

Conjugation spaces and equivariant Chern classes

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Abstract

Let η be a Real bundle, in the sense of Atiyah, over a space X . This is a complex vector bundle together with an involution which is compatible with complex conjugation. We use the fact that BU has a canonical structure of a conjugation space, as defined by Hausmann, Holm, and Puppe, to construct equivariant Chern classes in certain equivariant cohomology groups of X with twisted integer coefficients. We show that these classes determine the (non-equivariant) Chern classes of η , forgetting the involution on X , and the Stiefel-Whitney classes of the real bundle of fixed points.

Introduction

Let X be a topological space and η a complex bundle over X . A *Real structure* on η , in the sense of Atiyah [1], is given by a couple of compatible involutions on X and on the total space of η such that the latter is complex anti-linear. The machinery of K-theory applies to these bundles and yields $KR(X)$, called Real K-theory. It comes equipped with two natural transformations: the forgetful map $KR(X) \rightarrow KU(X)$ and the “fixed points map” $KR(X) \rightarrow KO(X^\tau)$, where X^τ denotes the subspace of fixed points. Since its introduction by Atiyah, Real K-theory has proved to be a useful tool that interpolates between complex K-theory and ordinary real K-theory. For instance it allows one to prove in a unified way both

*The authors are partially supported by FEDER/MEC grant MTM2010-20692. The second author would like to thank the MPI in Bonn for its hospitality.

Received by the editors May 2011 - In revised form in March 2012.

Communicated by Y. Félix.

2000 *Mathematics Subject Classification* : Primary 57R20, 55N91; Secondary 55N15; 55P92; 55R10.

Key words and phrases : Conjugation spaces, characteristic classes, equivariant Chern classes.

Bott periodicity phenomena. It is therefore natural to search for a theory of characteristic classes for Real bundles, called equivariant Chern classes for reasons that will be clear from their construction, and, ideally, these should interpolate between ordinary Chern classes and Stiefel-Whitney classes.

Kahn provided such a construction in [12]. The equivariant Chern classes $\tilde{c}_n(\eta)$ live in the equivariant cohomology $H_{C_2}^{2n}(X; \mathbf{Z}(n))$ with twisted integral coefficients, where the action of the cyclic group of order 2 on $\mathbf{Z}(n)$ is by multiplication by $(-1)^n$. When the involution on X is trivial, he shows that the mod 2 reduction of the equivariant Chern classes are the Stiefel-Whitney classes of η . His ideas have found applications in algebro-geometric contexts, such as the work of dos Santos and Lima-Filho, [7].

In this article we adopt a slightly different point of view. We take advantage of the existence of universal Real bundles, namely the canonical bundles over $BU(n)$ with involution induced by complex conjugation, to carry out our construction. The key ingredient which makes the analysis of the situation quite elementary is that $BU(n)$ is a so-called *spherical conjugation space*, a notion introduced by Hausmann, Holm, and Puppe in [10]. It turns out that the conjugation structure, which lives at the level of mod 2 cohomology, upgrades to the level of the (correct choice of) twisted integral coefficients. The bundle of fixed points in the universal Real bundle is the universal bundle for real K-theory with base space $BO(n)$, and this explains the relationship between our new characteristic classes and the Stiefel-Whitney classes.

Proposition 5.2. *Let η be a Real bundle over X and η^τ the associated real bundle of fixed points. Then the image of $\tilde{c}_n(\eta)$ in $H^*(X^\tau; \mathbf{F}_2)[u]$ is $\sum_{i=0}^n Sq^i(w_n(\eta^\tau))u^{n-i}$.*

As a particular case, we get back Kahn’s main result in [12] about Real bundles over spaces with trivial involution. Notwithstanding Kahn’s construction, we believe that the introduction of the “conjugation space” structure sheds some more light on the properties of the equivariant Chern classes and in particular on the conceptual reasons of the appearance of the twisting of the coefficients.

Acknowledgements. We would like to thank Jean-Claude Hausmann for introducing us to the beautiful theory of conjugation spaces.

1 Notation and basic facts about conjugation spaces

The cyclic group of order two is C_2 and \mathbf{F}_2 is the field of 2 elements. The graded ring $H^*(BC_2; \mathbf{Z})$ is isomorphic to the ring $\mathbf{Z}[a]/(2a)$ where a has degree 2 and the mod 2 cohomology $H^*(BC_2; \mathbf{F}_2)$ is isomorphic to $\mathbf{F}_2[u]$ a polynomial algebra on a generator u of degree 1. If X is a topological space equipped with a (left) C_2 -action (often denoted by τ), the Borel construction, or the homotopy orbits, X_{hC_2} is the space $EC_2 \times_{C_2} X$. The equivariant cohomology functor is defined on C_2 -spaces as the integral cohomology of the Borel construction (and likewise with mod 2 coefficients): $H_{C_2}^*(X; \mathbf{Z}) \cong H^*(X_{hC_2}; \mathbf{Z})$. The inclusion of the fixed points $X^\tau = X^{C_2}$ in X induces a restriction homomorphism $r : H_{C_2}^*(X; \mathbf{F}_2) \rightarrow H_{C_2}^*(X^\tau; \mathbf{F}_2)$.

The latter is isomorphic to $H^*(X; \mathbf{F}_2) \otimes \mathbf{F}_2[u]$ since the action of C_2 is trivial on the fixed points.

Let us give now the basic facts about conjugation spaces which we will need to construct equivariant Chern classes. A *conjugation space* is a C_2 -space together with an H^* -frame (κ, σ) , i.e.

- a) an additive isomorphism $\kappa : H^{2^*}(X; \mathbf{F}_2) \rightarrow H^*(X^\tau; \mathbf{F}_2)$ dividing degrees by 2,
- b) an additive section $\sigma : H^{2^*}(X; \mathbf{F}_2) \rightarrow H^{2^*}(X_{hC_2}; \mathbf{F}_2)$ of the restriction map $\rho : H^{2^*}(X_{hC_2}; \mathbf{F}_2) \rightarrow H^{2^*}(X; \mathbf{F}_2)$,

which satisfy the conjugation equation:

$$r \circ \sigma(x) = \kappa(x)u^m + lt_m$$

for all $x \in H^{2^*}(X; \mathbf{F}_2)$ and all $m \in \mathbf{N}$, where lt_m is a polynomial in the variable u of degree less than m .

A *conjugation cell* is a C_2 -space which is equivariantly homeomorphic to the unit disk in \mathbf{C}^n equipped with complex conjugation. A *spherical conjugation complex* is a CW-complex X constructed from conjugation cells with equivariant attaching maps. Hausmann, Holm, and Puppe prove in their foundational article [10] that any spherical conjugation complex is a conjugation space. In fact, the original motivating examples of conjugation spaces are the complex Grassmannians $Gr_{n,k}(\mathbf{C})$, which are spherical. In particular the spaces $BU(n)$, for any $n \geq 1$, and BU are spherical conjugation spaces. The main property of an H^* -frame that we will keep using in this article is the following.

Theorem 1.1 (Hausmann, Holm, Puppe [10]). *The morphisms κ and σ in an H^* -frame are ring homomorphisms.*

2 Why one has to twist the coefficients

We wish to construct equivariant Chern classes for Real bundles in equivariant cohomology groups with integral coefficients and our main requirement is that one recovers the classical Chern classes by forgetting the C_2 -action. Thus, a first naïve, but natural, place to look for such classes is in the integral equivariant cohomology $H_{C_2}^*(X; \mathbf{Z})$ with *trivial* coefficients. We will illustrate in a fundamental example why this does not work. It will show us at the same time how to calibrate the correct answer.

Example 2.1. Consider the sphere S^2 equipped with the reflection through the equatorial circle. This is a spherical conjugation space obtained by identifying the boundary of a conjugation disc to a point. The Hopf bundle over S^2 is naturally a Real bundle, so its first equivariant Chern class should correspond to a copy of the integers in $H_{C_2}^2(S^2; \mathbf{Z}) = H^2((S^2)_{hC_2}; \mathbf{Z})$.

But in this case it is easy to identify the Borel construction, since S^2 consists of an equatorial copy of S^1 with trivial action and two discs, the hemispheres,

transposed by τ . As a consequence $S_{hC_2}^2$ is equivalent to the half-smash $S^1 \times \mathbf{R}P^\infty$, hence $H_{C_2}^2(S^2; \mathbf{Z}) \cong \mathbf{Z}/2$. As there is no non-trivial homomorphism from $\mathbf{Z}/2$ to $H^2(S^2; \mathbf{Z}) \cong \mathbf{Z}$, there is no way to recover $c_2(\eta)$ from any class in $H_{C_2}^2(S^2; \mathbf{Z})$.

However, the Hopf bundle over S^4 can also be seen as a Real bundle over the conjugation sphere of dimension 4 and one computes here that $H_{C_2}^4(S^4; \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}/2$. There is room here for an equivariant Chern class!

A closer inspection of the examples shows that the difference can be stated as follows: on cells of dimension $2 \bmod 4$ the conjugation reverses the orientation and on cells of dimension $0 \bmod 4$ it preserves the orientation¹. This leads us to look for even degree Chern classes in the ordinary equivariant cohomology group $H_{C_2}^*(X; \mathbf{Z})$ and for odd degree Chern classes in the equivariant cohomology $H_{C_2}^*(X; \mathbf{Z}(1))$ with twisted coefficients $\mathbf{Z}(1)$, the C_2 -module \mathbf{Z} endowed with the change of sign action.

Let us denote by $\mathbf{Z}(n)$ the C_2 -module \mathbf{Z} where the action is given by multiplication by $(-1)^n$. It is straightforward to see that $\text{Hom}_{\mathbf{Z}}(\mathbf{Z}(i), \mathbf{Z}(j)) \cong \mathbf{Z}(i+j)$ and $\mathbf{Z}(i) \otimes_{\mathbf{Z}} \mathbf{Z}(j) \cong \mathbf{Z}(i+j)$. Because the module structure depends only on the parity of n , we only keep the modules $\mathbf{Z}(0)$ and $\mathbf{Z}(1)$ and notice the following.

Remark 2.2. Let X be a spherical conjugation complex. The cup product turns the direct sum $H_{C_2}^*(X; \mathbf{Z}(0)) \oplus H_{C_2}^*(X; \mathbf{Z}(1))$ into a commutative algebra, which is natural in X with respect to equivariant maps. We observe that the submodule

$$\mathcal{H}_{C_2}^*(X; {}^t\mathbf{Z}) = \bigoplus_{n \geq 0} H_{C_2}^{4n}(X; \mathbf{Z}(0)) \oplus \bigoplus_{n \geq 0} H_{C_2}^{4n+2}(X; \mathbf{Z}(1))$$

is a subalgebra.

We turn now to a more detailed analysis of the algebra $\mathcal{H}_{C_2}^*(X, {}^t\mathbf{Z})$ for a spherical conjugation complex. For this we compute the Leray-Serre spectral sequences with twisted coefficients $\mathbf{Z}(0)$ and $\mathbf{Z}(1)$ associated to the canonical fibration

$$X \hookrightarrow X_{hC_2} \twoheadrightarrow BC_2.$$

We have to compute first the cohomology $H^*(X, \mathbf{Z}(n))$ as C_2 -modules, [6, Section VII.7], where the action of C_2 is the diagonal one, induced by the action on $C_*^{cell}(X)$ and $\mathbf{Z}(n)$. We write $H^n(X; \mathbf{Z})$ for the non-equivariant cohomology of X with trivial action. A spherical conjugation complex has only even dimensional cells by construction so that one finds:

$$H^n(X; \mathbf{Z}(0)) = \begin{cases} \mathbf{Z}(0) & \text{if } n = 0, \\ 0 & \text{if } n \text{ is odd,} \\ H^n(X; \mathbf{Z}) \otimes \mathbf{Z}(0) & \text{if } n = 0 \bmod 4, \\ H^n(X; \mathbf{Z}) \otimes \mathbf{Z}(1) & \text{if } n = 2 \bmod 4. \end{cases}$$

$$H^n(X; \mathbf{Z}(1)) = \begin{cases} \mathbf{Z}(1) & \text{if } n = 0, \\ 0 & \text{if } n \text{ is odd,} \\ H^n(X; \mathbf{Z}) \otimes \mathbf{Z}(1) & \text{if } n = 0 \bmod 4, \\ H^n(X; \mathbf{Z}) \otimes \mathbf{Z}(0) & \text{if } n = 2 \bmod 4. \end{cases}$$

¹A conjugation cell of dimension $2n$ can be viewed as an open unit disk in \mathbf{C}^n with a C_2 -action induced by complex conjugation, in particular it has a canonical orientation.

As for the cohomology of BC_2 one has $H^*(BC_2; \mathbf{Z}(0)) \cong \mathbf{Z}[a]/(2a)$ where a has degree 2, and $H^*(BC_2; \mathbf{Z}(1)) \cong \bigoplus_{i \geq 0} \mathbf{Z}/2 \langle e_{2i+1} \rangle$ where e_{2i+1} has degree $2i + 1$. Both spectral sequences have a vertical periodicity pattern of order 4 starting from the first line on.

q						
0 mod 4	$\oplus \mathbf{Z}$	0	$\oplus \mathbf{Z}/2$	0	$\oplus \mathbf{Z}/2$	$H^4(X; \mathbf{Z}) \otimes H^p(BC_2; \mathbf{Z})$
odd	0	0	0	0	0	
2 mod 4	0	$\oplus \mathbf{Z}/2$	0	$\oplus \mathbf{Z}/2$	0	
odd	0	0	0	0	0	
0	\mathbf{Z}	0	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$	$H^0(X; \mathbf{Z}) \otimes H^p(BC_2; \mathbf{Z})$
	0	1	2	3	4	p

 Table 1: Spectral sequence for coefficients $\mathbf{Z}(0)$

q						
0 mod 4	0	$\oplus \mathbf{Z}/2$	0	$\oplus \mathbf{Z}/2$	0	$H^2(X; \mathbf{Z}) \otimes H^p(BC_2; \mathbf{Z})$
odd	0	0	0	0	0	
2 mod 4	$\oplus \mathbf{Z}$	0	$\oplus \mathbf{Z}/2$	0	$\oplus \mathbf{Z}/2$	
odd	0	0	0	0	0	
0	0	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$	0	
	0	1	2	3	4	

 Table 2: Spectral sequence for coefficients $\mathbf{Z}(1)$

Lemma 2.3. *The two spectral sequences $H^p(BC_2; H^q(X; \mathbf{Z}(1))) \Rightarrow H_{C_2}^{p+q}(X; \mathbf{Z}(1))$ and $H^p(BC_2; H^q(X; \mathbf{Z}(0))) \Rightarrow H_{C_2}^{p+q}(X; \mathbf{Z}(0))$ degenerate at the page E_2 . Moreover, $H_{C_2}^{2n}(X; \mathbf{Z}(n)) \cong H^{2n}(X; \mathbf{Z}) \oplus \bigoplus_{p < n} H^{2p}(X; \mathbf{Z}) \otimes \mathbf{Z}/2$.*

Proof. Reducing the coefficients modulo 2 gives a natural transformation of spectral sequences from either of the spectral sequences to the spectral sequence with trivial coefficients \mathbf{F}_2 , converging to $H^*(X_{hC_2}; \mathbf{F}_2)$. As X is a conjugation space we know that the latter degenerates at the page E_2 .

A direct computation, using the description of $H^*(X; \mathbf{Z}(n))$, shows that on non-zero elements, mod 2 reduction induces an isomorphism for any $p > 1$:

$$H^p(BC_2; H^q(X; \mathbf{Z}(n))) \xrightarrow{\cong} H^p(BC_2; H^q(X; \mathbf{F}_2)).$$

Since on the right hand side we have permanent cycles the left hand side has to be made of permanent cycles too. In particular all differentials originated in the

vertical axis, $p = 0$, must be trivial, so that $H^0(BC_2; H^q(X; \mathbf{Z}(n)))$ is also made of permanent cycles.

The comparison with the mod 2 reduction also allows us to compute the module structure of $H_{C_2}^{2n}(X, \mathbf{Z}(n))$. The surjection of the associated graded modules

$$H^{2n}(X; \mathbf{Z}) \oplus \bigoplus_{p < n} H^{2p}(X; \mathbf{Z}) \otimes \mathbf{Z}/2 \twoheadrightarrow H^{2n}(X; \mathbf{F}_2) \oplus \bigoplus_{p < n} H^{2p}(X; \mathbf{F}_2)$$

is an isomorphism except on the first factor. The latter graded module is isomorphic to $H_{C_2}^{2n}(X; \mathbf{F}_2)$ as there are no non-trivial extensions of \mathbf{F}_2 -vector spaces, and thus there can be no non-trivial extensions in the former (a copy of $\mathbf{Z}/2^k$ for some $k > 1$ would imply the presence of $\mathbf{Z}/2^{k-1}$ in the kernel of the mod 2 reduction map, which is impossible). ■

Combining the edge homomorphisms of both spectral sequences we get a restriction homomorphism to ordinary cohomology.

Lemma 2.4. *The restriction map $\tilde{\rho} : \mathcal{H}_{C_2}^*(X, {}^t\mathbf{Z}) \rightarrow H^*(X; \mathbf{Z})$ is surjective and a ring homomorphism.* ■

For a spherical conjugation complex there is a canonical integral lift for the cohomological section encompassed in the H^* -frame.

Theorem 2.5. *Let X be a spherical conjugation complex with H^* -frame (σ, κ) . Then there is a unique section $\tilde{\sigma}$ to the restriction map $\tilde{\rho} : \mathcal{H}_{C_2}^*(X, {}^t\mathbf{Z}) \rightarrow H^*(X; \mathbf{Z})$ such that the mod 2 reduction of the section is σ . Moreover the section $\tilde{\sigma}$ is a ring homomorphism which is natural with respect to equivariant maps between spherical complexes.*

Proof. As X has only cells in even dimension, $H^*(X; \mathbf{Z})$ is free in each degree. We have an isomorphism of modules by Lemma 2.3

$$\mathcal{H}_{C_2}^*(X, {}^t\mathbf{Z}) \cong H^*(BC_2; \mathbf{Z}) \otimes H^*(X; \mathbf{Z}) \cong \mathbf{Z}[a]/(2a) \otimes H^*(X; \mathbf{Z}).$$

We also know that $H_{C_2}^*(X; \mathbf{F}_2) \cong H^*(BC_2; \mathbf{F}_2) \otimes H^*(X; \mathbf{F}_2) \cong \mathbf{F}_2[u] \otimes H^*(X; \mathbf{F}_2)$ since X is a conjugation space.

Let K^* be the kernel of the (surjective) map $H^*(X; \mathbf{Z}) \rightarrow H^*(X; \mathbf{F}_2)$. Since the mod 2 reduction induces isomorphisms $H^*(BC_2; \mathbf{Z}) \rightarrow \mathbf{F}_2[u]$ in any positive even degree, we have a commutative diagram of groups:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K^* & \longrightarrow & \mathcal{H}_{C_2}^{2*}(X, {}^t\mathbf{Z}) & \xrightarrow{\text{red}} & H_{C_2}^{2*}(X; \mathbf{F}_2) & \longrightarrow & 0 \\ & & \parallel & & \downarrow \tilde{\rho} & & \downarrow \rho & & \\ 0 & \longrightarrow & K^* & \longrightarrow & H^*(X; \mathbf{Z}) & \longrightarrow & H^*(X; \mathbf{F}_2) & \longrightarrow & 0, \end{array}$$

which implies that the right-hand square is a pull-back square. The section σ splitting the restriction ρ determines hence a unique section $\tilde{\sigma} : H^*(X; \mathbf{Z}) \rightarrow \mathcal{H}_{C_2}^*(X, {}^t\mathbf{Z})$. It is a ring homomorphism because both σ and the mod 2 reduction are so. Finally, naturality follows from that of the section σ in an H^* -frame and the fact that the pullback diagram is natural in X . ■

A little more can be extracted from the proof: the collapse of the two spectral sequences allows one to prove a version of the classical Leray-Hirsch theorem for the algebra $\mathcal{H}_{C_2}^*(X, {}^t\mathbf{Z})$. The fibration $X_{hC_2} \rightarrow BC_2$ splits since the fixed point set X^τ is not empty. As it is in fact connected, any two sections are homotopic, which determines a canonical copy of $H^*(BC_2; \mathbf{Z})$ in $\mathcal{H}_{C_2}^*(X, {}^t\mathbf{Z})$.

Corollary 2.6 (Leray-Hirsch for the equivariant cohomology). *Let X be a spherical conjugation space, then the canonical map*

$$\begin{aligned} \mathbf{Z}[a]/(2a) \otimes H^*(X; \mathbf{Z}) &\longrightarrow \mathcal{H}_{C_2}^*(X, {}^t\mathbf{Z}) \\ r \otimes x &\longmapsto r \cup \tilde{\sigma}(x) \end{aligned}$$

is a $\mathbf{Z}[a]/(2a)$ -algebra isomorphism, natural in X . ■

3 Equivariant cohomology computations for BU

A Real bundle (in the sense of Atiyah) [1] over a space X equipped with an involution τ , is a complex bundle η over X together with an involution on the total space, compatible with τ and which is anti-linear on the fibers. The tautological bundle over $BU(n)$ is the universal Real bundle of rank n , and the tautological bundle over BU is a universal stable bundle, where the conjugation on the base space is induced by complex conjugation on the coefficients of complex matrices (see for instance [14] for an explicit proof of this fact). Our definition of equivariant Chern classes will rest on the definition of the universal equivariant Chern classes as elements in $\mathcal{H}_{C_2}^*(BU, {}^t\mathbf{Z})$. We will thus need a good understanding of how this algebra behaves with respect to the Whitney sum and restriction to fixed points. Theorem 2.5 applies in particular to the spherical conjugation complex BU , equipped with the C_2 -action coming from complex conjugation. Recall that the ordinary cohomology $H^*(BU; \mathbf{Z})$ is a polynomial algebra in the ordinary universal Chern classes c_n .

Definition 3.1. The classes $\tilde{c}_n = \tilde{\sigma}(c_n) \in H_{C_2}^{2n}(BU; \mathbf{Z}(n))$ are the *universal equivariant Chern classes*.

From Corollary 2.6 we get the expected structure for the equivariant cohomology algebra $\mathcal{H}_{C_2}^*(BU, {}^t\mathbf{Z})$: it is isomorphic to a polynomial algebra over $\mathbf{Z}[a]/(2a)$ on the equivariant Chern classes \tilde{c}_n for $n \geq 1$. This result is the analogue of Kahn's [12, Théorème 3].

Proposition 3.2. *The equivariant cohomology algebra $\mathcal{H}_{C_2}^*(BU, {}^t\mathbf{Z})$ is isomorphic to $\mathbf{Z}[a, \tilde{c}_1, \tilde{c}_2, \dots]/(2a)$. ■*

The following result concerns the relationship between the equivariant Chern classes living in the equivariant cohomology of BU and the Stiefel-Whitney classes in the mod 2 cohomology of BO . The subspace of fixed points in BU under complex conjugation is $BU^\tau = BO$. Reducing the coefficients in the last cohomology algebra modulo 2 yields a map of equivariant homology groups

$\mathcal{H}_{C_2}^*(BU, {}^t\mathbf{Z}) \rightarrow H_{C_2}^*(BU; \mathbf{F}_2)$, and restricting then to the fixed points we get a homomorphism:

$$\bar{r} : \mathcal{H}_{C_2}^*(BU, {}^t\mathbf{Z}) \rightarrow H^*(BO; \mathbf{F}_2)[u].$$

The conjugation equation allows us to compute the image of the equivariant Chern classes.

Proposition 3.3. *The image of \tilde{c}_n in $H^*(BO; \mathbf{F}_2)[u]$ is $\sum_{i=0}^n Sq^i(w_n)u^{n-i}$.*

Proof. Let us write \bar{c}_i for the mod 2 reduction of the universal Chern classes. By definition of $\tilde{\sigma}$ we have:

$$\bar{r}(\tilde{c}_n) = \bar{r}(\tilde{\sigma}(c_n)) = r(\sigma(\bar{c}_n))$$

Franz and Puppe, [8, Theorem 1.1], computed for us the effect of the restriction to the fixed points on the image of the section σ . Here

$$r(\sigma(\bar{c}_n)) = \sum Sq^i(\kappa(\bar{c}_n)u^{n-i}) = \sum Sq^i(w_n)u^{n-i}$$

since, via the “halving isomorphism” κ , the mod 2 Chern classes correspond to the Stiefel-Whitney classes of the fixed point bundle, $\kappa(\bar{c}_n) = w_n$, [10, Proposition 6.8]. \blacksquare

The last result we will need is the behavior of the universal equivariant Chern classes under Whitney sum. Let $\mu : BU \times BU \rightarrow BU$ be the H-structure map which induces the Whitney sum on complex bundles. The space $BU \times BU$, under the diagonal action, is also a spherical conjugation space, [10, Proposition 4.5], and μ is an equivariant map. The cross product is a morphism

$$\times : H_{C_2}^{2n}(BU; \mathbf{Z}(n)) \otimes H_{C_2}^{2m}(BU; \mathbf{Z}(m)) \rightarrow H_{C_2 \times C_2}^{2(n+m)}(BU \times BU; \mathbf{Z}(n) \otimes \mathbf{Z}(m))$$

and the diagonal inclusion $C_2 \rightarrow C_2 \times C_2$ induces a map:

$$\delta : (BU \times BU)_{hC_2} \longrightarrow (BU \times BU)_{h(C_2 \times C_2)} \simeq BU_{hC_2} \times BU_{hC_2}.$$

Lemma 3.4. *The map induced by the multiplication $\mu : BU \times BU \rightarrow BU$ sends the universal equivariant Chern class \tilde{c}_n to the cross product $\delta^*(\sum_{i=0}^n \tilde{c}_i \times \tilde{c}_{n-i})$.*

Proof. We have a pull-back square of $\mathbf{Z}[a]/(2a)$ -modules, as in the proof of Theorem 2.5:

$$\begin{array}{ccc} \mathcal{H}_{C_2}^*(BU \times BU; {}^t\mathbf{Z}) & \xrightarrow{\text{red}} & H_{C_2}^{2*}(BU \times BU; \mathbf{F}_2) \\ \tilde{\rho} \downarrow & & \rho \downarrow \\ H^*(BU \times BU; \mathbf{Z}) & \longrightarrow & H^*(BU \times BU; \mathbf{F}_2). \end{array}$$

To prove the lemma it remains to show that both $\tilde{\rho}$ and the mod 2 reduction send $\mu(\tilde{c}_n)$ and $\delta^*(\sum_{i=0}^n \tilde{c}_i \times \tilde{c}_{n-i})$ to the same element. Let us first compute the image under $\tilde{\rho}$:

$$(\tilde{\rho} \circ \mu)(\tilde{c}_n) = (\mu \circ \tilde{\rho})(\tilde{c}_n) = \mu(\tilde{\rho}(\tilde{\sigma}(c_n))) = \mu(c_n)$$

The non-equivariant computation of $\mu(c_n)$, [2, Theorem 1.4], identifies this with the cross product $\sum_{i=0}^n c_i \times c_{n-i}$. Therefore, the naturality of the cross-product implies that

$$\tilde{\rho}(\mu(\tilde{c}_n)) = \sum_{i=0}^n c_i \times c_{n-i} = \sum_{i=0}^n \tilde{\rho}(\tilde{c}_i) \times \tilde{\rho}(\tilde{c}_{n-i}) = \tilde{\rho} \left(\delta^* \left(\sum_{i=0}^n \tilde{c}_i \times \tilde{c}_{n-i} \right) \right).$$

To compare next the images under mod 2 reduction, we denote by \bar{c}_i the reduction of the Chern class c_i . Then, by naturality of the section σ in an H^* -frame, we compute

$$\delta^* \left(\sum_{i=0}^n \tilde{c}_i \times \tilde{c}_{n-i} \right) \text{mod } 2 = \delta^* \left(\sum_{i=0}^n \sigma(\bar{c}_i) \times \sigma(\bar{c}_{n-i}) \right) = \sigma \left(\sum_{i=0}^n \bar{c}_i \times \bar{c}_{n-i} \right).$$

We have seen above that $\mu(c_n)$ coincides with the cross product $\sum c_i \times c_{n-i}$, and hence so does $\mu(\bar{c}_n)$ with $\sum \bar{c}_i \times \bar{c}_{n-i}$. Therefore the mod 2 reduction of the cross product equals to

$$\sigma(\mu(\bar{c}_n)) = \mu(\sigma(\bar{c}_n)) = \mu(\tilde{\sigma}(c_n) \text{mod } 2) = \mu(\tilde{c}_n) \text{mod } 2. \quad \blacksquare$$

4 Classical and equivariant Chern classes

We are now ready to introduce equivariant Chern classes for Real bundles. Recall the axiomatic definition of Chern classes, as stated by Hirzebruch in [11, Chap. 1, p. 66].

- I (Existence) For every complex bundle η over a finite dimensional paracompact space B and every integer $i \geq 0$ there exists a *Chern class* $c_i(\eta)$ in $H^{2i}(B; \mathbf{Z})$. The class $c_0(\eta) = 1$ is the unit element.
- II (Naturality) If $f : B_1 \rightarrow B_2$ is a map of spaces and ξ is a complex bundle over B_2 , then $f^*(c_i(\xi)) = c_i(f^*(\xi))$ for all $i \geq 0$.
- III (Whitney sum) If $\eta = \eta_1 \boxplus \eta_2$ then $c(\eta) = c(\eta_1)c(\eta_2)$, where $c(-)$ is the total Chern class $\sum_{i=0}^{\infty} c_i(-)$.
- IV (Normalization) If η denotes the canonical bundle over $\mathbf{C}P^1$ then $c(\eta) = 1 + h$ where $h \in H^2(\mathbf{C}P^1; \mathbf{Z}) \cong \mathbf{Z}$ is the natural generator.

We want our equivariant Chern classes to live in the equivariant cohomology $\mathcal{H}_{C_2}^*(-; {}^t\mathbf{Z})$ and this forces us to change Axiom IV. Let us go back to the conjugation sphere S^2 examined in Example 2.1. The equivariant Leray-Hirsh Theorem, Corollary 2.6, asserts that $\mathcal{H}_{C_2}^*(S^2; {}^t\mathbf{Z})$ is isomorphic as a $\mathbf{Z}[a]/(2a)$ -module to $\mathbf{Z}[a]/(2a) \oplus \mathbf{Z}[a]/(2a)\langle \tilde{h} \rangle$, where \tilde{h} is of degree 2 and restricts to a generator h of $H^2(S^2; \mathbf{Z})$. The new form of Axiom IV is:

- IV' If η denotes the canonical bundle over $\mathbf{C}P^1 = S^2$, with the canonical Real structure, then $c(\eta) = 1 + \tilde{h}$, where 1 and \tilde{h} are generators of degree 0 and 2 of the $\mathbf{Z}[a]/(2a)$ -module $\mathcal{H}_{C_2}^*(S^2; {}^t\mathbf{Z})$.

Axioms I, II, III and IV' determine uniquely such equivariant Chern classes. The proof is analogous to that for classical Chern classes, [11, p. 58] and is left to the interested reader. Notice that this requires the use of the splitting principle, which follows from [1, Theorem 2.1]. We thus proceed with the construction of these classes and show they satisfy all four axioms.

Definition 4.1. Let η be a Real bundle over the space X with (equivariant) classifying map $f : X \rightarrow BU$. Consider the classes $\tilde{c}_n = \tilde{\sigma}(c_n) \in \mathcal{H}_{C_2}^*(BU; {}^t\mathbf{Z})$ and pull them back along f^* . The equivariant cohomology classes $\tilde{c}_n(\eta) = f^*(\tilde{c}_n) \in \mathcal{H}_{C_2}^*(X; {}^t\mathbf{Z})$ are called the *equivariant Chern classes* of η .

The following result is our version of Kahn's [12, Théorème 2].

Theorem 4.2. *The equivariant Chern classes satisfy Axioms I, II, III, and IV'.*

Proof. All axioms, except Axiom III, are merely routine. Axiom I follows from the fact that $\tilde{\sigma}$ is a ring homomorphism. In particular the total equivariant Chern class of the trivial bundle is 1. Naturality (Axiom II) is a direct consequence of the existence of a universal bundle, and the proof of Axiom IV' is essentially contained in the observation that lead us to modify Axiom IV. We are thus left with Axiom III about the total equivariant Chern class of a Whitney sum. As this class is defined by applying the section $\tilde{\sigma}$ to the usual Chern class, we have basically to check that the construction of the Whitney sum behaves well with respect to the H^* -frame of a conjugation space.

Let ξ and η be two Real bundles over a compact space X with classifying maps f and g respectively. Then, a classifying map for the Whitney sum $\xi \boxplus \eta$ is the composite $\mu \circ (f \times g) \circ \Delta$, where μ denotes as above the H -structure map on BU that gives rise to the Whitney sum, and Δ is the diagonal map for the space X . We have thus to compute the image of \tilde{c}_n through

$$\begin{aligned} H_{C_2}^{2n}(BU; \mathbf{Z}(n)) &\xrightarrow{\mu} H_{C_2}^{2n}(BU \times BU; \mathbf{Z}(n)) \xrightarrow{(f \times g)^*} H_{C_2}^{2n}(X \times X; \mathbf{Z}(n)) \\ &\xrightarrow{\Delta^*} H_{C_2}^{2n}(X; \mathbf{Z}(n)). \end{aligned}$$

From Lemma 3.4 we understand the first morphism on \tilde{c}_n : It is the cross product $\delta^*(\sum \tilde{c}_i \times \tilde{c}_{n-i})$. We are thus lead to compute $(f \times g)^*(\tilde{c}_i \times \tilde{c}_{n-i}) = \tilde{c}_i(\xi) \times \tilde{c}_{n-i}(\eta)$ to which we must apply the composite

$$\begin{aligned} H_{C_2}^{2i}(X; \mathbf{Z}(i)) \otimes H_{C_2}^{2(n-i)}(X; \mathbf{Z}(n-i)) &\xrightarrow{\times} H_{C_2 \times C_2}^{2n}(X \times X; \mathbf{Z}(i) \otimes \mathbf{Z}(n-i)) \\ &\xrightarrow{\delta^*} H_{C_2}^{2n}(X \times X; \mathbf{Z}(n)) \xrightarrow{\Delta^*} H_{C_2}^{2n}(X; \mathbf{Z}(n)). \end{aligned}$$

But this is the product in the cohomology of X_{hC_2} with twisted coefficients. Hence $\tilde{c}(\xi \boxplus \eta) = \tilde{c}(\xi)\tilde{c}(\eta)$. ■

5 Real bundles, complex bundles, and real bundles

In this short section we make the relation between equivariant Chern classes, classical Chern classes, and Stiefel-Whitney classes explicit. A Real bundle can always be considered as a complex bundle by forgetting the involution and the universal equivariant Chern classes have been constructed by applying a section to the universal Chern classes. The following proposition is thus obvious and recorded for completeness.

Proposition 5.1. *Let η be a Real bundle over X . Then the image of $\tilde{c}_n(\eta)$ via $\mathcal{H}_{C_2}^{2n}(X; {}^t\mathbf{Z}) \rightarrow H^{2n}(X; \mathbf{Z})$ is $c_n(\eta)$. ■*

We analyze now the relation between the equivariant Chern classes of a Real bundle over X and the Stiefel-Whitney classes of the associated real bundle over the fixed points X^τ . The following proposition generalizes [12, Theorem 4], which deals with spaces with trivial involution, as well as [10, Proposition 6.8] which treats the case of bundles over spherical conjugation spaces. It gives a description of the images of the equivariant Chern classes through the homomorphism

$$\bar{r} : \mathcal{H}_{C_2}^*(X; {}^t\mathbf{Z}) \xrightarrow{r} \mathcal{H}_{C_2}^*(X^\tau; {}^t\mathbf{Z}) \rightarrow H_{C_2}^*(X^\tau; \mathbf{F}_2) \cong H^*(X^\tau; \mathbf{F}_2)[u].$$

Proposition 5.2. *Let η be a Real bundle over X and η^τ the associated real bundle of fixed points. Then the image of $\tilde{c}_n(\eta)$ in $H^*(X^\tau; \mathbf{F}_2)[u]$ is $\sum_{i=0}^n Sq^i(w_n(\eta^\tau))u^{n-i}$.*

Proof. This follows at once from the analogous computation we have done for BU in Proposition 3.3. If $f : X \rightarrow BU$ is an equivariant map classifying the Real bundle η , the equivariant Chern classes are obtained by pulling-back the universal ones through f and the Stiefel-Whitney classes of η^τ are obtained by pulling-back the universal ones through $f^\tau : X^\tau \rightarrow BO$. ■

We recover, in our context, Kahn's main result result [12, Théorème 4] for Real bundles over spaces with trivial involution.

Corollary 5.3. *Let η be a Real bundle over a space X with trivial involution and η^τ the associated real bundle of fixed points. Then the mod 2 reduction of $\tilde{c}_n(\eta)$ in $H^*(X; \mathbf{F}_2)$ is $w_n(\eta)$. ■*

Remark 5.4. We have seen that the equivariant Chern classes of a Real bundle determine the classical Chern classes by forgetting the C_2 -action and the Stiefel-Whitney classes of the fixed point bundle by reducing mod 2. In fact, the pull-back diagram we have used in the proof of Theorem 2.5 shows that these two sets of classes determine the equivariant Chern classes, as long as one works with a Real bundle over a conjugation space. In particular, the equivariant Chern classes of a Real bundle over a point are all zero.

A Stiefel-Whitney classes and Thom spaces

A particular case where all the above applies is that of a conjugation manifold, for then the tangent bundle is a Real bundle. For this case our results show a nice interplay between the equivariant Chern classes of the conjugation manifolds, its classical Chern classes, as well as the the Stiefel-Whitney classes of the fixed submanifold (i.e. the tangent bundle on the fixed manifold). There is one set of classes missing from this picture, it is the Stiefel-Whitney classes of the conjugation manifold itself! We thus end this paper with some remarks about them. Let M be a conjugation manifold of dimension $2n$, and $N = M^\tau$ denotes the submanifold of fixed points. The isomorphism κ in the H^* -frame relates the mod 2 cohomology of M in even degrees with that of N . Recall that the *Wu classes* are defined as the unique classes $v_k \in H^k(M; \mathbf{F}_2)$ such that for all $x \in H^{2n-k}(M; \mathbf{F}_2)$, $v_k \cup x = Sq^k(x)$. We show that κ behaves well with respect to both Wu and Stiefel-Whitney classes, a result which has been proved as well by Hambleton and Hausmann in [9, Proposition 2.9].

Theorem A.1. *Let M be a conjugation manifold of dimension $2n$. Let v_*^M and w_*^M (resp. v_*^N and w_*^N) denote the Wu and Stiefel-Whitney class of M (resp. of N). Then, for any $k \geq 1$,*

$$\kappa(v_{2k}^M) = v_k^N \quad \text{and} \quad \kappa(w_{2k}^M) = w_k^N.$$

Proof. The isomorphism κ preserves cup products and Steenrod squares so that:

$$\begin{aligned} \kappa(v_{2k}^M) \cup x &= \kappa(v_{2k}^M \cup \kappa^{-1}(x)) \\ &= \kappa(Sq^{2k}(\kappa^{-1}(x))) \\ &= Sq^k(x) \end{aligned}$$

for all $x \in H^{n-k}(N; \mathbf{F}_2)$. The uniqueness of the Wu classes implies that $\kappa(v_{2k}^M) = v_k^N$ for all $k \in \mathbf{N}$. If v denotes the total Wu class, w the total Stiefel-Whitney class of the tangent bundle, and Sq the total Steenrod square, then it is known that $Sq(v) = w$. Again, as κ commutes with Steenrod squares, we see that

$$w^N = Sq(v^N) = Sq(\kappa(v^M)) = \kappa(Sq(v^M)) = \kappa(w^M). \quad \blacksquare$$

Another way to relate the Stiefel-Whitney classes of M with those of N would be to analyze their construction via Thom spaces. We will not go through all the details, but indicate why both the tangent bundle and the Thom space of a conjugation manifold is still equipped with a conjugation structure. Of course the tangent bundle of M is not anymore a closed space, but as it is homotopy equivalent to M , one can check directly that the cohomological conditions are trivially satisfied as they match those satisfied by M .

Theorem A.2. *Let M be a conjugation manifold and N the fixed submanifold, then $TN = TM^\tau$.*

Proof. First choose a C_2 -invariant Riemann metric on M . A fixed element in TM is necessarily an element of $TM|_N = TN \oplus \nu_N$, where ν_N denotes the normal bundle of N , and the direct sum is orthogonal and compatible with the action

of C_2 . We know from the equivariant tubular neighborhood theorem that there exists a neighborhood of the zero section of ν_N , say \mathcal{V}_N such that the exponential map

$$\exp : \mathcal{V}_N \rightarrow M$$

is injective, and maps onto an open tubular neighborhood of N in M . Moreover, as the underlying metric is invariant, this map is equivariant with respect to the natural actions on \mathcal{V}_N and M . In particular, if there exists a vector in $TM|_N$ not in TN that is fixed, then, from the above orthogonal decomposition we get a non-zero vector in ν_N that is fixed, and therefore, via the exponential map, a fixed point outside N , a contradiction. ■

It follows that the natural compactification of TM , namely the Thom space $Th(M)$ is a conjugation space (not a conjugation manifold). The next result is an extension to the tangent bundle of M of a previous result of Hausmann, Holm, and Puppe concerning Real bundles over conjugation spaces (see [10, p. 946]).

Corollary A.3. *Let M be a conjugation manifold and N the fixed submanifold, then the Thom space $Th(M)$ is a conjugation space with fixed subspace $Th(N)$.*

Proof. The proof is exactly the same as [10, Proposition 6.4]. ■

For a manifold with involution, the Stiefel-Whitney numbers of the manifold are determined by the Stiefel-Whitney numbers of the fixed point submanifolds and of their normal bundles (see for instance [13]). In view of the above result for conjugation manifolds we have a slightly stronger result, compare also with [9, Corollary 2.14].

Corollary A.4. *Let M_1 and M_2 be conjugation manifolds with fixed submanifolds N_1 and N_2 respectively. Then M_1 and M_2 are (non-equivariantly) cobordant if and only if N_1 and N_2 are cobordant.*

Remark A.5. Brown and Peterson determined in [3] (see also the more complete versions [4] and [5]) all relations between Stiefel-Whitney classes which hold in a given degree for any n -dimensional manifold. In our search for obstructions to realizability of conjugation manifolds with a given fixed point submanifold, see also [15], we noticed that there are no obstructions to be found in terms of such relations. More precisely, if R is a relation involving only even degree Stiefel-Whitney classes of $2n$ -dimensional manifolds, then the corresponding relation r obtained by halving all degrees will also be true for any n -dimensional manifold.

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