $0 \leq x_i \leq 1, i = 1, ..., k$.

In conclusion, for all $\|q\| \leq d$ we have

$$\|P_{n,r}(e_k)(q) - e_k(q)\| \leq \frac{k^2}{n} d^k, \text{ for all } k, n \in \mathbb{N},$$

which implies the estimate in (i).

(ii) Let q be such that $\|q\| \leq d$. By Theorems 4.5.3, 4.5.4 in [3], we can write the Cauchy formula and integrate on a specific complex plane $\mathbb{C}_I$. Thus we can choose $I = I_q$. Let γ be the circle of radius $r_1 > d$ and center 0 in the plane $\mathbb{C}_{I_q}$.

For any $v \in \gamma$, we have $|v - q| \geq r_1 - d$, and by the Cauchy's formula it follows that for all $\|q\| \leq d$ and $n \in \mathbb{N}$, we have

$$\|\partial_s^p P_{n,r}(f)(q) - \partial_s^p f(q)\| \\ = \frac{p!}{2\pi} \left\| \int_\gamma [S^{-1}(v,q)(q-\overline{v})^{-1}]^{p+1}(q-\overline{v})^{(p+1)*} dv_{I_q}(P_{n,r}(f)(v) - f(v)) \right\|$$

and since v and q commute we have

$$[S^{-1}(v,q)(q-\overline{v})^{-1}]^{p+1}(q-\overline{v})^{(p+1)*} = (v-q)^{-(p+1)},$$

thus we obtain:

$$\|\partial_s^p P_{n,r}(f)(q) - \partial_s^p f(q)\| = \frac{p!}{2\pi} \left\| \int_\gamma (v-q)^{-(p+1)} dv_{I_q}(P_{n,r}(f)(v) - f(v)) \right\| \\ \leq \frac{M_{r_1}(f)}{n} \frac{p!}{2\pi} \frac{2\pi r_1}{(r_1 - d)^{p+1}},$$

which proves (ii) and the theorem. ■

We now present a Voronovskaja-type theorem for the quaternionic case.

Theorem 3.2. *Let $R > 1$, $\mathbb{B}_R = \{q \in \mathbb{H}; \|q\| < R\}$ and let us suppose that $f : \mathbb{B}_R \to \mathbb{H}$ is W-analytic in $\mathbb{B}_R$, that is we can write $f(q) = \sum_{k=0}^{\infty} q^k c_k$, for all $q \in \mathbb{B}_R$. For any $d \in [1, R)$ we have*

$$\left\| P_{n,r}(f) - f + \frac{e_2 \partial_s^2 f}{n} + \frac{e_1 \partial_s f}{n} \right\|_d \leq \frac{A_d(f)}{n^2}, n \in \mathbb{N}$$

where $A_d(f) = \sum_{k=1}^{\infty} \|c_k\| k^4 d^k < \infty$, $e_k(q) = q^k$.

Proof. Since the left slice derivatives of a power series representing a W-analytic function coincide with its formal derivatives (see e.g. [9] , p. 127), we can write $\partial_s f(q) = \sum_{k=1}^{\infty} q^{k-1} c_k k$ and $\partial_s^2 f(q) = \sum_{k=2}^{\infty} q^{k-2} c_k k(k-1)$.

Therefore, denoting

$$\|E_{k,n}(q)\| = \left\| P_{n,r}(e_k)(q) - e_k(q) + \frac{q^k k(k-1)}{n} + \frac{q^k k}{n} \right\|,$$

for all $\|q\| \leq d$ we get

$$\left\| P_{n,r}(f)(q) - f(q) + \frac{e_2(q) \partial_s^2 f(q)}{n} + \frac{e_1(q) \partial_s f(q)}{n} \right\| \\ \leq \sum_{k=0}^{\infty} \|E_{k,n}(q)\| \, \|c_k\| = \sum_{k=1}^{n} \|E_{k,n}(q)\| \, \|c_k\| + \sum_{k=n+1}^{\infty} \|E_{k,n}(q)\| \, \|c_k\| \\ = \sum_{k=1}^{n} \|E_{k,n}(q)\| \, \|c_k\| + \sum_{k=n+1}^{\infty} \left\| -q^k + \frac{q^k k(k-1)}{n} + \frac{q^k k}{n} \right\| \|c_k\|.$$

But for $\|q\| \leq d$ we have

$$\sum_{k=n+1}^{\infty} \|c_k\| \left\| -q^k + \frac{q^k k(k-1)}{n} + \frac{q^k k}{n} \right\| = \sum_{k=n+1}^{\infty} \|c_k\| d^k \left\| -1 + \frac{k(k-1)}{n} + \frac{k}{n} \right\| \leq$$
$$\sum_{k=n+1}^{\infty} \|c_k\| d^k \frac{k^2}{n} \leq \sum_{k=n+1}^{\infty} \|c_k\| \frac{k}{n} d^k \frac{k^2}{n} = \frac{1}{n^2} \sum_{k=n+1}^{\infty} \|c_k\| k^3 d^k \leq \frac{1}{n^2} \sum_{k=n+1}^{\infty} \|c_k\| k^4 d^k. \tag{3.2}$$

Therefore, it remains to estimate $\|E_{k,n}(q)\|$ for $\|q\| \leq d$ and $0 \leq k \leq n$. Since it is immediate that $E_{0,n}(q) = 0$, it suffices to consider $1 \leq k \leq n$. We obtain

$$\begin{aligned} \|E_{k,n}(q)\| &= \left\| q^k \frac{(n!)^2}{(n-k)!(n+k)!} - q^k + \frac{q^k k(k-1)}{n} + \frac{q^k k}{n} \right\| \\ &= \|q\|^k \left\| \frac{(n!)^2}{(n-k)!(n+k)!} - 1 + \frac{k^2}{n} \right\|. \end{aligned} \tag{3.3}$$

But by mathematical induction can be proved (see relationship (3.1), p. 184 in [5])

$$0 \leq \frac{(n!)^2}{(n-k)!(n+k)!} - 1 + \frac{k^2}{n} \leq \frac{k^4}{n^2}, \text{ for all } k = 1, 2, ..., n \text{ and } n \in \mathbb{N}. \tag{3.4}$$

Replacing this inequality in (3.3) and using (3.2) we obtain the theorem. ■

Now we are in position to obtain the exact degree of approximation by $P_{n,r}(f)(q)$. Firstly, we present a lower estimate of the approximation error in Theorem 3.1, (i).

Theorem 3.3. *Let $R > 1$, $\mathbb{B}_R = \{q \in \mathbb{H}; \|q\| < R\}$ and let us suppose that $f : \mathbb{B}_R \to \mathbb{H}$ is W-analytic in $\mathbb{B}_R$, that is we can write $f(q) = \sum_{k=0}^{\infty} q^k c_k$, for all $q \in \mathbb{B}_R$. If f is not a constant function, then for any $d \in [1, R)$ we have*

$$\|\|P_{n,r}(f) - f\|\|_d \geq \frac{C_d(f)}{n}, n \in \mathbb{N},$$

where the constant $0 < C_d(f) < \infty$ depends only on f and d.

Proof. For all $q \in \mathbb{B}_R$ and $n \in \mathbb{N}$ we have

$$P_{n,r}(f)(q) - f(q) =$$
$$\frac{1}{n}\left\{ -[q^2 \partial_s^2 f(q) + q\partial_s f(q)] + \frac{1}{n}\left[n^2 \left(P_{n,r}(f)(q) - f(q) + \frac{q^2 \partial_s^2 f(q)}{n} + \frac{q\partial_s f(q)}{n} \right)\right]\right\}.$$

We will apply to this identity the following obvious property :

$$\|\|F + G\|\|_d \geq | \, \|\|F\|\|_d - \|\|G\|\|_d \, | \geq \|\|F\|\|_d - \|\|G\|\|_d.$$

It follows

$$\begin{aligned} &\|\|P_{n,r}(f) - f\|\|_d \\ &\quad \geq \frac{1}{n}\left\{ \left\|\left\| e_2 \partial_s^2 f + e_1 \partial_s f \right\|\right\|_d - \frac{1}{n}\left[n^2 \left\|\left\| P_{n,r}(f) - f + \frac{e_2 \partial_s^2 f}{n} + \frac{e_1 \partial_s f}{n} \right\|\right\|_d \right]\right\}. \end{aligned}$$

Taking into account that by hypothesis f is not a constant function in $\mathbb{B}_R$, we get $\|e_2\partial_s^2 f + e_1\partial_s f\|_d > 0$. Indeed, supposing the contrary it follows that $q^2\partial_s^2 f(q) + q\partial_s f(q) = 0$ for all $q \in \overline{\mathbb{B}}_r$. By using the uniqueness of the power series for f (see e.g. [10], Theorem 2.7) and by identifying the coefficients in the above equation, it easily follows that $c_k = 0$ for all $k \geq 1$. This means that $f(q) = c_0$, for all $q \in \overline{\mathbb{B}}_d$, which contradicts the hypothesis. Theorem 3.2 implies that

$$n^2 \left\|P_{n,r}(f) - f + \frac{e_2}{n}\partial_s^2 f + \frac{e_1}{n}\partial_s f\right\|_d \leq A_d(f).$$

Therefore, there exists an index n_0 depending only on f and d, such that for all $n \geq n_0$ we have

$$\left\|e_2\partial_s^2 f + e_1\partial_s f\right\|_d - \frac{1}{n}\left[n^2 \left\|P_{n,r}(f) - f + \frac{e_2}{n}\partial_s^2 f + \frac{e_1}{n}\partial_s f\right\|_d\right] \geq \frac{1}{2}\left\|e_2\partial_s^2 f + e_1\partial_s f\right\|_d,$$

which immediately implies

$$\|P_{n,r}(f) - f\|_d \geq \frac{1}{2n}\left\|e_2\partial_s^2 f + e_1\partial_s f\right\|_d, \forall n \geq n_0.$$

For $n \in \{1, ..., n_0 - 1\}$ we obviously have $\|P_{n,r}(f) - f\|_d \geq \frac{M_{d,n}(f)}{n}$ with $M_{d,n}(f) = n\|P_{n,r}(f) - f\|_d > 0$, which finally implies $\|P_{n,r}(f) - f\|_d \geq \frac{C_d(f)}{n}$ for all n, where $C_d(f) = \min\{M_{d,1}(f), \ldots, M_{d,n_0-1}(f), \frac{1}{2}\left\|e_2\partial_s^2 f + e_1\partial_s f\right\|_d\}$. This completes the proof. ■

By Theorem 3.3 and Theorem 3.1, (i), we immediately get the following result, in which the equivalence $a_n \sim b_n$, $n \in \mathbb{N}$ means that there exist two constants $c_1, c_2 > 0$ independent of n, such that $c_1 b_n \leq a_n \leq c_2 b_n$ for all $n \in \mathbb{N}$.

Theorem 3.4. *Let $R > 1$, $\mathbb{B}_R = \{q \in \mathbb{H}; \|q\| < R\}$ and let us suppose that $f : \mathbb{B}_R \to \mathbb{H}$ is W-analytic in $\mathbb{B}_R$. If f is not a constant function in $\mathbb{B}_R$, then for any $d \in [1, R)$ we have*

$$\|P_{n,r}(f) - f\|_d \sim \frac{1}{n}, n \in \mathbb{N},$$

where the constants in the equivalence $\sim$ depend only on f and d.

In the case of approximation by the slice derivatives of $P_{n,r}(f)(q)$ we have

Theorem 3.5. *Let $\mathbb{B}_R = \{q \in \mathbb{H}; \|q\| < R\}$ be with $R > 1$ and let us suppose that $f : \mathbb{B}_R \to \mathbb{H}$ is W-analytic in $\mathbb{B}_R$, i.e. $f(q) = \sum_{k=0}^{\infty} q^k c_k$, for all $q \in \mathbb{B}_R$. Also, let $1 \leq d < r_1 < R$ and $p \in \mathbb{N}$ be fixed. If f is not a polynomial of degree $\leq p - 1$, then we have*

$$\|\partial_s^p P_{n,r}(f) - \partial_s^p f\|_d \sim \frac{1}{n},$$

where the constants in the equivalence $\sim$ depend on f, d, r_1 and p.

Proof. Reasoning as in the proof of Theorem 3.1, let $q \neq 0$ be such that $\|q\| \leq d$ and consider the complex plane $\mathbb{C}_{I_q^*}$. Let γ be the circle of radius $r_1 > d \geq 1$ and center 0 in the plane $\mathbb{C}_{I_q^*}$. For any $v \in \gamma$, we have $|v - q| \geq r_1 - d$, and by the Cauchy's formula it follows that for all $\|q\| \leq d$ and $n \in \mathbb{N}$, we have

$$\partial_s^p P_{n,r}(f)(q) - \partial_s^p f(q) = \\ = \frac{p!}{2\pi} \int_\gamma [S^{-1}(v,q)(q - \overline{v})^{-1}]^{p+1} (q - \overline{v})^{(p+1)*} dv_{I_q^*} (P_{n,r}(f)(v) - f(v)),$$

where $I_q^* := I_q$ for q not real, and $I_q^* = I$ arbitrary in $\mathbb{S}$ for $q \in \mathbb{R} \setminus \{0\}$.

Taking into account Theorem 3.1, (ii), we just have to prove the lower estimate for $\|\partial_s^p P_n(f) - \partial_s^p f\|_d$. For this purpose, as in the proof of Theorem 3.3, for all $v \in \gamma$ and $n \in \mathbb{N}$ we have

$$P_{n,r}(f)(v) - f(v) =$$

$$\frac{1}{n} \left\{ -[v^2 \partial_s^2 f(v) + v \partial_s f(v)] + \frac{1}{n} \left[n^2 \left(P_{n,r}(f)(v) - f(v) + \frac{v^2}{n} \partial_s^2 f(v) + \frac{v \partial_s f(v)}{n} \right) \right] \right\},$$

which can be replaced in the above Cauchy's formula. Since v and q belong to the same complex plane we have:

$$\partial_s^p P_{n,r}(f)(q) - \partial_s^p f(q) = \frac{1}{n} \left\{ \frac{p!}{2\pi} \int_\gamma -(v - q)^{p+1} dv_{I_q^*} [v^2 \partial_s^2 f(v) + v \partial_s^2 f(v)] \right. \\ \left. + \frac{1}{n} \frac{p!}{2\pi} \int_\gamma (v - q)^{p+1} dv_{I_q^*} \left[n^2 \left(P_{n,r}(f)(v) - f(v) + \frac{v^2}{n} \partial_s^2 f(v) + \frac{v}{n} \partial_s f(v) \right) \right] \right\} \\ = \frac{1}{n} \left\{ \partial_s^p \left[-q^2 \partial_s^2 f(q) - q \partial_s f(q) \right] \right. \\ \left. + \frac{1}{n} \frac{p!}{2\pi} \int_\gamma (v - q)^{p+1} dv_{I_q^*} \left[n^2 \left(P_{n,r}(f)(v) - f(v) + \frac{v^2}{n} \partial_s^2 f(v) + \frac{v}{n} \partial_s f(v) \right) \right] \right\}.$$

By taking the norm $\| \cdot \|_d$ it follows

$$\|\partial_s^p P_{n,r}(f) - \partial_s^p f\|_d \geq \frac{1}{n} \left\{ \left\| \partial_s^p \left[e_2 \partial_s^2 f + e_1 \partial_s f \right] \right\|_d \right. \\ \left. - \frac{1}{n} \left\| \frac{p!}{2\pi} \int_\gamma (v - q)^{p+1} dv_{I_q^*} \left[n^2 \left(P_n(f)(v) - f(v) + \frac{v^2}{n} \partial_s^2 f(v) + \frac{v}{n} \partial_s f \right) \right] \right\|_d \right\},$$

and using Theorem 2.2 we obtain

$$\left\| \frac{p!}{2\pi} \int_\gamma (v - q)^{p+1} dv_{I_q^*} \left[n^2 \left(P_{n,r}(f)(v) - f(v) + \frac{v^2}{n} \partial_s^2 f(v) + \frac{v}{n} \partial_s^2 f \right) \right] \right\|_d \\ \leq \frac{p!}{2\pi} \frac{2\pi r_1 n^2}{(r_1 - d)^{p+1}} \left\| P_{n,r}(f) - f + \frac{e_2}{n} \partial_s^2 f + \frac{e_1}{n} \partial_s^2 f \right\|_{r_1} \leq \frac{A_{r_1}(f) p! r_1}{(r_1 - d)^{p+1}}.$$

The hypothesis on f implies that $\left\| \partial_s^p \left[e_2 \partial_s^2 f + e_1 \partial_s f \right] \right\|_d > 0$. Indeed, supposing the contrary it follows that $q^2 \partial_s^2 f(q) + q \partial_s f(q) = Q_{p-1}(q)$, for all $q \in \overline{\mathbb{B}}_r$, where $Q_{p-1}(q) = \sum_{j=1}^{p-1} A_j q^j$ necessarily is a polynomial of degree $\leq p - 1$, vanishing at $q = 0$. Denoting $\partial_s f(q) = g(q)$ the above differential equation becomes $q^2 \partial_s g(q) + q g(q) = Q_{p-1}(q)$, for all $q \in \overline{\mathbb{B}}_d$. Let us now look for a W-analytic solution of the form $g(q) = \sum_{j=0}^{\infty} q^j \alpha_j$: by replacing in the differential equation, and by the identification of coefficients we easily obtain that $g(q)$ necessarily is a polynomial of degree $\leq p-2$. Thus $f(q)$ necessarily is a polynomial of degree $\leq p-1$, in contradiction with the hypothesis. Finally, reasoning exactly as in the proof of Theorem 3.3, we immediately get the desired conclusion. ■

Remark 3.6. Similar results could be easily adapted for the left convolution operator of the de la Vallée-Poussin type, $P_{n,l}(f)(q)$ attached to left W-analytic functions. Also, in a similar way, approximation results for other choices of the trigonometric kernel $K_n(u)$ can be obtained, like for those of Fejér, Riesz-Zygmund, Jackson and Beatson (see them in e.g. Chapter 3 of the book [5], where the corresponding complex convolutions were studied).

Remark 3.7. Using Lemma 4.1.7, p. 117 (Splitting Lemma), Corollary 4.3.4, p. 121 and Corollary 4.3.6, p. 121, all in [3], it is not difficult to prove that for quaternionic W-analytic functions with all the coefficients real (this subclass is denoted by $\mathcal{N}$), the de la Vallée Poussin polynomials of quaternion variable given by (3.1) preserve some geometric properties, like what happens in the complex case. More precisely, the de la Vallée Poussin quaternion polynomials given by (3.1) preserve the starlikeness and the convexity of $f \in \mathcal{N}$, where for $f : \mathbb{B}_1 \to \mathbb{H}$ normalized by $f(0) = 0$ and $\partial_s(f)(0) \neq 0$, the starlikeness (convexity) is understood in the sense that for all $0 < r \leq 1$, $f(\mathbb{B}_r)$ are starlike (convex, respectively) sets in $\mathbb{R}^4$. We recall here that $A \subset \mathbb{R}^4$ is called starlike with respect to the origin 0, if for any point $p \in A$, the Euclidean segment determined by 0 and p entirely belongs to A and that $A \subset \mathbb{R}^4$ is called convex if for all $p, q \in A$, the Euclidean segment joining p and q entirely belongs to A. In other words, because we can easily find many examples of starlike (or convex) functions $f \in \mathcal{N}$ by simply replacing $z \in \mathbb{C}$ with $q \in \mathbb{H}$ in the Taylor's expansion with all the coefficients real numbers, of a starlike (convex, respectively) function of complex variable, by (3.1) we can easily construct polynomials of quaternion variable with nice geometric properties, if these polynomials are interpreted as transformations from $\mathbb{R}^4$ to $\mathbb{R}^4$. More details on these geometrical aspects will be given in the forthcoming paper [8].

4 Approximation by nonpolynomial quaternion convolutions

In this section we deal in details with the approximation properties of the convolution based on the classical Gauss-Weierstrass kernel given by $K_t(u) = e^{-u^2/(2t)}$, $u \in \mathbb{R}$. Note that here $t > 0$ is a real parameter which replaces the natural parameter $n \in \mathbb{N}$ in the definition of the trigonometric kernels.

Replacing in the formula (2.1) for the convolution operator in Definition 2.1, $a = -\infty$, $b = +\infty$, α_n by $\alpha_t = \frac{1}{\sqrt{2\pi t}}$, and $K_n(u)$ by $K_t(u) = e^{-u^2/(2t)}$, for a W-analytic function $f : \mathbb{B}_R \to \mathbb{H}$, $\mathbb{B}_R = \{q \in \mathbb{H}; \|q\| < R\}$, we obtain the right Gauss-Weierstrass convolution operator of quaternion variable

$$W_{t,r}(f)(q) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} f(qe^{I_q u}) e^{-u^2/(2t)} du, \qquad q \in \mathbb{H} \setminus \mathbb{R}, q = re^{I_q a} \in \mathbb{B}_R,$$

$$W_{t,r}(f)(q) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} f(qe^{Iu}) e^{-u^2/(2t)} du, \; q \in \mathbb{R} \setminus \{0\}, q = re^{Ia} \in \mathbb{B}_R, \; a = 0 \text{ or } \pi,$$
$$W_{t,r}(f)(0) = f(0), \tag{4.1}$$

where $I \in \mathbb{S}$ is fixed (but arbitrary).

Reasoning exactly as in the proof of Theorem 2.2, for $f(q) = \sum_{k=0}^{\infty} q^k c_k$ and taking into account that $A_{k,t} = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-u^2/(2t)} \cos(ku) du = e^{-k^2 t/2}$, we have

$$W_{t,r}(f)(q) = \sum_{k=0}^{\infty} q^k c_k A_{k,t} = \sum_{k=0}^{\infty} q^k c_k e^{-k^2 t/2}, \qquad q \in \mathbb{B}_R, \; t > 0. \tag{4.2}$$

Theorem 4.1. *Let $R > 1$, $\mathbb{B}_R = \{q \in \mathbb{H}; \|q\| < R\}$ and let us suppose that $f : \mathbb{B}_R \to \mathbb{H}$ is W-analytic in $\mathbb{B}_R$, that is we can write $f(q) = \sum_{k=0}^{\infty} q^k c_k$, for all $q \in \mathbb{B}_R$.*

(i) For any $d \in [1, R)$ we have

$$|||W_{t,r}(f) - f|||_d \le \frac{t}{2} M_d(f), \qquad t > 0,$$

where $M_d(f) = \sum_{k=1}^{\infty} \|c_k\| k^2 d^k < \infty$.

(ii) If $1 \le d < R$ and $p \in \mathbb{N}$, then for all $I \in \mathbb{S}$ we have

$$|||\partial_s^p W_{t,r}(f) - \partial_s^p f|||_d \le \frac{t}{2} M_{d,p}(f), \qquad t > 0,$$

where $M_{d,p}(f) = \sum_{k=p}^{\infty} d^{k-p} \|c_k\| k^3 (k-1)...(k-p+1) < \infty$.

Proof. (i) By (4.2), for all $\|q\| \le d$ we have (since $\|\cdot\|$ is a multiplicative norm)

$$\|W_{t,r}(f)(q) - f(q)\| \le \sum_{k=0}^{\infty} \|q^k\| \, \|c_k\| \left| e^{-k^2 t/2} - 1 \right| \le \sum_{k=0}^{\infty} d^k \, \|c_k\| \left| e^{-k^2 t/2} - 1 \right|.$$

Now, denoting $h(t) = e^{-k^2 t/2}$, and since $h(0) = 1$, by the mean value theorem, there exists a point $\xi \in (0, t)$ such that

$$\left| e^{-k^2 t/2} - 1 \right| = |h'(\xi)| t = \frac{t}{2} k^2 e^{-k^2 \xi/2} \le \frac{t}{2} k^2,$$

which replaced above implies

$$\|W_{t,r}(f)(q) - f(q)\| \le \frac{t}{2} \sum_{k=0}^{\infty} r^k \|c_k\| k^2.$$

Passing to supremum after $\|q\| \le d$, the estimate in the statement follows.

(ii) Taking into account the formula for the slice derivative of the quaternion power series (see e.g. [9], p. 127) by (4.2), reasoning as above we get

$$\|\partial_s^p W_{t,r}(f)(q) - \partial_s^p f(q)\| = \| \sum_{k=p}^{\infty} q^{k-p} c_k k(k-1)...(k-p+1)(e^{-k^2 t/2} - 1) \|$$
$$\le \frac{t}{2} \sum_{k=p}^{\infty} d^{k-p} k^3 (k-1)...(k-p+1),$$

which proves the theorem. ∎

Also, the following exact estimate holds.

Theorem 4.2. *Let $R > 1$, $\mathbb{B}_R = \{q \in \mathbb{H}; \|q\| < R\}$ and let us suppose that $f : \mathbb{B}_R \to \mathbb{H}$ is W-analytic in $\mathbb{B}_R$, that is we can write $f(q) = \sum_{k=0}^{\infty} q^k c_k$, for all $q \in \mathbb{B}_R$. If f is not constant function for $j = 0$ and not a polynomial of degree $\leq j-1$ for $j \in \mathbb{N}$, then for all $I \in \mathbb{S}$ we have*

$$\|\partial_s^j W_{t,r}(f) - \partial_s^j f\|_d \sim t,$$

where the constants in the equivalence depend only on f, d and j. Here the equivalence $a(t) \sim b(t)$ means that there exists two absolute constants $C_1 > 0$ and $C_2 > 0$ such that $0 \leq C_1 a(t) \leq b(t) \leq C_2 a(t)$, for all $t > 0$.

Proof. Taking into account the upper estimate in Theorem 4.1, (i), it remains to prove a lower estimate for $\|\partial_s^j W_{t,r}(f) - \partial_s^j f\|_d$.

For this goal, consider the trigonometric form of $q \neq 0$, $q = \|q\| e^{I_q \varphi}$ (see the Introduction) and choose a $q := de^{I_q\varphi}$ and $p \in \mathbb{N} \bigcup \{0\}$. Let $j \in \mathbb{N} \bigcup \{0\}$. We get

$$\frac{1}{2\pi} e^{-I_q p\varphi}[\partial_s^j f(q) - \partial_s^j W_{t,r}(f)(q)]$$
$$= \frac{1}{2\pi} \sum_{k=j}^{\infty} d^{k-j} e^{I_q\varphi(k-j-p)} c_k k(k-1)...(k-j+1)[1 - e^{-k^2 t/2}].$$

Integrating from $-\pi$ to π, with some computations, we obtain

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-I_q p\varphi}[\partial_s^j f(q) - \partial_s^j W_{t,r}(f)(q)] d\varphi$$
$$= \frac{1}{2\pi} \sum_{k=j}^{\infty} d^{k-j} \int_{-\pi}^{\pi} e^{I_q\varphi(k-j-p)} d\varphi c_k k(k-1)...(k-j+1)[1 - e^{-k^2 t/2}]$$
$$= d^p c_{j+p}(j+p)(j+p-1)...(p+1)[1 - e^{-(j+p)^2 t/2}].$$

Taking into account that $\|e^{I_q\varphi}\| = 1$ and passing above to $\|\cdot\|$, we easily obtain

$$\|a_{j+p}\|(j+p)(j+p-1)...(p+1)d^p[1 - e^{-(j+p)^2 t/2}] \leq \|\partial_s^j f - \partial_s^j(f)\|_d.$$

First consider $j = 0$ and denote $V_t = \inf_{1 \leq p}(1 - e^{-p^2 t/2})$. We get $V_t = 1 - e^{-t/2}$ and by the mean value theorem applied to $h(x) = e^{-x/2}$ on $[0, t]$ there exists $\eta \in (0, t)$ such that for all $t \in (0, 1]$ we have

$$V_t = h(0) - h(t) = (-t)h'(\eta) = (t/2)e^{-\eta/2} \geq (t/2)e^{-t/2} \geq \frac{e^{-1/2}}{2} t \geq t/4.$$

By the above lower estimate for $\|W_{t,r}(f) - f\|_d$, for all $p \geq 1$ and $t \in (0, 1]$ it follows

$$\frac{4\|W_{t,r}(f) - f\|_d}{t} \geq \frac{\|W_{t,r}(f) - f\|_d}{V_t} \geq \frac{\|W_{t,r}(f) - f\|_d}{1 - e^{-p^2 t/2}} \geq \|c_p\| d^p.$$

This implies that if there exists a subsequence $(t_k)_k$ in $(0,1]$ with $\lim_{k\to\infty} t_k = 0$ and such that $\lim_{k\to\infty} \frac{|||W_{t_k,r}(f)-f|||_d}{t_k} = 0$, then $c_p = 0$ for all $p \geq 1$, that is f is constant on $\overline{\mathbb{B}_d}$.

Therefore, if f is not a constant then $\inf_{t\in(0,1]} \frac{|||W_{t,r}(f)-f|||_d}{t} > 0$, which implies that there exists a constant $C_d(f) > 0$ such that $\frac{|||W_{t,r}(f)-f|||_d}{t} \geq C_d(f)$, for all $t \in (0,1]$, that is

$$|||W_{t,r}(f) - f|||_d \geq C_d(f)t, \text{ for all } t \in (0,1].$$

Now, consider $j \geq 1$ and denote $V_{j,t} = \inf_{p\geq 0}(1 - e^{-(p+j)^2t/2})$. Evidently that we have $V_{j,t} \geq \inf_{p\geq 1}(1 - e^{-p^2t/2}) \geq t/4$.

Reasoning as in the case of $j = 0$ we obtain

$$\frac{4|||\partial_s^j W_{t,r}(f) - \partial_s^j f|||_d}{t} \geq \frac{|||\partial_s^j W_{t,r}(f) - \partial_s^j f|||_d}{V_{j,t}} \geq \|c_{j+p}\| \frac{(j+p)!}{p!} d^p,$$

for all $p \geq 0$ and $t \in (0,1]$.
This implies that if there exists a subsequence $(t_k)_k$ in $(0,1]$ with $\lim_{k\to\infty} t_k = 0$ and such that $\lim_{k\to\infty} \frac{|||\partial_s^j W_{t_k,r}(f)-\partial_s^j f|||_d}{t_k} = 0$ then $c_{j+p} = 0$ for all $p \geq 0$, that is f is a polynomial of degree $\leq j-1$ on $\overline{\mathbb{B}_d}$.

Therefore, $\inf_{t\in(0,1]} \frac{|||\partial_s^j W_{t,r}(f)-\partial_s^j f|||_d}{t} > 0$ when f is not a polynomial of degree $\leq j-1$, which implies that there exists a constant $C_{d,j}(f) > 0$ such that $\frac{|||\partial_s^j W_{t,r}(f)-\partial_s^j f|||_d}{t} \geq C_{d,j}(f)$, for all $t \in (0,1]$, that is

$$|||\partial_s^j W_{t,r}(f) - \partial_s^j f|||_d \geq C_{d,j}(f)t, \text{ for all } t \in (0,1],$$

which proves the theorem. ∎

Let $\mathcal{A}_r(\mathbb{B}_R)$ be the quaternionic right Banach space of the W-analytic functions on $\mathbb{B}_R$, continuous on $\overline{\mathbb{B}_R}$. The space $\mathcal{A}_r(\mathbb{B}_R)$ is endowed with the uniform norm given by $|||f|||_R = \max\{\|f(u)\|; u \in \overline{\mathbb{B}_R}\}$. To conclude this section we prove that the right Gauss-Weierstrass convolution of quaternion variable defines a contraction semigroup on the quaternionic right Banach space with respect to the uniform norm on $\mathcal{A}_r(\mathbb{B}_R)$.

Theorem 4.3. *Let $f \in \mathcal{A}_r(\mathbb{B}_R)$, $f(q) = \sum_{k=0}^{\infty} q^k c_k$, $q \in \mathbb{B}_R$.*
(i) For all $t > 0$, $W_{t,r}(f) \in \mathcal{A}_r(\mathbb{B}_R)$ and

$$W_{t,r}(f)(q) = \sum_{k=0}^{\infty} q^k c_k e^{-k^2t/2}, \quad \textit{for all } q \in \mathbb{B}_R.$$

(ii) For all $q \in \overline{\mathbb{B}_R}$, $t > 0$, the following estimate holds :

$$\|W_{t,r}(f)(q) - f(q)\| \leq C_R \omega_1(f;\sqrt{t})_{\overline{\mathbb{B}_R}},$$

where

$$\omega_1(f;\delta)_{\overline{\mathbb{B}_R}} = \sup\{\|f(u) - f(v)\|;\ \|u - v\| \leq \delta,\ u,v \in \overline{\mathbb{B}_R}\}.$$

and $C_R > 0$ is a constant independent of t and f.

(iii) We have :

$$\|W_{t,r}(f)(q) - W_{s,r}(f)(q)\| \le C_s|\sqrt{t} - \sqrt{s}|, \text{ for all } q \in \overline{\mathbb{B}_R},\ t \in V_s \subset (0, +\infty),$$

where $C_s > 0$ is a constant depending on f, independent of q and t and V_s is any neighborhood of s.

(iv) The operator $W_{t,r}:\ \mathcal{A}_r(\mathbb{B}_R) \to \mathcal{A}_r(\mathbb{B}_R)$ is contractive, that is

$$|||W_{t,r}(f)|||_R \le |||f|||_R, \text{ for all } t > 0,\ f \in \mathcal{A}_r(\mathbb{B}_R).$$

(v) $(W_{t,r}, t \ge 0)$ is a (C_0)-contraction semigroup of linear operators on the real Banach space $(\mathcal{A}_r(\mathbb{B}_R), ||| \cdot |||_R)$ and the unique solution $u(t,q)$ (that belongs to $\mathcal{A}_r(\mathbb{B}_R)$, for each fixed $t > 0$) of the Cauchy problem (for a kind of heat equation in t and φ)

$$\frac{\partial u}{\partial t}(t,q) = \frac{1}{2}\frac{\partial^2 u}{\partial \varphi^2}(t,q), \qquad (t,q) \in (0,+\infty) \times \mathbb{B}_R,\ q = he^{I_q \varphi},\ q \ne 0, h = \|q\|,$$
$$u(0,q) = f(q), \qquad q \in \overline{\mathbb{B}_R},\ f \in \mathcal{A}_r(\mathbb{B}_R),$$

is given by $u(t,q) = W_{t,r}(f)(q)$.

Proof. (i) The fact that $W_{t,r}(f)$ is W-analytic in $\mathbb{B}_R$ follows from the relationship (4.2). It remains to prove the continuity in $\overline{\mathbb{B}_R}$. For this purpose, let $q_0, q_n \in \overline{\mathbb{B}_R}$ be such that $\lim_{n\to\infty} q_n = q_0$, that is $\lim_{n\to\infty} \|q_n - q_0\| = 0$.

Suppose first that q_0 is not a real quaternion. Then, without loss of generality, we may suppose that q_n are not real quaternions, for all $n \in \mathbb{N}$. In this case, denoting $q_n = r_n e^{I_{q_n} a_n}$ and $q_0 = r_0 e^{I_{q_0} a_0}$, by the definition of trigonometric form it easily follows that as $n \to \infty$, we get $a_n \to a_0$ and $\|I_{q_n} - I_{q_0}\| \to 0$.

Now, since

$$\begin{aligned}
\|q_n e^{I_{q_n} u} - q_0 e^{I_{q_0} u}\| &= \|(q_n - q_0)\cos(u) + (q_n I_{q_n} - q_0 I_{q_0})\sin(u)\| \\
&\le \|q_n - q_0\| + \|q_n I_{q_n} - q_0 I_{q_0}\| = \|q_n - q_0\| + \|q_n I_{q_n} - q_n I_{q_0} + q_n I_{q_0} - q_0 I_{q_0}\| \\
&\le \|q_n - q_0\| + \|q_n\|\|I_{q_n} - I_{q_0}\| + \|q_n - q_0\|\|I_{q_0}\| \\
&= 2\|q_n - q_0\| + \|q_n\|\|I_{q_n} - I_{q_0}\|,
\end{aligned}$$

we easily get

$$\begin{aligned}
&\|W_{t,r}(f)(q_n) - W_{t,r}(f)(q_0)\| \\
&\quad \le \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \|f(q_n e^{I_{q_n} u}) - f(q_0 e^{I_{q_0} u})\| e^{-u^2/(2t)}\, du \\
&\quad \le \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1(f; \|q_n e^{I_{q_n} u} - q_0 e^{I_{q_0} u}\|)_{\overline{\mathbb{B}_R}} e^{-u^2/(2t)}\, du \\
&\quad \le \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1(f; 2\|q_n - q_0\| + \|q_n\|\|I_{q_n} - I_{q_0}\|)_{\overline{\mathbb{B}_R}} e^{-u^2/(2\pi t)}\, du \\
&\quad = \omega_1(f; 2\|q_n - q_0\| + \|q_n\|\|I_{q_n} - I_{q_0}\|)_{\overline{\mathbb{B}_R}}.
\end{aligned}$$

Passing to limit with $n \to \infty$ and taking into account that it is easy to show that $\omega_1(f;\delta)_{\overline{\mathbb{B}_R}}$ keeps all the usual properties of a modulus of continuity for a real-valued functions of real variable, including the property that if f is continuous on its compact domain of definition then $\lim_{\delta \searrow 0} \omega_1(f;\delta)_{\overline{\mathbb{B}_R}} = 0$, it follows that $W_{t,r}(f)(q_n)$ converges to $W_{t,r}(f)(q_0)$, as $n \to \infty$.

Now, let us suppose that $q_0 \in \mathbb{R} \setminus \{0\}$ and assume firstly $q_0 > 0$.

If all q_n are real quaternions, then clearly we can suppose that $q_n > 0$ for all $n \in \mathbb{N}$ and writing $q_0 = \|q_0\|(\cos(0) + I\sin(0))$, $q_n = \|q_n\|(\cos(0) + I\sin(0))$, with arbitrary $I \in \mathbb{S}$, we immediately obtain

$$\|q_n e^{Iu} - q_0 e^{Iu}\| = \|(q_n - q_0)\cos(u)\| = |q_n - q_0|,$$

which implies

$$\begin{aligned}
&\|W_{t,r}(f)(q_n) - W_{t,r}(f)(q_0)\| \\
&\quad \leq \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \|f(q_n e^{Iu}) - f(q_0 e^{Iu})\| e^{-u^2/(2t)}\, du \\
&\quad \leq \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1(f; \|q_n e^{Iu} - q_0 e^{Iu}\|)_{\overline{\mathbb{B}_R}} e^{-u^2/(2t)}\, du \\
&\quad = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1(f; \|q_n - q_0\|)_{\overline{\mathbb{B}_R}} e^{-u^2/(2\pi t)}\, du = \omega_1(f; \|q_n - q_0\|)_{\overline{\mathbb{B}_R}}.
\end{aligned}$$

This again implies $\lim_{n\to\infty} W_{t,r}(f)(q_n) = W_{t,r}(f)(q_0)$.

If all the q_n are not real quaternions, we can write $q_n = r_n e^{I_{q_n} a_n}$ and $q_0 = q_0(\cos(0) + I\sin(0))$, with arbitrary $I \in \mathbb{S}$. Therefore, we can choose $I = I_{q_n}$ and write $q_0 = q_0(\cos(0) + I_{q_n}\sin(0)) = q_0 e^{I_{q_n} 0}$ and in the definition of $W_{t,r}(f)(q_0)$ we can choose $I = I_{q_n}$, which implies

$$\|q_n e^{I_{q_n} u} - q_0 e^{I_{q_n} u}\| = \|q_n - q_0\| \, \|e^{I_{q_n} u}\| = \|q_n - q_0\|,$$

and reasoning as above, it follows

$$\begin{aligned}
\|W_{t,r}(f)(q_n) - W_{t,r}(f)(q_0)\| &\leq \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \|f(q_n e^{I_{q_n} u}) - f(q_0 e^{I_{q_n} u})\| e^{-u^2/(2t)}\, du \\
&\leq \omega_1(f; \|q_n - q_0\|)_{\overline{\mathbb{B}_R}}.
\end{aligned}$$

Therefore, in this case too we obtain $\lim_{n\to\infty} W_{t,r}(f)(q_n) = W_{t,r}(f)(q_0)$.

If $q_0 < 0$, we reason exactly as above, with the only difference that we can write $q_0 = \|q_0\|(\cos(\pi) + I\sin(\pi))$, with arbitrary $I \in \mathbb{S}$.

In conclusion, $W_{t,r}(f)$ is continuous at any $q_0 \in \overline{\mathbb{B}_R}$, since f is continuous on $\overline{\mathbb{B}_R}$.

(ii) For $\|q\| \leq R$, q non real quaternion, we easily obtain

$$
\begin{aligned}
||W_{t,r}(f)(q) - f(q)|| &\leq \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \|f(qe^{I_q u}) - f(q)\| e^{-u^2/(2t)}\, du \\
&\leq \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \omega_1(f; R\|1 - e^{I_q u}\|)_{\overline{\mathbb{B}_R}} e^{-u^2/(2t)}\, du \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1\left(f; 2R\left|\sin\frac{u}{2}\right|\right)_{\overline{\mathbb{B}_R}} e^{-u^2/(2t)}\, du \\
&\leq \frac{2R+1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1(f; |u|)_{\overline{\mathbb{B}_R}} e^{-u^2/(2t)}\, du \\
&\leq \frac{2R+1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \omega_1(f; \sqrt{t})_{\overline{\mathbb{B}_R}} \left(\frac{|u|}{\sqrt{t}} + 1\right) e^{-u^2/(2t)}\, du \\
&= (2R+1)\left[\omega_1(f; \sqrt{t})_{\overline{\mathbb{B}_R}} + \frac{\omega_1(f; \sqrt{t})_{\overline{\mathbb{B}_R}}}{\sqrt{t}\sqrt{2\pi t}} \int_0^{\infty} 2ue^{-u^2/(2t)}\, du\right].
\end{aligned}
$$

Since $\int_0^{\infty} 2ue^{-u^2/(2t)} du = 2t \int_0^{\infty} e^{-v}\, dv = 2t$, we infer

$$
\begin{aligned}
&\|W_{t,r}(f)(q) - f(q)\| \\
&\quad \leq (2R+1)\left[\omega_1(f; \sqrt{t})_{\overline{\mathbb{B}_R}} + \left(\omega_1(f; \sqrt{t})_{\overline{\mathbb{B}_R}}\right) \frac{2t}{t\sqrt{2\pi}}\right] \leq C_R \omega_1(f; \sqrt{t})_{\overline{\mathbb{B}_R}}.
\end{aligned}
$$

Now, for $\|q\| \leq R$, $q \in \mathbb{R} \setminus \{0\}$, we fix an arbitrary $I \in \mathbb{S}$ and we can write

$$
||W_{t,r}(f)(q) - f(q)|| \quad \leq \quad \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \|f(qe^{Iu}) - f(q)\| e^{-u^2/(2t)}\, du
$$

and reasoning exactly as in the case of q non real, we arrive at the same upper estimate. Finally, for $q = 0$ we get $||W_{t,r}(f)(q) - f(q)|| = 0$, which all together imply the estimate in (ii) for all $q \in \mathbb{B}_R$.

(iii) From the definition of $W_{t,r}(f)(q)$ in (4.1), for all $q \in \mathbb{B}_R$ we easily get

$$
\|W_{t,r}(f)(q) - W_{s,r}(f)(q)\| \leq \frac{\|\|f\|\|_R}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left|\frac{e^{-u^2/t}}{\sqrt{t}} - \frac{e^{-u^2/s}}{\sqrt{s}}\right| du.
$$

First, let us denote $\sqrt{t} = a$, $\sqrt{s} = b$. Applying now the mean value theorem, there exists a value $c \in (a, b)$, such that

$$
\left|\frac{e^{-u^2/a^2}}{a} - \frac{e^{-u^2/b^2}}{b}\right| = |a - b| e^{-u^2/c^2} \left[\frac{2u^2}{c^4} - \frac{1}{c^2}\right],
$$

which together with the fact that $\int_{-\infty}^{+\infty} e^{-u^2/(2c)} < \infty$, $\int_{-\infty}^{+\infty} u^2 e^{-u^2/(2c)} < \infty$, it immediately implies the desired inequality for $W_{t,r}$.

(iv) Since $\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-u^2/(2t)} du = 1$, we deduce

$$
\|W_{t,r}(f)(q)\| \leq \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \|f(qe^{I_q^* u})\| e^{-u^2/(2t)} du \leq \|\|f\|\|_R,\ q \in \overline{\mathbb{B}_R},
$$

where $I_q^* := I_q$ if q is not real, and $I_q^* := I$-arbitrary in $\mathbb{S}$ if $q \in \mathbb{R} \setminus \{0\}$. Together with $W_{t,r}(f)(0) = f(0)$ all these easily imply $\|\|W_{t,r}(f)\|\|_R \leq \|\|f\|\|_R$.

(v) Let $f \in \mathcal{A}_r(\mathbb{B}_R)$, that is, $f(q) = \sum_{k=0}^{\infty} q^k a_k$, $q \in \mathbb{B}_R$. If $q \in \mathbb{B}_R$, $q = de^{I_q \varphi}$, $0 < d < R$, then by (i), we can write $W_{t,r}(f)(q) = \sum_{k=0}^{\infty} d^k e^{kI_q\varphi} c_k e^{-k^2 t/2}$. It easily follows that $W_{t+s,r}(f)(q) = W_{s,r}[W_{t,r}(f)](q)$, for all t, $s > 0$. If q is on the boundary of $\mathbb{B}_R$, then we may take a sequence $(q_n)_{n\in\mathbb{N}}$ of points in $\mathbb{B}_R$ such that $lim_{n\to\infty} q_n = q$ and we apply the above relationship and the continuity property from (i). Furthermore, denoting $W_{t,r}(f)(q)$ by $T(t)(f)(q)$, it is easy to check that the property $\lim_{t\searrow 0} T(t)(f) = f$, the continuity of $T(\cdot)$ and its contraction property follow from (ii), (iii) and (iv), respectively. Finally, all these facts together show that $(W_{t,r}, t \geq 0)$ is a (C_0)-contraction semigroup of linear operators on $\mathcal{A}_r(\mathbb{B}_R)$.

Consequently, since the above series representation for $W_{t,r}(f)(q)$ is uniformly and absolutely convergent in any compact ball included in $\mathbb{B}_R$, it can be differentiated term by term, with respect to t and φ. We then easily obtain that

$$\frac{\partial W_{t,r}(f)(q)}{\partial t} = \frac{1}{2}\frac{\partial^2 W_{t,r}(f)(q)}{\partial \varphi^2}.$$

Finally, from the same series representation, it is easy to check that

$$W_{0,r}(f)(q) = f(q), \qquad q \in \overline{\mathbb{B}_R}.$$

We also note that in the differential equation we must take $q \neq 0$ simply because $q = 0$ has not polar representation, that is, $q = 0$ cannot be represented as a function of φ. This completes the proof of the theorem. ■

Remark 4.4. Similar results can easily be adapted for the left convolution operator of Gauss-Weierstrass type, $W_{n,l}(f)(q)$ attached to left W-analytic functions.

Moreover, one may obtain similar results by choosing different kernels, like the Picard kernel $K_t(u) = e^{-|u|t}$, the Poisson-Cauchy kernel $K_t(u) = \frac{1}{u^2+t^2}$, and many others (see them in e.g. Chapter 3 of the book [5], where the corresponding complex convolutions were studied).

5 Approximation by convolution operators of a paravector variable

We now discuss how the results obtained in the preceding sections can be extended to a more general setting. Let us consider the real Clifford algebra $\mathbb{R}_n$ over n imaginary units $e_1, \ldots, e_n$ satisfying the relations $e_i e_j + e_j e_i = -2\delta_{ij}$. An element in the Clifford algebra will be denoted by $\sum_A e_A x_A$ where $A = i_1 \ldots i_r$, $i_\ell \in \{1, 2, \ldots, n\}$, $i_1 < \ldots < i_r$ is a multi-index and $e_A = e_{i_1} e_{i_2} \ldots e_{i_r}$, $e_\emptyset = 1$. As it is well known, $\mathbb{R}_1 = \mathbb{C}$, $\mathbb{R}_2 = \mathbb{H}$; for $n \geq 3$, $\mathbb{R}_n$ contains zero divisors. In the Clifford algebra $\mathbb{R}_n$, we can identify the element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with a so called paravector in the Clifford algebra $\mathbf{x} = x_0 + x_1 e_1 + \ldots + x_n e_n$. Given an element $a = \sum_A a_A e_A \in \mathbb{R}_n$ we define its norm as $\|a\| = (\sum_A a_A^2)^{\frac{1}{2}}$. The norm is not multiplicative, in fact for any two elements $a, b \in \mathbb{R}_n$ we have

$$\|ab\| \leq C_n \|a\| \, \|b\| \tag{5.1}$$

where C_n is a constant depending only on the dimension of the Clifford algebra $\mathbb{R}_n$. Moreover, we have $C_n \leq 2^{n/2}$. The norm is however multiplicative for example when a is a paravector or, in particular, a real number.

One can extend a notion generalizing holomorphy to functions defined on open sets $U \subseteq \mathbb{R}^{n+1}$ (with the above identification) and with values in a Clifford algebra. The most studied notion of (hyper)holomorphy in this setting leads to the so-called monogenic functions, see [1]. However, the set of monogenic functions does not contain the power of the paravector variable $\mathbf{x}^m$ not even for $m = 1$. The set of slice monogenic functions, see [4, Definition 2.1] and [3], includes power series. Roughly speaking, a real differentiable function $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ is slice monogenic if its restriction complex plane $\mathbb{C}_I$ is in the kernel of the corresponding Cauchy-Riemann operator. Here I is an element in the sphere of unit 1-vectors. For the class of slice monogenic functions we can repeat, with suitable modifications, the results mentioned in section 1 on slice regular functions. Let $\mathbb{B}_R = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| < R\}$. We say that $f : \mathbb{B}_R \to \mathbb{R}_n$ is (right) W-analytic in $\mathbb{B}_R$ if $f(\mathbf{x}) = \sum_{k=0}^{\infty} \mathbf{x}^k c_k$, where $c_k \in \mathbb{R}_n$, for all k, for all $\mathbf{x} \in \mathbb{B}_R$. Such a function f is slice monogenic in $\mathbb{B}_R$.

Given a W-analytic function f on $\mathbb{B}_R$, we can define the right convolution operator of paravector variable by mimicking Definition 2.1. Then we can consider the de la Vallée-Poussin convolution operator of a paravector variable for a W-analytic function f as above.

One can prove the generalization of Theorem 3.1 to this setting by noting that it is based on the validity of the Cauchy formula, on inequalities on norms and (5.1). Similarly, we can state and prove also the following Voronovskaja-type theorem, see Theorem 3.2, in which, as a consequence of (5.1), the inequality we obtain is:

$$\left\| \left| P_{m,r}(f) - f + \frac{e_2 \cdot \partial_s^2 f}{m} + \frac{e_1 \partial_s f}{m} \right\| \right|_d \leq C_n \frac{A_d(f)}{m^2}, m \in \mathbb{N},$$

where C_n is a constant depending on the dimension of $\mathbb{R}_n$. Most importantly, we can obtain the analogue of Theorem 3.5. It is crucial to note that its proof is based on lower estimates. Thus one has to verify that the fact the norm in $\mathbb{R}_n$ is not multiplicative has no influence.

One may also consider the approximation properties of the convolution obtained by using the Gauss-Weierstrass kernel. With the techniques illustrated in section 4, it is possible to show that the analogues of Theorem 4.2 and 4.3 hold also in this setting. Note, once more, that the fact that in the Clifford algebra $\mathbb{R}_n$, $n \geq 3$ one has lesser properties than $\mathbb{H}$ does not hinder the extension of the proofs given in the quaternion case. This is not assured, in general.

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