Note on New General Integral Operators of p-valent Functions

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Abstract

We study new general integral operators of p-valent functions by giving sufficient conditions of p-valently starlikeness, p-valently close-to-convexness, uniformly p-valent close-to-convexness and strongly starlikeness of order τ $(0 < \tau \leq 1)$ in U (open unit disk). We end our investigation with an example from literature and some other references.

1 Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j,\ a_j \geq 0, p \in \{1, 2, \ldots\},\ z \in U \tag{1}$$

or

$$f(z) = z^p - \sum_{j=p+1}^{\infty} a_j z^j,\ a_j \geq 0, p \in \{1, 2, \ldots\},\ z \in U\,, \tag{2}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. We note that $\mathcal{A}_1 = \mathcal{A}$.

A function $f \in \mathcal{A}_p$ is said to be p-valently starlike of order β $(0 \leq \beta < p)$ iff

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (z \in U)\,.$$

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We denote by $\mathcal{S}_p^\star(\beta)$ the class of all functions from $\mathcal{A}_p$ which satisfy the condition above. On the other hand, a function $f \in \mathcal{A}_p$ is said to be p-valently convex of order β $(0 \leq \beta < p)$ if and only if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \quad (z \in U)\,.$$

Let $\mathcal{K}_p(\beta)$ be the class of all p-valently convex functions of order β in U. Furthermore, a function $f(z) \in \mathcal{A}_p$ is said to be in the class $\mathcal{C}_p(\beta)$ of p-valently close-to-convex functions of order β $(0 \leq \beta < p)$ in U iff

$$Re\left(\frac{f'(z)}{z^{p-1}}\right) > \beta \quad (z \in U)\,.$$

It is easy to be seen that $\mathcal{S}_p^\star(0) = \mathcal{S}_p^\star$, $\mathcal{K}_p(0) = \mathcal{K}_p$ and $\mathcal{C}_p(0) = \mathcal{C}_p$ are, respectively, the classes of p-valently starlike, p-valently convex and p-valently close-to-convex functions in U. We note also that $S_1^\star = S^\star$, $\mathcal{K}_1 = \mathcal{K}$ and $\mathcal{C}_1 = \mathcal{C}$ are, respectively, the well known classes of starlike, convex and close-to-convex functions in U.

A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{UC}_p(\beta)$ of uniformly p-valent close-to-convex functions of order β $(0 \leq \beta < p)$ in U iff

$$Re\left(\frac{zf'(z)}{g(z)} - \beta\right) \geq \left|\frac{zf'(z)}{g(z)} - p\right| \quad (z \in U)\,,$$

for some $g(z) \in \mathcal{US}_p(\beta)$, where $\mathcal{US}_p(\beta)$ is the class of uniformly p-valent starlike functions of order β $(-1 \leq \beta < p)$ in U that satisfy

$$Re\left(\frac{zf'(z)}{f(z)} - \beta\right) \geq \left|\frac{zf'(z)}{f(z)} - p\right| \quad (z \in U)\,. \tag{3}$$

The uniformly starlike functions are firstly introduced in [8].

2 Preliminary results

Definition 2.1. *[2] Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. We denote by $D_\lambda^\beta f(z)$ the linear operator defined by*

$$D_\lambda^\beta : A \to A\,, \quad D_\lambda^\beta f(z) = z + \sum_{j=n+1}^{\infty} [1 + (j-1)\lambda]^\beta a_j z^j\,. \tag{4}$$

Remark 2.1. *The following linear operator of complex functions with negative coefficients is introduced in ([1]) :*

$$D_\lambda^\beta : A \to A\,, \quad D_\lambda^\beta f(z) = z - \sum_{j=n+1}^{\infty} [1 + (j-1)\lambda]^\beta a_j z^j\,. \tag{5}$$

The neighborhoods with respect to the class of functions defined using the operator (5) is studied in [5].

Remark 2.2. *Let consider the following operator of the functions $f \in S$, $S = \{f \in \mathcal{A} : f$ is univalent in $U\}$:*

$$D^{n,\beta}_{\lambda_1,\lambda_2} f(z) = (h * \psi_1 * f)(z) = z \pm \sum_{k\geq 2} \frac{[1-\lambda_1(k-1))]^{\beta-1}}{[1-\lambda_2(k-1))]^{\beta}} \cdot \frac{1+c}{k+c} \cdot C(n,k) \cdot a_k \cdot z^k , \tag{6}$$

where $C(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$; $(n)_k$ is the Pochammer symbol; $k \geq 2$, $c \geq 0$ and $Re\{c\} \geq 0$; $z \in U$.

Remark 2.3. *If we denote by $(x)_k$ the Pochammer symbol, we define it as follows:*

$$(x)_k = \begin{cases} 1 & for\ k = 0,\ x \in \mathbb{C} \setminus \{0\} \\ x(x+1)(x+2)\cdot \ldots \cdot (x+k-1) & for\ k \in \mathbb{N} - \{0\}\ and\ x \in \mathbb{C}. \end{cases}$$

Let consider the following integral operators:

$$I^1(z) = \left\{ \beta \int_0^z t^{\beta\delta-1} \cdot \prod_{j=1}^{p} \left[\frac{((D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(t^n)')^{2\gamma_1-1}}{t^\sigma} \right]^{\delta_j^1} \cdot \left[\frac{(D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(t^n))^{2\gamma_2-1}}{t^\sigma} \right]^{\delta_j^2} dt \right\}^{\frac{1}{\beta}} , \tag{7}$$

where α, γ_1, γ_2, $\beta \in \mathbb{C}$, $Re\alpha = a > 0$ and $D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z) \in \mathcal{A}$, λ_1, λ_2, $\kappa \geq 0$, $\sigma \in \mathbb{R}$, $j = \overline{1,p}$, $p \in \mathbb{N}$, $D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$ of form (6) and

$$I^2(z) = \left\{ \chi \int_0^z t^{\chi\delta-1} \prod_{j=1}^{p} \left[\frac{((D^{\beta}_{\lambda} f_j(t^n)')^{2\gamma_1-1}}{t^\sigma} \right]^{\delta_j^1} \left[\frac{(D^{\beta}_{\lambda} f_j(t^n))^{2\gamma_2-1}}{t^\sigma} \right]^{\delta_j^2} dt \right\}^{\frac{1}{\chi}} , \tag{8}$$

where α, γ_1, γ_2, $\chi \in \mathbb{C}$, $Re\alpha = a > 0$ and $D^{\beta}_{\lambda} f_j(z) \in \mathcal{A}$, $\beta \geq 0$, $\lambda \geq 0$, $\sigma \in \mathbb{R}$, $D^{\beta}_{\lambda} f(z^n)$ of form (5).

We will make use of the following Lemmas in order to derive our main results.

Lemma 2.1. *[10] If $f \in \mathcal{A}_p$ satisfies*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < p + \frac{1}{4} \quad (z \in U),$$

then f is p-valently starlike in U.

Lemma 2.2. *[6] If $f \in \mathcal{A}_p$ satisfies*

$$\left| \frac{zf''(z)}{f'(z)} + 1 - p \right| < p + 1 \quad (z \in U),$$

then f is p-valently starlike in U.

Lemma 2.3. *[13] If $f \in \mathcal{A}_p$ satisfies*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} < p+\frac{a+b}{(1+a)(1-b)} \quad (z \in U),$$

where $a > 0$, $b \geq 0$ and $a+2b \leq 1$, then f is p-valently close-to-convex in U.

Lemma 2.4. *[3] If $f \in \mathcal{A}_p$ satisfies*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} < p+\frac{1}{3} \quad (z \in U),$$

then f is uniformly p-valent close-to-convex in U.

Lemma 2.5. *[14] If $f \in \mathcal{A}_p$ satisfies*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{p}{4}-1 \quad (z \in U),$$

then

$$Re\sqrt{1+\frac{zf'(z)}{f(z)}} > \frac{\sqrt{p}}{2} \quad (z \in U).$$

Lemma 2.6. *[11] If $f \in \mathcal{A}_p$ satisfies*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > p-\frac{\tau}{2} \quad (z \in U),$$

then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| > \frac{\pi}{2}\tau \quad (z \in U).$$

Let consider the following integral operators:

$$I_p^1(z) = \left\{\beta\int_0^z pt^{p-1}\cdot\prod_{j=1}^m\left[\frac{((D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(t^n)')^{2\gamma_1-1}}{t^p}\right]^{\delta_j^1}\cdot\left[\frac{(D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(t^n))^{2\gamma_2-1}}{t^p}\right]^{\delta_j^2}dt\right\}^{\frac{1}{\beta}} \tag{9}$$

and

$$I_p^2(z) = \left\{\chi\int_0^z pt^{p-1}\prod_{j=1}^m\left[\frac{((D^{\beta}_{\lambda}f_j(t^n)')^{2\gamma_1-1}}{t^p}\right]^{\delta_j^1}\left[\frac{(D^{\beta}_{\lambda}f_j(t^n))^{2\gamma_2-1}}{t^p}\right]^{\delta_j^2}dt\right\}^{\frac{1}{\chi}}. \tag{10}$$

We derive new general integral operators from (9) (or (10)), for which we study the sufficient conditions of p-valently starlikeness, p-valently close-to-convexness, uniformly p-valent close-to-convexness and strongly starlikeness of order τ $(0 < \tau \leq 1)$ in U, giving also several examples that prove its relevance.

3 Main Results

We consider the following operators:

$$I_p^3(z) = \left\{ \beta \int_0^z pt^{p-1} \cdot \prod_{j=1}^{m} \left[\frac{((D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(t^n)'')^{2\gamma_1 - 1}}{pt^{p-1}} \right]^{\delta_j^1} \cdot \left[\frac{[(D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(t^n))']^{2\gamma_2 - 1}}{pt^{p-1}} \right]^{\delta_j^2} dt \right\}^{\frac{1}{\beta}} \tag{11}$$

with respect to the general integral operator $I_p^1(z)$ of form (9) and

$$I_p^4(z) = \left\{ \chi \int_0^z pt^{p-1} \prod_{j=1}^{m} \left[\frac{((D^{\beta}_{\lambda} f_j(t^n)'')^{2\gamma_1 - 1}}{pt^{p-1}} \right]^{\delta_j^1} \left[\frac{[(D^{\beta}_{\lambda} f_j(t^n))']^{2\gamma_2 - 1}}{pt^{p-1}} \right]^{\delta_j^2} dt \right\}^{\frac{1}{\chi}} \tag{12}$$

with respect to the general integral operator $I_p^2(z)$ of form (11).

Further, we give sufficient conditions for the operator $I_p^3(z)$ of form (11) to be p-valently starlike, p-valently close-to-convex, uniformly p-valent close-to-convex and strongly starlike of order τ $(0 < \tau \leq 1)$ in U.

Sufficient conditions for the operator $I_p^3(z)$

For further simplification, we note the integral operator $I_p^3(z)$ of form (11) as follows:

$$I_p^3(z) = \left\{ \beta \int_0^z pt^{p-1} \cdot \prod_{\substack{j=1 \\ a\in\{1,2\}}}^{m} \left[\frac{((D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(t^n))^{2\gamma_a - 1}}{pt^{p-1}} \right]^{\delta_j^a} dt \right\}^{\frac{1}{\beta}}, \tag{13}$$

where $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f_j(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))''$ and $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f_j(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))'$, $z \in U$.

We firstly study the sufficient conditions for the operator $I_p^3(z)$ to be in the class $\mathcal{S}_p^\star$.

Theorem 3.1. *Let δ_j^a, $\gamma_a \in \mathbb{C}$, $a \in \{1,2\}$, $j = \overline{1,m}$. If $f_j \in \mathcal{A}_p$ for all $j = \overline{1,m}$, satisfies*

$$Re \left[1 + \frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a - 1}]''}{[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a - 1}]'} \right] < p + \frac{1}{4 \cdot \sum\limits_{\substack{j=1 \\ a\in\{1,2\}}}^{m} \delta_j^a} \quad (z \in U), \tag{14}$$

where $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f_j(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))''$ and $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f_j(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))'$, $z \in U$, $D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$ is of form (6), then $I_p^3(z)$ is p-valently starlike in U.

Proof.

From (9) we see that $I_p^3(z) \in \mathcal{A}_p$. Moreover, by differentiating (13) logarithmically, multiplying by z and adding 1 , we have

$$1+\frac{z[I_p^3(z)]''}{[I_p^3(z)]'}=\frac{1-\beta}{\beta}\cdot\frac{\left[\int_0^z pt^{p-1}\cdot\prod^m{}_{\substack{j=1\\a\in\{1,2\}}}\left[\frac{((D^{n,\kappa,a}_{\lambda_1,\lambda_2}f_j(t^n))^{2\gamma_a-1}}{pt^p}\right]^{\delta_j^a}dt\right]'}{\int_0^z pt^{p-1}\cdot\prod^m{}_{\substack{j=1\\a\in\{1,2\}}}\left[\frac{((D^{n,\kappa,a}_{\lambda_1,\lambda_2}f_j(t^n))^{2\gamma_a-1}}{pt^{p-1}}\right]^{\delta_j^a}dt} \tag{15}$$

$$+p\left(1-\sum_{\substack{j=1\\a\in\{1,2\}}}^{m}\delta_j^a\right)+\sum_{\substack{j=1\\a\in\{1,2\}}}^{m}\delta_j^a(2\gamma_a-1)\cdot nz^{n-1}\cdot\frac{z(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f_j(z^n))'}{D^{n,\kappa,a}_{\lambda_1,\lambda_2}f_j(z^n)},$$

where $D^{n,\kappa,1}_{\lambda_1,\lambda_2}f_j(z^n)=(D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n))''$ and $D^{n,\kappa,2}_{\lambda_1,\lambda_2}f_j(z^n)=(D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n))'$, $z\in U$, $D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n)$ is of form (6).

After we take the real part of (15) and consider its conditions, we obtain the following

$$Re\left(1+\frac{z[I_p^3(z)]''}{[I_p^3(z)]'}\right)\le\frac{1-\beta}{\beta}+p\left(1-\sum_{\substack{j=1\\a\in\{1,2\}}}^{m}\delta_j^a\right)+$$
$$\sum_{\substack{j=1\\a\in\{1,2\}}}^{m}\delta_j^a\cdot Re\left[\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f_j(z^n))^{2\gamma_a-1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f_j(z^n))^{2\gamma_a-1}}\right]. \tag{16}$$

From (14) and (16) we obtain that

$$Re\left(1+\frac{z[I_p^3(z)]''}{[I_p^3(z)]'}\right)<\frac{1-\beta}{\beta}+p\left(1-\sum_{\substack{j=1\\a\in\{1,2\}}}^{m}\delta_j^a\right)+\sum_{\substack{j=1\\a\in\{1,2\}}}^{m}\delta_j^a\cdot\left(p+\frac{1}{4\cdot\sum_{\substack{j=1\\a\in\{1,2\}}}^{m}\delta_j^a}\right) \tag{17}$$
$$=\frac{1-\beta}{\beta}+p+\frac{1}{4}.$$

We apply Lemma 2.1 to an integral operator of order $\frac{1}{\beta}$ and we obtain immediately that $I_p^3(z)\in\mathcal{S}_p^\star$.

Remark 3.1. *If $\beta=1$, we apply Lemma 2.1 to (17) and we directly obtain that $I_p^3(z)\in\mathcal{S}_p^\star$ $(\beta=1)$, where (17) becomes $Re\left(1+\frac{z[I_p^3(z)]''}{[I_p^3(z)]'}\right)<p+\frac{1}{4}$.*

Remark 3.2. *Let $\delta_j^1 = 0$ and $\beta \in \mathbb{C}$. Then, for all $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, we obtain the following integral operator:*

$$I_p^{31}(z) = \left\{ \beta \int_0^z pt^{p-1} \cdot \prod_{j=1}^{m} \left[\frac{(D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(t^n))^{2\gamma_2 - 1}}{pt^{p-1}} \right]^{\delta_j^2} dt \right\}^{\frac{1}{\beta}} . \quad (18)$$

On the other hand, if $\delta_j^2 = 0$, we obtain the following integral operator:

$$I_p^{32}(z) = \left\{ \beta \int_0^z pt^{p-1} \cdot \prod_{j=1}^{m} \left[\frac{(D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(t^n))^{2\gamma_2 - 1}}{pt^{p-1}} \right]^{\delta_j^2} dt \right\}^{\frac{1}{\beta}} . \quad (19)$$

Corollary 3.1. *a) Let $\delta_j^1 = 0$, δ_j^2, $\gamma_2 \in \mathbb{C}$, $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$. If $f_j \in \mathcal{A}_p$ for all $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, satisfy*

$$Re \left[1 + \frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))^{2\gamma_2 - 1}]''}{([(D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))^{2\gamma_2 - 1}]'} \right] < p + \frac{1}{4 \cdot \sum_{j=1}^{m} \delta_j^2} \quad (z \in U),$$

where $D_{\lambda_1,\lambda_2}^{n,\kappa,1} f_j(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))''$ and $D_{\lambda_1,\lambda_2}^{n,\kappa,2} f_j(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))'$, $z \in U$, $D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n)$ is of form (6), then $I_p^{31}(z)$ of form (18) is p-valently starlike in U.

b) Let $\delta_j^2 = 0$, δ_j^1, $\gamma_1 \in \mathbb{C}$, $j = \overline{1,m}$. If $f_j \in \mathcal{A}_p$, for all $j = \overline{1,m}$, satisfy

$$Re \left[1 + \frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))^{2\gamma_2 - 1}]''}{([(D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))^{2\gamma_2 - 1}]'} \right] < p + \frac{1}{4 \cdot \sum_{j=1}^{m} \delta_j^1} \quad (z \in U),$$

where $D_{\lambda_1,\lambda_2}^{n,\kappa,1} f_j(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))''$ and $D_{\lambda_1,\lambda_2}^{n,\kappa,2} f_j(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))'$, $z \in U$, $D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n)$ is of form (6), then $I_p^{32}(z)$ of form (19) is p-valently starlike in U.

Furthermore, if we take $j = p = 1, \forall j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, $\delta_1^1 = \delta^1 \in \mathbb{C}$, $\delta_1^2 = \delta^2 \in \mathbb{C}$ and $D_{\lambda_1,\lambda_2}^{n,\kappa} f_1(z^n) = D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$ in Theorem 3.1, we obtain the result below.

Corollary 3.2. *If $f \in \mathcal{A}$, $a \in \{1,2\}$, satisfies the condition*

$$Re \left[1 + \frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n))^{2\gamma_2 - 1}]''}{([(D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n))^{2\gamma_2 - 1}]'} \right] < 1 + \frac{1}{4 \prod_{a \in \{1,2\}} \delta^a} \quad (z \in U),$$

then $\left\{ \beta \int_0^z pt^{p-1} \cdot \prod_{a \in \{1,2\}} \left[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(t^n))^{2\gamma_a - 1} \right]^{\delta^a} dt \right\}^{\frac{1}{\beta}}$ is starlike in U, for any $\gamma_1, \gamma_2 \in \mathbb{C}$, $D_{\lambda_1,\lambda_2}^{n,\kappa,1} f(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n))''$ and $D_{\lambda_1,\lambda_2}^{n,\kappa,2} f(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n))'$, where $D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$ is of form (6), $j = 1$, $z \in U$.

Moreover, if we take $\delta_1^1 = \delta_1^2 = \delta \in \mathbb{C}$ in Corollary 3.2, we have the following result:

Corollary 3.3. *If $f \in \mathcal{A}$, $\delta \in \mathbb{C}$ and $a \in \{1,2\}$, satisfies the condition*

$$Re\left[1 + \frac{z[(D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_2 - 1}]''}{([(D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n))^{2\gamma_2 - 1}]'}\right] < 1 + \frac{1}{4\delta} \quad (z \in U),$$

then $\int\limits_0^z \prod\limits_{a\in\{1,2\}} \left[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f(t^n))^{2\gamma_a - 1}\right]^{\delta} dt$ *is starlike in U, for any γ_1, $\gamma_2 \in \mathbb{C}$, $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n))''$ and $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f(z^n))'$, where $D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$ is of form (6), $j = 1$, $z \in U$.*

Theorem 3.2. *Let δ_j^a, $\gamma_a \in \mathbb{C}$, $a \in \{1,2\}$, $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$. If $f_j \in \mathcal{A}_p$ for all $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, satisfy the condition*

$$\left|1 + \frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a - 1}]''}{[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a - 1})'}\right| < \frac{p+1}{4 \cdot \sum\limits_{\substack{j=1 \\ a\in\{1,2\}}}^{m} \delta_j^a} - p + 1 \quad (z \in U), \tag{20}$$

where $\sum\limits_{\substack{j=1 \\ a\in\{1,2\}}}^{m} \delta_j^a > 1$, $D^{n,\kappa,1}_{\lambda_1,\lambda_2} f_j(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))''$ and $D^{n,\kappa,2}_{\lambda_1,\lambda_2} f_j(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n))'$, $z \in U$, $z \in U$, $D^{n,\kappa}_{\lambda_1,\lambda_2} f_j(z^n)$ is of form (6), then $I_p^1(z)$ is p-valently starlike in U.

Proof.

From the relation (15) and the hypothesis (20), we obtain the following:

$$\begin{aligned}
\left|1 + \frac{z[I_p^3(z)]''}{[I_p^3(z)]'}\right| &\le \frac{1-\beta}{\beta} + \left|\sum_{\substack{j=1 \\ a\in\{1,2\}}}^{m} \delta_j^a \left(\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a - 1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a - 1}} - (p-1)\right)\right| \\
&< \frac{1-\beta}{\beta} + \sum_{\substack{j=1 \\ a\in\{1,2\}}}^{m} \delta_j^a \left|\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a - 1}]'}{(D^{n,\kappa,a}_{\lambda_1,\lambda_2} f_j(z^n))^{2\gamma_a - 1}} - p + 1\right| \\
&< \frac{1-\beta}{\beta} + (p-1) \sum_{\substack{j=1 \\ a\in\{1,2\}}}^{m} \delta_j^a + \sum_{\substack{j=1 \\ a\in\{1,2\}}}^{m} \delta_j^a \left(\frac{p+1}{4 \cdot \sum\limits_{\substack{j=1 \\ a\in\{1,2\}}}^{m} \delta_j^a} - (p-1)\right) \\
&< \frac{1-\beta}{\beta} + p + 1.
\end{aligned}$$

Using Lemma 2.2 for an integral operator of order $\frac{1}{\beta}$, we get immediately that $I_p^3(z) \in \mathcal{S}_p^{\star}$.

Remark 3.3. *If $\beta = 1$, we apply Lemma 2.2 to the relation above and we obtain that $I_p^3(z) \in \mathcal{S}_p^\star$ $(\beta = 1)$ under the condition (20), where $\left|1 + \frac{z[I_p^3(z)]''}{[I_p^3(z)]'}\right| < p+1$, $z \in U$.*

Letting $j = p = 1$, $\delta_1^1 = \delta^1 \in \mathbb{C}$, $\delta_1^2 = \delta^2 \in \mathbb{C}$ and $D_{\lambda_1,\lambda_2}^{n,\kappa} f_1(z^n) = D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$ in Theorem 3.2, we have the following corollary.

Corollary 3.4. *If $f \in \mathcal{A}$, $a \in \{1,2\}$, satisfies the condition below*

$$\left|\frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f_j(z^n))^{2\gamma_a-1}]''}{[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f_j(z^n))^{2\gamma_a-1}]'}\right| < \frac{2}{\sum\limits_{a\in\{1,2\}} \delta^a} \qquad (z \in U),$$

then $\left\{\beta \int\limits_0^z \prod\limits_{a\in\{1,2\}} \left[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(t^n))^{2\gamma_a-1}\right]^{\delta^a} dt\right\}^{\frac{1}{\beta}}$ is starlike in U, for any γ_1, $\gamma_2 \in \mathbb{C}$, $D_{\lambda_1,\lambda_2}^{n,\kappa,1} f(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n))''$ and $D_{\lambda_1,\lambda_2}^{n,\kappa,2} f(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n))'$, where $D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$ is of form (6), $(j = 1)$, $z \in U$.

Moreover, if we take $\delta_1^1 = \delta_1^2 = \delta \in \mathbb{C}$ in Corollary 3.4, we have the following result:

Corollary 3.5. *If $f \in \mathcal{A}$, $\delta \in \mathbb{C}$ and $a \in \{1,2\}$, satisfies the condition*

$$\left|\frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f_j(z^n))^{2\gamma_a-1}]''}{[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f_j(z^n))^{2\gamma_a-1}]'}\right| < \frac{2}{\delta} \qquad (z \in U),$$

then $\left\{\beta \int\limits_0^z \prod\limits_{a\in\{1,2\}} \left[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(t^n))^{2\gamma_a-1}\right]^{\delta} dt\right\}^{\frac{1}{\beta}}$ is starlike in U, for any γ_1, $\gamma_2 \in \mathbb{C}$, $D_{\lambda_1,\lambda_2}^{n,\kappa,1} f(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n))''$ and $D_{\lambda_1,\lambda_2}^{n,\kappa,2} f(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n))'$, where $D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$ is of form (6), $(j = 1)$, $z \in U$.

Next, we apply Lemma 2.3 and Lemma 2.4 in order to obtain sufficient conditions for I_p^3 to be p-valently close-to-convex and uniformly p-valent close to convex in U.

Theorem 3.3. *Let δ_j^r, $\gamma_r \in \mathbb{C}$, $r \in \{1,2\}$, $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$. If $f_j \in \mathcal{A}_p$ for all $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, satisfy*

$$Re\left[1 + \frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa,r} f_j(z^n))^{2\gamma_r-1}]''}{[(D_{\lambda_1,\lambda_2}^{n,\kappa,r} f_j(z^n))^{2\gamma_r-1}]'}\right] < p + \frac{a+b}{(1+a)(1-b)\cdot \sum\limits_{\substack{j=1\\ r\in\{1,2\}}}^{m} \delta_j^r} \qquad (z \in U), \tag{21}$$

where $a > 0$, $b \geq 0$, $a + 2b \leq 1$, $D_{\lambda_1,\lambda_2}^{n,\kappa,1} f_j(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))''$ and $D_{\lambda_1,\lambda_2}^{n,\kappa,2} f_j(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))'$, $z \in U$, $z \in U$, $D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n)$ is of form (6), then $I_p^3(z)$ is p-valently close-to-convex in U.

Proof. Using (16) and (21), we apply Lemma 2.3 to an integral operator of order $\frac{1}{\beta}$ and we have that $I_p^1(z) \in \mathcal{C}_p(\alpha)$ $(0 \le \alpha < p)$.

Letting $j = p = 1$, $\delta_1^1 = \delta^1 \in \mathbb{C}$, $\delta_1^2 = \delta^2 \in \mathbb{C}$ and $D_{\lambda_1,\lambda_2}^{n,\kappa} f_1(z^n) = D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$ in Theorem 3.3, we have:

Corollary 3.6. *If $f \in \mathcal{A}$ and $r \in \{1,2\}$, satisfies the following condition*

$$Re\left[1+\frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa,r} f(z^n))^{2\gamma_r-1}]''}{[(D_{\lambda_1,\lambda_2}^{n,\kappa,r} f(z^n))^{2\gamma_r-1}]'}\right] < p + \frac{a+b}{(1+a)(1-b)\cdot \sum\limits_{a\in\{1,2\}}^{m} \delta^a} \qquad (z \in U),$$

where $a > 0$, $b \ge 0$, $a+2b \le 1$, then $\left\{\beta \int\limits_0^z \prod\limits_{r\in\{1,2\}} \left[(D_{\lambda_1,\lambda_2}^{n,\kappa,r} f(t^n))^{2\gamma_r-1}\right]^{\delta^r} dt\right\}^{\frac{1}{\beta}}$ is close-to-convex in U, for any γ_1, $\gamma_2 \in \mathbb{C}$, $D_{\lambda_1,\lambda_2}^{n,\kappa,1} f(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n))''$ and $D_{\lambda_1,\lambda_2}^{n,\kappa,2} f(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n))'$, where $D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$ is of form (6), $(j=1)$, $z \in U$.

Moreover, if we take $\delta_1^1 = \delta_1^2 = \delta \in \mathbb{C}$ in Corollary 3.6, we have the following result:

Corollary 3.7. *If $f \in \mathcal{A}$, $\delta \in \mathbb{C}$, $r \in \{1,2\}$, satisfies the condition*

$$Re\left[\frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa,r} f(z^n))^{2\gamma_r-1}]'}{(D_{\lambda_1,\lambda_2}^{n,\kappa,r} f(z^n))^{2\gamma_r-1}}\right] < p + \frac{a+b}{(1+a)(1-b)\cdot\delta} \qquad (z \in U),$$

where $a > 0$, $b \ge 0$, $a+2b \le 1$, then $\left\{\beta \int\limits_0^z \prod\limits_{r\in\{1,2\}} \left[(D_{\lambda_1,\lambda_2}^{n,\kappa,r} f(t^n))^{2\gamma_r-1}\right]^{\delta} dt\right\}^{\frac{1}{\beta}}$ is close-to-convex in U, for any γ_1, $\gamma_2 \in \mathbb{C}$, $D_{\lambda_1,\lambda_2}^{n,\kappa,1} f(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n))'$ and $D_{\lambda_1,\lambda_2}^{n,\kappa,2} f(z^n) = D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$, $D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$ is of form (6), $(j=1)$, $z \in U$.

Theorem 3.4. *Let δ_j^a, $\gamma_a \in \mathbb{C}$, $r \in \{1,2\}$, $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, . If $f_j \in \mathcal{A}_p$ for all $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, satisfy*

$$Re\left[1+\frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa,r} f_j(z^n))^{2\gamma_r-1}]''}{[(D_{\lambda_1,\lambda_2}^{n,\kappa,r} f_j(z^n))^{2\gamma_r-1}]'}\right] < p + \frac{1}{3\cdot \sum\limits_{\substack{j=1\\ a\in\{1,2\}}}^{m} \delta_j^a} \qquad (z \in U), \tag{22}$$

where $D_{\lambda_1,\lambda_2}^{n,\kappa,1} f_j(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))''$ and $D_{\lambda_1,\lambda_2}^{n,\kappa,2} f_j(z^n) = (D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n))'$, $z \in U$, $z \in U$, $D_{\lambda_1,\lambda_2}^{n,\kappa} f_j(z^n)$ is of form (6), then $I_p^3(z)$ is uniformly p-valent close-to-convex in U.

Proof. From (16), (22) and applying Lemma 2.4 to an integral operator of order $\frac{1}{\beta}$, we get that $I_p^3(z) \in \mathcal{UC}_p(\beta)$.

Letting $j = p = 1$, $\delta_1^1 = \delta^1 \in \mathbb{C}$, $\delta_1^2 = \delta^2 \in \mathbb{C}$ and $D_{\lambda_1,\lambda_2}^{n,\kappa} f_1(z^n) = D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$ in Theorem 3.4, we have the result below.

Corollary 3.8. *If $f \in \mathcal{A}$, $a \in \{1,2\}$, satisfies the following condition*

$$Re\left[1+\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}]''}{[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}]'}\right] < p+\frac{1}{3\cdot\sum\limits_{a\in\{1,2\}}^{m}\delta^a} \quad (z\in U),$$

then $\left\{\beta\int\limits_0^z\prod\limits_{a\in\{1,2\}}\left[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(t^n))^{2\gamma_a-1}\right]^{\delta^a}dt\right\}^{\frac{1}{\beta}}$ *is uniformly close-to-convex in U, for any γ_1, $\gamma_2\in\mathbb{C}$, $D^{n,\kappa,1}_{\lambda_1,\lambda_2}f(z^n)=(D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n))''$ and $D^{n,\kappa,2}_{\lambda_1,\lambda_2}f(z^n)=(D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n))'$, where $D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n)$ is of form (6), $(j=1)$, $z\in U$.*

Moreover, if we take $\delta_1^1=\delta_1^2=\delta\in\mathbb{C}$ in Corollary 3.8, we have the following result:

Corollary 3.9. *If $f\in\mathcal{A}$, $\delta\in\mathbb{C}$, $a\in\{1,2\}$, satisfies the condition*

$$Re\left[1+\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}]''}{[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}]'}\right] < p+\frac{1}{3\delta} \quad (z\in U),$$

then $\left\{\beta\int\limits_0^z\prod\limits_{a\in\{1,2\}}\left[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(t^n))^{2\gamma_a-1}\right]^{\delta}dt\right\}^{\frac{1}{\beta}}$ *is uniformly close-to-convex in U, for any γ_1, $\gamma_2\in\mathbb{C}$, $D^{n,\kappa,1}_{\lambda_1,\lambda_2}f(z^n)=(D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n))''$ and $D^{n,\kappa,2}_{\lambda_1,\lambda_2}f(z^n)=(D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n))'$, $D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n)$ is of form (6), $(j=1)$, $z\in U$.*

Theorem 3.5. *Let δ_j^a, $\gamma_a\in\mathbb{C}$, $a\in\{1,2\}$, $j=\overline{1,m}$. If $f_j\in\mathcal{A}_p$, for all $j=\overline{1,m}$, $m\in\mathbb{N}-\{0\}$, satisfy*

$$Re\left[1+\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f_j(z^n))^{2\gamma_a-1}]''}{[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f_j(z^n))^{2\gamma_a-1}]'}\right] > p-\frac{3p+4}{4\cdot\sum\limits_{\substack{j=1\\a\in\{1,2\}}}^{m}\delta_j^a} \quad (z\in U), \tag{23}$$

then

$$Re\sqrt{\frac{z[I_p^1(z)]'}{I_p^1(z)}} > \frac{\sqrt{p}}{2} \quad (z\in U),$$

where $D^{n,\kappa,1}_{\lambda_1,\lambda_2}f_j(z^n)=(D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n))''$ and $D^{n,\kappa,2}_{\lambda_1,\lambda_2}f_j(z^n)=(D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n))'$, $z\in U$, $D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n)$ is of form (6), $z\in U$.

Proof. From (16) and (23) we get that

$$Re\left(1+\frac{z[I_p^3(z)]''}{[I_p^3(z)]'}\right) > p\left(1-\sum_{\substack{j=1\\a\in\{1,2\}}}^{m}\delta_j^a\right)+\sum_{\substack{j=1\\a\in\{1,2\}}}^{m}\delta_j^a\cdot\left(p-\frac{3p+4}{4\sum\limits_{\substack{j=1\\a\in\{1,2\}}}^{m}\delta_j^a}\right)=\frac{p}{4}-1.$$

We apply Lemma 2.5 and we obtain that

$$Re\sqrt{\frac{z[I_p^3(z)]'}{I_p^3(z)}} > \frac{\sqrt{p}}{2} \quad (z \in U)\,.$$

Letting $j = p = 1$, $\delta_1^1 = \delta^1 \in \mathbb{C}$, $\delta_1^2 = \delta^2 \in \mathbb{C}$ and $D_{\lambda_1,\lambda_2}^{n,\kappa} f_1(z^n) = D_{\lambda_1,\lambda_2}^{n,\kappa} f(z^n)$ in Theorem 3.5, we have the following corollary.

Corollary 3.10. *If $f \in \mathcal{A}$, $a \in \{1,2\}$, satisfies the following condition*

$$Re\left[1 + \frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(z^n))^{2\gamma_a - 1}]''}{[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(z^n))^{2\gamma_a - 1}]'}\right] > -\frac{3}{4 \cdot \sum\limits_{a\in\{1,2\}}^{m} \delta^a} \quad (z \in U)\,,$$

then

$$Re\sqrt{\frac{z[I_1^3(z)]'}{I_p^3(z)}} > \frac{1}{2} \quad (z \in U)\,,$$

Moreover, if we consider that $\delta_1^1 = \delta_1^2 = \delta \in \mathbb{C}$ in Corollary 3.10, we obtain the next result.

Corollary 3.11. *If $f \in \mathcal{A}$, $\delta \in \mathbb{C}$, $a \in \{1,2\}$, satisfies the condition*

$$Re\left[1 + \frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(z^n))^{2\gamma_a - 1}]''}{[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(z^n))^{2\gamma_a - 1}]'}\right] > -\frac{3}{4 \cdot \sum\limits_{a\in\{1,2\}}^{m} \delta} \quad (z \in U)\,,$$

then

$$Re\sqrt{\frac{z[I_1^3(z)]'}{I_p^3(z)}} > \frac{1}{2} \quad (z \in U)\,,$$

Furthermore, if we take $\delta = 1$ in Corollary 3.11, we have the following result:

Corollary 3.12. *If $f \in \mathcal{A}$, $\delta \in \mathbb{C}$, $a \in \{1,2\}$, satisfies the condition*

$$Re\left[1 + \frac{z[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(z^n))^{2\gamma_a - 1}]''}{[(D_{\lambda_1,\lambda_2}^{n,\kappa,a} f(z^n))^{2\gamma_a - 1}]'}\right] > -\frac{3}{4} \quad (z \in U)\,,$$

then

$$Re\sqrt{\frac{z[I_1^3(z)]'}{I_p^3(z)}} > \frac{1}{2} \quad (z \in U)\,,$$

Remark 3.4. *The sufficient conditions for the operator $I_p^4(z)$ of form (12) can be obtained in a similar way.*

Strong starlikness of the integral operator $I_p^3(z)$

Theorem 3.6. *Let δ_j^a, $\gamma_a \in \mathbb{C}$, $a \in \{1,2\}$, $j = \overline{1,m}$. If $f_j \in \mathcal{A}_p$, for all $j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, satisfy*

$$Re\left[1+\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f_j(z^n))^{2\gamma_a-1}]''}{[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f_j(z^n))^{2\gamma_a-1}]'}\right] > p-\frac{\gamma}{2\cdot\sum\limits_{\substack{j=1\\a\in\{1,2\}}}^{m}\delta_j^a} \quad (z\in U), \tag{24}$$

then I_p^3 is strongly starlike of order γ $(0<\gamma\le 1)$ in U, where $D^{n,\kappa,1}_{\lambda_1,\lambda_2}f_j(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n))''$ and $D^{n,\kappa,2}_{\lambda_1,\lambda_2}f_j(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n))'$, $z\in U$, $z\in U$, $D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n)$ is of form (6).

Proof. We apply Lemma 2.6 twice and use the inequality (16), which follows that I_p^3 is strongly starlike of order γ $(0<\gamma\le 1)$ in U.

Letting $j=p=1$, $\delta_1^1=\delta^1\in\mathbb{C}$, $\delta_1^2=\delta^2\in\mathbb{C}$ and $D^{n,\kappa}_{\lambda_1,\lambda_2}f_1(z^n) = D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n)$ in Theorem 3.6, we obtain the next result.

Corollary 3.13. *If $f\in\mathcal{A}$, $a\in\{1,2\}$, satisfies the condition*

$$Re\left[1+\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}]''}{[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}]'}\right] > 1-\frac{\gamma}{2\cdot\sum\limits_{a\in\{1,2\}}^{m}\delta^a} \quad (z\in U),$$

then $\left\{\beta\int\limits_0^z\prod\limits_{a\in\{1,2\}}\left[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(t^n))^{2\gamma_a-1}\right]^{\delta^a}dt\right\}^{\frac{1}{\beta}}$ is strongly starlike of order γ $(0<\gamma\le 1)$ in U, where $D^{n,\kappa,2}_{\lambda_1,\lambda_2}f(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n))'$ and $D^{n,\kappa,1}_{\lambda_1,\lambda_2}f(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n))'$, $D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n)$ is of form (6), $z\in U$.

Moreover, if we take $\delta_1^1=\delta_1^2=\delta\in\mathbb{C}$ in Corollary 3.13, we have the following result:

Corollary 3.14. *If $f\in\mathcal{A}$, $\delta\in\mathbb{C}$, $a\in\{1,2\}$, satisfies the condition*

$$Re\left[1+\frac{z[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}]''}{[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(z^n))^{2\gamma_a-1}]'}\right] > 1-\frac{\gamma}{2\delta} \quad (z\in U),$$

then $\left\{\beta\int\limits_0^z\prod\limits_{a\in\{1,2\}}\left[(D^{n,\kappa,a}_{\lambda_1,\lambda_2}f(t^n))^{2\gamma_a-1}\right]^{\delta}dt\right\}^{\frac{1}{\beta}}$ is strongly starlike of order γ $(0<\gamma\le 1)$ in U, where $D^{n,\kappa,2}_{\lambda_1,\lambda_2}f(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n))'$, $D^{n,\kappa,1}_{\lambda_1,\lambda_2}f(z^n) = (D^{n,\kappa}_{\lambda_1,\lambda_2}f(z^n))'$, $D^{n,\kappa}_{\lambda_1,\lambda_2}f_j(z^n)$ is of form (6), $z\in U$.

Remark 3.5. *The strongly starlikness condition for the integral operator $I_p^4(z)$ of form (12) can be obtained in a similar way.*

Example 3.1. *Let* $\beta = 0$ *in* $D_\lambda^\beta f(z)$ *of form (4) or (5), where* $n = 1$. *So, we have that* $D_\lambda^0 f(z) = f(z)$, $\forall \lambda \geq 0$, $f(z) \in \mathcal{A}_p$. *We will use this form of the integral operator, where the function* f *is of form (1) with respect to the integral operator (12). For further simplification, we consider that* $\gamma_1 = \gamma_2 = 1$, *and* $\delta = 1$

If $\chi = 1$, $\delta_j = 0$, $\forall j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$ *and we consider* $\delta_j^2 = \alpha_j$, $\forall j = \overline{1,m}$, $m \in \mathbb{N} - \{0\}$, *we obtain the operator* $G_p(z) = \int_0^z pt^{p-1} \prod_{j=1}^{n} \left(\frac{f_j'(t)}{pt^{p-1}} \right)^{\alpha_j} dt$, *which is already studied in [7].*

Remark 3.6. *There are other integral operators of p-valent functions in literature (e.g. see[4], [9], [12]) for whom the sufficient conditions of p-valently starlikeness, p-valently close-to-convexness, uniformly p-valent close-to-convexness and strongly starlikeness of order* τ $(0 < \tau \leq 1)$ *in U (open unit disk) is covered by our present work.*

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References

[1] M. Acu, I. Dorca, S. Owa, *On some starlike functions with negative coefficients*, Proceedings of the International Conference on Theory and Applications of Mathematics and Informatics, ICTAMI 2011, Alba Iulia (2011), 101 - 112.

[2] M. Acu, S. Owa, *Note on a class of starlike functions*, Proceeding Of the International Short Joint Work on Study on Calculus Operators in Univalent Function Theory - Kyoto (2006), 1 - 10.

[3] H.A. Al-Karsani AND S.S. Al-Hajiry, *A note on certain inequalities for p-valent functions*, J. Inequal. Pure Appl. Math., 9(3) (2008), Art. 90.

[4] D. Breaz, *Integral operators on univalent function spaces*, Ed. Acad. Române, Bucureşti, (2004), 159 - 203.

[5] I. Dorca, M. Acu, and D. Breaz, *Note on Neighborhoods of Some Classes of Analytic Functions with Negative Coefficients*, ISRN Mathematical Analysis, vol. 2011, Article ID 610549, (2011), 7 pages.

[6] J. Dziok, *Applications of the Jack lemma*, Acta Math. Hungar., 105(1-2) (2004), 93 - 102.

[7] B. A. Frasin, *Integral Operators of p-valent Functions*, J. Inequal. Pure Appl. Math., 10(4) (2009), Art. 109.

[8] A.W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl., 55 (1991), 364 - 370.

[9] Y.J. Kim AND E.P. Merkes, *On an integral of powers of a spirallike function*, Kyungpook Mathematical Journal, 12 (1972), 249 - 252.

[10] M. Nunokawa, *On the multivalent functions*, Indian J. Pure Appl. Math., 20 (1989), 577 - 582.

[11] M. Nunokawa, S. Owa and A. Ikeda, *On the strongly starlikeness of multivalently convex functions of order α*, IJMMS, 28(1) (2001), 51 - 55.

[12] N.N. Pascu and V. Pescar, *On the integral operators of Kim-Merkes and Pfaltzgraff*, Mathematica, 32(55)(2) (1990), 185 - 192.

[13] R.K. Raina AND I.B. Bapna, *Inequalities defining certain subclasses of analytic functions involving fractional calculus operators*, J. Inequal. Pure Appl. Math., 5(2) (2004), Art. 28.

[14] S. Owa, *On the Nunokawas conjecture for multivalent functions*, Bull. Austral. Math. Soc., 41 (1990), 301 - 305.

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