Notions of Möbius inversion

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Abstract

Möbius inversion, originally a tool in number theory, was generalized to posets for use in group theory and combinatorics. It was later generalized to categories in two different ways, both of which are useful. We provide a unifying abstract framework. This allows us to compare and contrast the two theories of Möbius inversion for categories, and advance each of them. Among several side benefits is an improved understanding of the following fact: the Euler characteristic of the classifying space of a (suitably finite) category depends only on its underlying graph.

Introduction

The history of Möbius inversion begins with August Ferdinand Möbius (1790–1868), the basic aspects of whose work on this can be described in modern terms as follows. Consider sequences $\alpha(1), \alpha(2), \ldots$ of complex numbers. Any two sequences α, β have a convolution product $\alpha * \beta$, defined by

$$(\alpha * \beta)(n) = \sum_{k,m\colon\, km=n} \alpha(k)\beta(m).$$

This product has a unit, and the constant sequence $\zeta = (1, 1, \ldots)$ has a convolution inverse: the classical Möbius function μ, given by a well-known formula involving prime factorizations. It has many uses in elementary and not-so-elementary number theory. For example, every sequence α determines a formal

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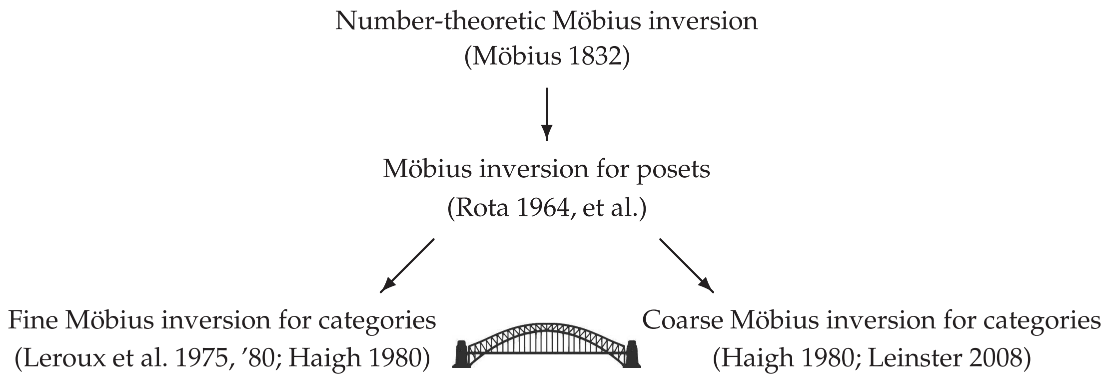

Figure 1: Simplified history of Möbius inversion. This paper builds a bridge between the two notions of Möbius inversion for categories. Notably missing from the diagram are the finite difference calculus and the theories of Möbius inversion developed by Cartier and Foata [7], Dür [10] and Lück [24].

Dirichlet series $\sum_{n=1}^{\infty} \alpha(n)/n^s$, where s is a formal variable. Convolution of sequences corresponds to multiplication of Dirichlet series. The constant sequence ζ corresponds to the Riemann zeta function, and the relationship between ζ and μ can be expressed as

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = 1 \Big/ \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

In the mid-twentieth century, it was realized that Möbius inversion could usefully be defined for general partially ordered sets, the original case being the set of positive integers ordered by divisibility. This insight is usually associated with the name of Gian-Carlo Rota [31]. Although Rota was not (as he made clear) the first to generalize Möbius inversion to posets, he was responsible for harnessing its power to solve problems in enumerative combinatorics.

Rota’s theory was subsequently generalized by multiple people in multiple directions, but two particularly concern us here. Both are theories of Möbius inversion for categories (Fig. 1).

The first was developed independently by Pierre Leroux and collaborators and by John Haigh. (Leroux published a short announcement in 1975 [23]. The full account, joint with Content and Lemay, appeared in 1980 [9], as did Haigh’s paper [13].) The second was also introduced by Haigh (in Section 3 of [13]), in just a dozen lines of text. It was developed more fully by the author [20] as part of the theory of Euler characteristic of categories.

A comparison of the two theories would be hopelessly confusing if both were referred to as ‘Möbius inversion’. We therefore introduce new terminology. The first type of Möbius inversion is called ‘fine’, and the second ‘coarse’. These same adjectives are applied systematically throughout; for example, the finiteness condition used in the fine theory is renamed ‘fine finiteness’, and its coarse counterpart ‘coarse finiteness’. This makes various relationships clear. The names are apt: the fine Möbius function of a category is a more refined invariant, more sensitive to the category’s structure than the coarse one. And there are far more categories for which the coarse Möbius function is well-defined than the fine one: it is like a weed that grows almost anywhere, compared to a fine but delicate flower. See Examples 1.2 and 1.4, and Theorem 1.6.

This paper proves results connecting the two theories, points out essential differences between them, and advances each one further. But more importantly, it provides a single abstract setting in which all of this takes place. As we shall see, the two theories, together with a third intermediate one, arise from the inclusions of categories

$$\mathbf{1} \hookrightarrow \mathbf{2} \hookrightarrow \mathbf{Set}$$

in a uniform manner (Fig. 2).

We begin with a review of fine and coarse Möbius inversion for categories, introducing the new terminology (Section 1). The basic theorem connecting them is stated. We see that the fine and coarse theories can help each other: for instance, Corollaries 1.7, 1.8 and 3.10 are all stated purely in terms of fine Möbius inversion, but proved using the coarse theory.

We also explore in Section 1 the following curious phenomenon. Every small category gives rise to a topological space, its classifying space or geometric realization. It is a fact that (under finiteness assumptions) the Euler characteristic of that space is independent of the composition in the category. The theory of coarse Möbius inversion sheds light on this.

The abstract framework is introduced in Section 2. The key is the covariant functoriality of the incidence algebra construction. As the framework is developed, a third level naturally emerges, between coarse and fine. This allows the coarse theory, previously confined for the most part to finite categories, to be extended to infinite categories (Section 3). There we generalize one of the main theorems of Rota's original paper [31].

Coarse Möbius inversion also makes sense for enriched categories (Section 4). This fact has already been exploited in investigations of geometric measure in metric spaces, as will be explained.

The remaining sections are contributions to the fine theory. The incidence algebra construction is functorial in both the covariant and contravariant senses, and in Section 5, we prove a Beck–Chevalley theorem enabling the two to be unified. In Section 6, we prove a new characterization of the 'Möbius categories' of Leroux.

There are two appendices. Setting up the coarse theory for infinite categories requires a nontrivial result on inverse matrices, proved in Appendix A. Finally, Appendix B creates an abstract home for the notion of functor with unique lifting of factorizations, important for Möbius inversion. The concept developed there, 'pullback-homomorphism', may also be of more general interest.

Related work I will not attempt to survey the large body of work on Möbius inversion for posets; see [32] for a good overview. Lawvere and Menni's paper [19] contains further pointers to the literature on fine Möbius inversion for categories. Coarse Möbius inversion is used in the theory of the Euler characteristic of a category, which since the original paper [20] has been developed and applied by Berger and Leinster [5], Fiore, Lück and Sauer [11, 12], Jacobsen and Møller [15], and Noguchi [27, 28, 29, 30]. Sections 3 and 4 of the present work expand on points covered briefly in Sections 4 and 2, respectively, of [20]. We do not touch here on the theory of Möbius inversion developed by Cartier and Foata [7] for use in combinatorics, nor that of Dür [10] or Lück [24].

Many of the finiteness conditions arising in Möbius inversion for categories were explored by Mitchell [26], as was the incidence algebra construction. The question of which finite directed graphs admit a category structure, implicitly raised by Lemma 3.4, has recently been answered by Allouch [1, 2].

Notation Given a small category $\mathbf{A}$, we write $\mathbf{A}_0$ for its set of objects and $\mathbf{A}_1$ for the set of all maps or morphisms in $\mathbf{A}$. We often write $a \in \mathbf{A}$ to mean $a \in \mathbf{A}_0$, and we write $\mathbf{A}(a, b)$ for the set of maps from a to b. Given a finite set X, we write $\#X$ for its cardinality.

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1 Fine and coarse Möbius inversion

Here we review the two types of Möbius inversion for categories, introducing systematic new terminology. We then state the basic result connecting the fine and coarse theories.

A **rig** (or semiring) is a ri*n*g without *n*egatives: a set equipped with a commutative monoid structure $(+, 0)$ and a monoid structure $(\cdot, 1)$, the latter distributing over the former. We take rig to mean *commutative* rig: one whose multiplication is commutative. Similarly, **ring** means commutative ring. A rig is **trivial** if it has only one element. For a natural number n and a rig k, we often use n to denote the element $n \cdot 1 = 1 + \cdots + 1$ of k.

A **module** over a rig k is a commutative monoid acted on by k, in the evident sense; **algebras** over k are defined similarly (and not assumed to be commutative). When k is a ring, k-modules are the same whether k is regarded as a rig or as a ring. The same goes for algebras.

Fine Möbius inversion

The convolution of the opening paragraph involved a sum, and that sum is well-defined because each positive integer has only finitely many factorizations into two parts. Similarly, when developing Möbius inversion for categories, we need to impose the following finiteness condition. A category $\mathbf{A}$ is **finely finite** if for each map $f : a \to b$ in $\mathbf{A}$, there are only finitely many diagrams

$$a \xrightarrow{g} c \xrightarrow{h} b$$

in $\mathbf{A}$ whose composite is f.

Let $\mathbf{A}$ be a finely finite category and k a rig. The **fine incidence algebra** $k\mathbf{A}$ is the set of all functions $\mathbf{A}_1 \to k$, made into a k-algebra as follows. Its k-module structure is pointwise. The multiplication $*$ is given by

$$(\alpha * \beta)(f) = \sum_{hg=f} \alpha(g)\, \beta(h),$$

where $\alpha, \beta \in k\mathbf{A}$ and $f \in \mathbf{A}_1$. (Fine finiteness guarantees that the sum is finite.) The multiplicative unit δ is given by $\delta(1_a) = 1$ whenever $a \in \mathbf{A}$, and $\delta(f) = 0$ otherwise.

The fine incidence algebra has a special element: the **fine zeta function** $\zeta_{\mathbf{A}}$, defined by

$$\zeta_{\mathbf{A}}(f) = 1$$

for all $f \in \mathbf{A}_1$. We say that $\mathbf{A}$ has **fine Möbius inversion** over k if $\zeta_{\mathbf{A}}$ has a multiplicative inverse in $k\mathbf{A}$, which is called the **fine Möbius function** $\mu_{\mathbf{A}} = \zeta_{\mathbf{A}}^{-1} \in k\mathbf{A}$.

These terms are all new; let us compare them with previous usage. Where we call a category finely finite, Leroux et al. [9] say that it 'has finite decompositions of degree 2'. What we call the fine incidence algebra and fine Möbius function, they simply call the incidence algebra and Möbius function. They also have a definition of 'Möbius category'. Being a Möbius category is a stronger condition than having fine Möbius inversion. The precise relationship is determined in Section 6, but we will not need the concept of Möbius category elsewhere.

Haigh [13] removes the possibility of infinite sums by a different strategy: he imposes no finiteness conditions on $\mathbf{A}$, but considers only those functions $\alpha \colon \mathbf{A}_1 \to k$ such that $\alpha(f) = 0$ for all but finitely many maps f. He calls the resulting algebra the 'category algebra'; it only has a multiplicative identity if $\mathbf{A}$ is finite. He calls a finite category $\mathbf{A}$ a 'Möbius category' if it has fine Möbius inversion, in conflict with the usage of Leroux et al.

Both Haigh and Leroux et al. take k to be a ring, not a general rig.

Example 1.1. Let A be a partially ordered set, viewed as a category. It is finely finite if and only if it is **locally finite**: for all $a, b \in A$, the set $\{c \in A \mid a \leq c \leq b\}$ is finite. (Rota's theory proceeds on this assumption.) The fine incidence algebra is the set of functions

$$\{(a, b) \in A \times A \mid a \leq b\} \to k,$$

and the fine Möbius function μ_A, if it exists, is characterized by the equations

$$\sum_{c \colon a \leq c \leq b} \mu_A(a, c) = \sum_{c \colon a \leq c \leq b} \mu_A(c, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

($a, b \in A$). Hall [14] showed that when k is a ring, the Möbius function exists and is given by

$$\mu_A(a, b) = \sum_{n \in \mathbb{N}} (-1)^n \cdot \#\{\text{chains } a = a_0 < \cdots < a_n = b\}.$$

For example, if A is the poset of positive integers ordered by divisibility, and $k = \mathbb{Z}$, then $\mu_A(a, b) = \mu(b/a)$, where the μ on the right-hand side is the classical Möbius function.

Example 1.2. A group, viewed as a one-object category, is finely finite if and only if it is finite. The fine incidence algebra kG of a finite group G is its group algebra. No group has fine Möbius inversion, except when it or k is trivial.

Coarse Möbius inversion

A category is **finite**—or **coarsely finite**, for emphasis—if it has only finitely many objects and arrows. For now, coarse Möbius inversion will be defined only for finite categories. We will see how to relax this assumption in Section 3.

Let $\mathbf{A}$ be a finite category and k a rig. The **coarse incidence algebra** $k_c\mathbf{A}$ is the set of all functions $\mathbf{A}_0 \times \mathbf{A}_0 \to k$, made into a k-algebra as follows. Its k-module structure is pointwise. The multiplication $*$ is given by

$$(\alpha * \beta)(a, b) = \sum_{c \in \mathbf{A}} \alpha(a, c)\, \beta(c, b)$$

($\alpha, \beta \in k_c\mathbf{A}, a, b \in \mathbf{A}$). The multiplicative unit is the Kronecker δ, defined by $\delta(a, b) = 1$ if $a = b$ and $\delta(a, b) = 0$ otherwise. If a total order is chosen on the n objects of $\mathbf{A}$, then $k_c\mathbf{A}$ is just the algebra of $n \times n$ matrices over k.

The coarse incidence algebra has a special element: the **coarse zeta function** $\zeta_{\mathbf{A}}$, defined for $a, b \in \mathbf{A}$ by

$$\zeta_{\mathbf{A}}(a, b) = \#(\mathbf{A}(a, b)) \in k.$$

We say that $\mathbf{A}$ has **coarse Möbius inversion** over k if $\zeta_{\mathbf{A}}$ has a multiplicative inverse in $k_c\mathbf{A}$. The **coarse Möbius function** is then $\mu_{\mathbf{A}} = \zeta_{\mathbf{A}}^{-1} \in k_c\mathbf{A}$.

In [20], the algebra $k_c\mathbf{A}$ is only considered in the case $k = \mathbb{Q}$. What we call coarse Möbius inversion and the coarse Möbius function here are simply called Möbius inversion and the Möbius function there. The same is true in [5] and [21].

Example 1.3. Let A be a finite partially ordered set. The coarse incidence algebra is the set of functions $A \times A \to k$. It contains the fine incidence algebra as a subalgebra, consisting of those $\alpha\colon A \times A \to k$ such that $\alpha(a, b) = 0$ whenever $a \not\leq b$. The fine and coarse zeta functions, viewed as elements of the coarse incidence algebra, are equal. Hence when A has fine Möbius inversion (e.g. when k is a ring), the fine and coarse Möbius functions are also equal.

No confusion should be caused by writing $\zeta_{\mathbf{A}}$ for both the fine and coarse zeta functions. When we write '$\zeta_{\mathbf{A}}(f)$', the $\zeta_{\mathbf{A}}$ in question must be the fine one; when we write '$\zeta_{\mathbf{A}}(a, b)$', it must be the coarse one. A priori there could be an ambiguity when $\mathbf{A}$ is a poset, since there we might use (a, b) to denote the unique map $a \to b$. But the previous example shows that in that case, the two meanings of $\zeta_{\mathbf{A}}(a, b)$ agree. The same goes for the fine and coarse Möbius functions $\mu_{\mathbf{A}}$. Moreover, when $\mathbf{A}$ is understood, we write them as just ζ and μ.

Example 1.4. Let M be a finite monoid. Then $k_cM = k$, and $\zeta = \#M \in k$. So, for instance, if k is a field of characteristic 0 then every finite monoid has coarse Möbius inversion over k. Contrast Example 1.2.

A category with coarse Möbius inversion over a nontrivial rig must be **skeletal**, that is, isomorphic objects must be equal. (For if not, the matrix ζ would have two identical rows.) But since every category is equivalent to some skeletal category, this is not a serious restriction. Large classes of finite skeletal categories have coarse Möbius inversion over $\mathbb{Q}$: all posets, groupoids and monoids, all categories containing no nontrivial idempotents, and all categories admitting an epi-mono factorization system. See [20] for details.

The **Euler characteristic** of a finite category $\mathbf{A}$ with coarse Möbius inversion is $\chi(\mathbf{A}) = \sum_{a,b \in \mathbf{A}} \mu_{\mathbf{A}}(a, b)$. (This can be taken as a definition, although in fact Euler characteristic can be defined under weaker hypotheses [20].) The name is largely justified by the following fact. Let $\mathbf{A}$ be a finite skeletal category containing no nontrivial endomorphisms. Write $|N\mathbf{A}|$ for its classifying space, that is, the geometric realization of its simplicial nerve $N\mathbf{A}$. Proposition 2.11 of [20] states that $\chi(\mathbf{A}) = \chi(|N\mathbf{A}|)$. Further results in [20] relate the Euler characteristic of categories to other invariants of size: the Euler characteristics of graphs, posets and orbifolds, the cardinality of sets, and the Baez–Dolan cardinality of groupoids [3].

The coarse Möbius function of a category does not depend on its composition, just its underlying directed graph. The same is therefore true of Euler characteristic. Of course, every nontrivial invariant throws away *some* information, but to throw away the composition of a category might be thought extravagant.

Nevertheless, there is an important precedent. Consider homotopically tame spaces—say, finite CW-complexes. Any such space X can be built up from a stock of points, intervals, disks, etc., by gluing them together, and it hardly needs saying that the topology of X depends entirely on *how* they are glued together. But the Euler characteristic does not. Topologically important as Euler characteristic is, it is independent of gluing.

The result on classifying spaces implies:

Proposition 1.5. *Let $\mathbf{A}$ and $\mathbf{A}'$ be finite skeletal categories containing no nontrivial endomorphisms. If they have the same underlying directed graph then $\chi(|N\mathbf{A}|) = \chi(|N\mathbf{A}'|)$.*

Now, the theory of group homology is set up so that the homology of a group is equal to the homology of its classifying space. If we wish the analogous statement to be true of Euler characteristic of categories (under finiteness conditions), Proposition 1.5 *forces* it to be independent of composition.

One could, nonetheless, develop the theories of coarse Möbius inversion and Euler characteristic for arbitrary directed graphs. Many of the results in [20] and [5] involve categorical concepts: automorphisms, epi-mono factorization, equivalences, adjunctions, fibrations, ... In principle, it must be possible to rephrase them purely in terms of graphs, but it is not yet clear that it is fruitful to do so. Perhaps the following situation is comparable. Limits in a category $\mathscr{C}$ are usually phrased in terms of a functor $\mathbf{I} \to \mathscr{C}$, even though the definition of limit does not use the category structure on $\mathbf{I}$. One could therefore rephrase all results about limits in terms of graphs $\mathbf{I}$; but it is not clear that this is a useful step to take.

Comparison between fine and coarse

In the interests of describing the relationship between fine and coarse Möbius inversion as soon as possible, we first state a result under unnecessarily restrictive hypotheses. It first appeared as Proposition 3.6 of Haigh [13], and was also stated at the end of Section 4 of [20]. The unrestricted form appears as Theorem 3.9 below.

Fix a rig k.

Theorem 1.6 (Haigh)**.** *Let* $\mathbf{A}$ *be a finite category. If* $\mathbf{A}$ *has fine Möbius inversion over* k *then* $\mathbf{A}$ *also has coarse Möbius inversion over* k*, given for* $a, b \in \mathbf{A}$ *by*

$$\mu_{\mathbf{A}}(a,b) = \sum_{f \in \mathbf{A}(a,b)} \mu_{\mathbf{A}}(f).$$

This can easily be proved by a direct calculation, but a proof also arises naturally in our abstract development (Section 2).

The following corollary is due to Matías Menni (private communication, 2010).

Corollary 1.7 (Menni)**.** *Let* $\mathbf{A}$ *and* $\mathbf{A}'$ *be finite categories with fine Möbius inversion over* k*. Suppose that* $\mathbf{A}$ *and* $\mathbf{A}'$ *have the same underlying directed graph. Then for all objects* a, b*,*

$$\sum_{f\colon a \to b} \mu_{\mathbf{A}}(f) = \sum_{f\colon a \to b} \mu_{\mathbf{A}'}(f).$$

Proof. By Theorem 1.6, $\mathbf{A}$ and $\mathbf{A}'$ have coarse Möbius inversion and the equation is equivalent to $\mu_{\mathbf{A}}(a,b) = \mu_{\mathbf{A}'}(a,b)$. This is true because the coarse Möbius function of a category depends only on its underlying graph. ∎

Corollary 1.8. *Let* $\mathbf{A}$ *and* $\mathbf{A}'$ *be finite categories with fine Möbius inversion over* k*. Suppose that* $\mathbf{A}$ *and* $\mathbf{A}'$ *have the same underlying directed graph. Then*

$$\sum_{f \in \mathbf{A}_1} \mu_{\mathbf{A}}(f) = \sum_{f \in \mathbf{A}'_1} \mu_{\mathbf{A}'}(f).$$

The two sides of this equation are the Euler characteristics of $\mathbf{A}$ and $\mathbf{A}'$. But note that Corollaries 1.7 and 1.8, while proved using the coarse theory, refer only to the theory of *fine* Möbius inversion.

2 Functoriality

We have seen that each sufficiently finite category $\mathbf{A}$ gives rise to a k-algebra $k\mathbf{A}$, for each rig k. Here we show how this process can be made functorial in $\mathbf{A}$. Although we deal primarily with fine incidence algebras, the coarse ones enter naturally as the story unfolds.

There is a well-known way to make $\mathbf{A} \mapsto k\mathbf{A}$ functorial in the *contravariant* sense, using functors with unique lifting of factorizations. This is discussed in Section 5, but is not needed to achieve the main aims of this paper. Instead, we make $\mathbf{A} \mapsto k\mathbf{A}$ into a *covariant* functor.

Let $\mathbf{A}$ and $\mathbf{B}$ be finely finite categories. Let $F\colon \mathbf{A} \to \mathbf{B}$ be a functor with **finite fibres**, meaning that for each $g \in \mathbf{B}_1$, the set $\{f \in \mathbf{A}_1 \mid Ff = g\}$ is finite. (This implies the analogous condition on objects.) There is an induced k-linear map

$$F_! \colon k\mathbf{A} \to k\mathbf{B}$$

defined for $\alpha \in k\mathbf{A}$ and $g \in \mathbf{B}_1$ by

$$(F_!\alpha)(g) = \sum_{f\colon Ff=g} \alpha(f).$$

This covariant functoriality was introduced by Content, Lemay and Leroux [9]. The following result is close to their Proposition 5.6.

Proposition 2.1. *Let $\mathbf{A}$ and $\mathbf{B}$ be finely finite categories, and let $F\colon \mathbf{A} \to \mathbf{B}$ be a functor with finite fibres. Then $F_!\colon k\mathbf{A} \to k\mathbf{B}$ is an algebra homomorphism for all rigs k if and only if F is bijective on objects.*

Proof. First consider preservation of identities. For each map g in $\mathbf{B}$, we have

$$(F_!\delta_{\mathbf{A}})(g) = \sum_{f\colon Ff=g} \delta_{\mathbf{A}}(f) = \#\{a \in \mathbf{A}_0 \mid 1_{F(a)} = g\} \in k.$$

If g is not an identity then $(F_!\delta_{\mathbf{A}})(g) = 0 = \delta_{\mathbf{B}}(g)$. If g is an identity, say $g = 1_b$, then

$$(F_!\delta_{\mathbf{A}})(1_b) = \#\{a \in \mathbf{A}_0 \mid Fa = b\}$$

and $\delta_{\mathbf{B}}(1_b) = 1$. Hence if F is bijective on objects then $F_!\delta_{\mathbf{A}} = \delta_{\mathbf{B}}$. Conversely, if $F_!\delta_{\mathbf{A}} = \delta_{\mathbf{B}}$ for $k = \mathbb{Z}$ then F is bijective on objects.

A straightforward calculation shows that if F is injective on objects then $F_!$ preserves binary multiplication. ∎

Write $\mathbf{Cat}_!$ for the category whose objects are finely finite categories and whose maps are bijective-on-objects functors with finite fibres. There is a functor $\mathbf{Cat}_! \to k\text{-}\mathbf{Alg}$ defined by $\mathbf{A} \mapsto k\mathbf{A}$ and $F \mapsto F_!$.

Example 2.2. Given a category $\mathbf{A}$, write $C\mathbf{A}$ for the codiscrete category with the same objects as $\mathbf{A}$; thus, there is precisely one map $a \to b$ in $C\mathbf{A}$ for each pair (a, b) of objects. There is a unique identity-on-objects functor $\mathbf{A} \to C\mathbf{A}$. Assume now that $\mathbf{A}$ is (coarsely) finite. Then $C\mathbf{A}$ is finely finite and $\mathbf{A} \to C\mathbf{A}$ has finite fibres.

The coarse incidence algebra of $\mathbf{A}$ is the fine incidence algebra of the codiscrete category on $\mathbf{A}$:

$$k_c\mathbf{A} \cong k(C\mathbf{A}).$$

So the functor $\mathbf{A} \to C\mathbf{A}$ induces a homomorphism of k-algebras

$$\Sigma\colon k\mathbf{A} \to k_c\mathbf{A}.$$

Explicitly,

$$(\Sigma\alpha)(a, b) = \sum_{f \in \mathbf{A}(a,b)} \alpha(f) \tag{1}$$

($\alpha \in k\mathbf{A}$, $a, b \in \mathbf{A}$). The image under Σ of the fine zeta function $\zeta_{\mathbf{A}} \in k\mathbf{A}$ is the coarse zeta function $\zeta_{\mathbf{A}} \in k_c\mathbf{A}$. This proves Haigh's comparison theorem:

$$\begin{tikzcd}[column sep=small] & \mathbf{Set} \arrow[dl, bend right] \arrow[dr, bend left] & \\ \mathbf{2} \arrow[ur, hook, bend right, "\top"] \arrow[rr, bend left] & & \mathbf{1} \arrow[ul, hook, bend left, "\top"'] \arrow[ll, hook, bend left, "\perp"] \end{tikzcd}$$
(a)

$$\begin{tikzcd}[column sep=small] & \mathbf{Cat} \arrow[dl, bend right] \arrow[dr, bend left] & \\ \mathbf{Poset} \arrow[ur, hook, bend right, "\top"] \arrow[rr, bend left] & & \mathbf{Set} \arrow[ul, hook, bend left, "\top"'] \arrow[ll, hook, bend left, "\perp"] \end{tikzcd}$$
(b)

$$\begin{tikzcd}[column sep=small] & \mathbf{A} \arrow[dl, "\text{unit}"'] \arrow[dr, "\text{unit}"] & \\ P\mathbf{A} \arrow[rr, hook] & & C\mathbf{A} \end{tikzcd}$$
(c)

$$\begin{tikzcd}[column sep=small] & k\mathbf{A} \arrow[dl, "\Sigma"'] \arrow[dr, "\Sigma"] & \\ k_{\mathrm{p}}\mathbf{A} \arrow[rr, hook] & & k_{\mathrm{c}}\mathbf{A} \end{tikzcd}$$
(d)

Figure 2: Abstract origins of the three incidence algebras.

Proof of Theorem 1.6 $\Sigma\colon k\mathbf{A} \to k_{\mathrm{c}}\mathbf{A}$ is an algebra homomorphism mapping $\zeta_{\mathbf{A}} \in k\mathbf{A}$ to $\zeta_{\mathbf{A}} \in k_{\mathrm{c}}\mathbf{A}$, so it also maps $\mu_{\mathbf{A}} = \zeta_{\mathbf{A}}^{-1} \in k\mathbf{A}$ to $\mu_{\mathbf{A}} = \zeta_{\mathbf{A}}^{-1} \in k_{\mathrm{c}}\mathbf{A}$.

A **preorder** on a set is a reflexive transitive binary relation. The 2-categories of preordered and partially ordered sets are equivalent, so the difference between the two types of structure is inessential; both will be referred to as 'posets'.

Example 2.3. Let $\mathbf{A}$ be a small category. There is a preorder on the set of objects of $\mathbf{A}$ defined by $a \leq b$ if and only if there is at least one map $a \to b$. Denote the resulting poset by $P\mathbf{A}$. There is a unique identity-on-objects functor $\mathbf{A} \to P\mathbf{A}$.

In order for this to induce a homomorphism $k\mathbf{A} \to k(P\mathbf{A})$, and in order for the algebras $k\mathbf{A}$ and $k(P\mathbf{A})$ to be defined at all, some finiteness conditions must hold. We defer precise discussion of those conditions to the next section, temporarily making the simplifying assumption that $\mathbf{A}$ is coarsely finite. This suffices.

Write

$$k_{\mathrm{p}}\mathbf{A} = k(P\mathbf{A}).$$

Thus, $k_{\mathrm{p}}\mathbf{A}$ consists of the functions $\{(a,b) \in \mathbf{A}_0 \times \mathbf{A}_0 \mid \mathbf{A}(a,b) \neq \emptyset\} \to k$. It can also be seen as a subalgebra of $k_{\mathrm{c}}\mathbf{A}$:

$$k_{\mathrm{p}}\mathbf{A} \cong \{\alpha \in k_{\mathrm{c}}\mathbf{A} \mid \mathbf{A}(a,b) = \emptyset \Rightarrow \alpha(a,b) = 0\}. \tag{2}$$

The functor $\mathbf{A} \to P\mathbf{A}$ induces a homomorphism $\Sigma\colon k\mathbf{A} \to k_{\mathrm{p}}\mathbf{A}$, given by equation (1) above. So we have a commutative triangle of k-algebras as in Fig. 2(d).

This commutative triangle also arises inexorably from very simple origins, by applying standard categorical constructions. We start with the inclusions of categories

$$\mathbf{1} \hookrightarrow \mathbf{2} \hookrightarrow \mathbf{Set}.$$

Here $\mathbf{2}$ is the full subcategory of $\mathbf{Set}$ consisting of the empty set $\emptyset$ and the one-element set 1, and $\mathbf{1}$ is the subcategory consisting of 1 alone. Both inclusions have left adjoints, giving the commutative triangle of Fig. 2(a). Moreover, all the categories have finite products and all the functors preserve them. So we may apply the 2-functor $\mathscr{V} \mapsto \mathscr{V}$-$\mathbf{Cat}$, giving the commutative triangle of Fig. 2(b). The adjunction $\mathbf{Poset} \rightleftarrows \mathbf{Cat}$ induces a monad P on $\mathbf{Cat}$, the adjunction $\mathbf{Set} \rightleftarrows \mathbf{Cat}$ induces a monad C on $\mathbf{Cat}$, and the adjunction $\mathbf{Poset} \rightleftarrows \mathbf{Set}$ induces a map of monads $P \to C$. So for each $\mathbf{A} \in \mathbf{Cat}$, we obtain a commutative triangle as in Fig. 2(c).

The categories $P\mathbf{A}$ and $C\mathbf{A}$, and all three functors, are the same as in the explicit descriptions above. In particular, the functors are bijective on objects. So assuming that $\mathbf{A}$ is finite, we may take fine incidence algebras throughout, and the result is the commutative triangle of Fig. 2(d).

3 Möbius inversion for infinite categories

Here we extend the theory of coarse Möbius inversion to a class of infinite categories. The relationship between coarse and fine Möbius inversion, stated in Theorem 1.6, persists.

Fix a rig k. Assume that k has **characteristic zero**, in the sense that 0 is the only natural number n satisfying $n \cdot 1 = 0 \in k$.

The finiteness condition that we are about to introduce can be motivated both pragmatically and abstractly.

Pragmatically, we seek the minimal finiteness conditions on a category $\mathbf{A}$ allowing the apparatus of coarse Möbius inversion to be set up. First, for $\zeta_{\mathbf{A}}$ to make sense, the homsets of $\mathbf{A}$ must be finite. Second, if $\zeta_{\mathbf{A}}$ is to belong to an incidence algebra with the usual kind of convolution product, then in particular $\zeta_{\mathbf{A}} * \zeta_{\mathbf{A}}$ must be defined; and since we have no way of handling infinite sums, we require that for each $a, b \in \mathbf{A}$, there are only finitely many $c \in \mathbf{A}$ such that $\zeta_{\mathbf{A}}(a,c)\zeta_{\mathbf{A}}(c,b) \neq 0$. For that to be true over all rigs, for each a, b there can be only finitely many c such that there exist maps $a \to c \to b$.

We will see that these two requirements suffice.

Definition 3.1. Let a and b be objects of a category $\mathbf{A}$. The **patch** $[a,b]_{\mathbf{A}}$ is the full subcategory of $\mathbf{A}$ with objects $\{c \in \mathbf{A} \mid \text{there exist maps } a \to c \to b\}$.

(A patch might also be called a 'coarse interval', and the intervals of [19] 'fine intervals'.)

Lemma 3.2. *The following conditions on a category* $\mathbf{A}$ *are equivalent:*

i. *for all* $a, b \in \mathbf{A}$*, the patch* $[a,b]_{\mathbf{A}}$ *is a finite category*

ii. $\mathbf{A}$ *is finely finite and has finite homsets*

iii. *for all* $a, b \in \mathbf{A}$*, the set* $\{c \in \mathbf{A} \mid \text{there exist maps } a \to c \to b\}$ *is finite, and* $\mathbf{A}$ *has finite homsets.*

A category $\mathbf{A}$ is **patch-finite** if it satisfies the equivalent conditions of Lemma 3.2. For example, a poset A is patch-finite if and only if it is locally finite (Example 1.1).

We have met three finiteness conditions: coarse, patch and fine. They are not ad hoc. To see how they arise systematically, recall from Section 2 that the inclusions $\mathbf{1} \hookrightarrow \mathbf{2} \hookrightarrow \mathbf{Set}$ give rise to three monads Q on $\mathbf{Cat}$, namely, C, P and the identity. To make $k(Q\mathbf{A})$ into an algebra, we require $Q\mathbf{A}$ to be finely finite. To furnish $k(Q\mathbf{A})$ with a zeta function, we want to transport the zeta function of $k\mathbf{A}$ along the unit map $\mathbf{A} \to Q\mathbf{A}$, and for that we require $\mathbf{A} \to Q\mathbf{A}$ to have finite fibres. So: to make the basic definitions, we require $Q\mathbf{A}$ to be finely finite and $\mathbf{A} \to Q\mathbf{A}$ to have finite fibres.

In the case $Q = \mathrm{id}$, this just says that $\mathbf{A}$ is finely finite. In the case $Q = P$, it says that $\mathbf{A}$ is patch-finite (by Lemma 3.2(iii)). In the case $Q = C$, it says that $\mathbf{A}$ is coarsely finite. In fact, the three conditions are successively stronger:

$$\text{coarsely finite} \ \Rightarrow\ \text{patch-finite} \ \Rightarrow\ \text{finely finite}.$$

Let $\mathbf{A}$ be a patch-finite category. Then the algebra $k_{\mathrm{p}}\mathbf{A} = k(P\mathbf{A})$ is defined and the induced map $\Sigma\colon k\mathbf{A} \to k_{\mathrm{p}}\mathbf{A}$ is a homomorphism; the **coarse zeta function** $\zeta_{\mathbf{A}} \in k_{\mathrm{p}}\mathbf{A}$ is the image under Σ of $\zeta_{\mathbf{A}} \in k\mathbf{A}$. Explicitly, $k_{\mathrm{p}}\mathbf{A}$ is the submodule of $k_{\mathrm{c}}\mathbf{A}$ specified in (2), and the product on $k_{\mathrm{p}}\mathbf{A}$ is given by

$$(\alpha * \beta)(a, b) = \sum_{c \in [a,b]_{\mathbf{A}}} \alpha(a, c)\, \beta(c, b)$$

($\alpha, \beta \in k_{\mathrm{p}}\mathbf{A}$, $a, b \in \mathbf{A}$). (There is no product defined on the larger module $k_{\mathrm{c}}\mathbf{A}$ unless $\mathbf{A}$ is finite.) As before, the zeta function is given explicitly by $\zeta_{\mathbf{A}}(a, b) = \#(\mathbf{A}(a, b)) \in k$.

For example, when A is a locally finite poset, $k_{\mathrm{p}}A$ is the classical incidence algebra.

Definition 3.3. A patch-finite category $\mathbf{A}$ has **coarse Möbius inversion** if $\zeta_{\mathbf{A}} \in k_{\mathrm{p}}\mathbf{A}$ is invertible. In that case, its **coarse Möbius function** is $\mu_{\mathbf{A}} = \zeta_{\mathbf{A}}^{-1} \in k_{\mathrm{p}}\mathbf{A}$.

Prima facie, we should have used different terminology: 'patch Möbius inversion/function'. After all, when $\mathbf{A}$ is a finite category, $k_{\mathrm{p}}\mathbf{A}$ is in general a *proper* subalgebra of $k_{\mathrm{c}}\mathbf{A}$, so one might think that it would be easier to invert $\zeta_{\mathbf{A}}$ in $k_{\mathrm{c}}\mathbf{A}$ than in $k_{\mathrm{p}}\mathbf{A}$. It is a nontrivial fact that it makes no difference (Corollary 3.6). Definition 3.3 is therefore consistent with the definitions for finite categories.

Lemma 3.4. *Let $\mathbf{A}$ be a finite category, $n \geq 0$, and $a_0, \ldots, a_n \in \mathbf{A}$. Then*

$$\zeta(a_0, a_1) \cdots \zeta(a_{n-1}, a_n) \neq 0 \;\Rightarrow\; \zeta(a_0, a_n) \neq 0.$$

Proof. Since k has characteristic zero, an equivalent statement is that if the set $\mathbf{A}(a_0, a_1) \times \cdots \times \mathbf{A}(a_{n-1}, a_n)$ is nonempty then so is $\mathbf{A}(a_0, a_n)$. But since $\mathbf{A}$ is a category, there is a map from the first set to the second, and the result follows. ■

Theorem 3.5. *Let $\mathbf{A}$ be a finite category with coarse Möbius inversion over k. Let $a, b \in \mathbf{A}$. Then $\zeta_{\mathbf{A}}(a, b) = 0 \Rightarrow \mu_{\mathbf{A}}(a, b) = 0$.*

Proof. In the terminology of Appendix A, Lemma 3.4 states that $\zeta_{\mathbf{A}}$ is transitive. The result follows from Theorem A.4 on inverse matrices. ■

Corollary 3.6. *Let $\mathbf{A}$ be a finite category. The coarse zeta function of $\mathbf{A}$ is invertible in $k_{\mathrm{p}}\mathbf{A}$ if and only if it is invertible in $k_{\mathrm{c}}\mathbf{A}$.*

Consider, for example, a finite poset A. The algebra $kA = k_{\mathrm{p}}A$ consists of the k-valued functions on pairs $(a, b) \in A \times A$ such that $a \leq b$, whereas the algebra $k_{\mathrm{c}}A$ consists of the k-valued functions on *all* pairs (a, b). When k is a ring, the zeta function is always invertible in kA (and therefore in $k_{\mathrm{c}}A$), by the formula in Example 1.1. But for other rigs, it might not be invertible in kA, and Corollary 3.6 then implies that it is not invertible in the larger algebra $k_{\mathrm{c}}A$ either. These and earlier remarks tell us, in short, that the results on categorical Möbius inversion presented here give no more for posets than was already known to Rota et al.

When a patch-finite category has coarse Möbius inversion, its Möbius function is determined 'locally', that is, patchwise:

Proposition 3.7. *Let* $\mathbf{A}$ *be a patch-finite category. Then* $\mathbf{A}$ *has coarse Möbius inversion if and only if each patch* $[a,b]_{\mathbf{A}}$ *does. In that case, the coarse Möbius function of each patch* $[a,b]_{\mathbf{A}}$ *is the restriction of that of* $\mathbf{A}$.

This was stated without proof in the case of finite $\mathbf{A}$ as Corollary 4.3 of [20].

Proof. First suppose that $\mathbf{A}$ has coarse Möbius inversion, with coarse Möbius function $\mu \in k_{\mathrm{p}}\mathbf{A}$. Let $a, b \in \mathbf{A}$. We have to prove that for all $x, y \in [a,b]_{\mathbf{A}}$,

$$\sum_{z \in [a,b]_{\mathbf{A}}} \mu(x,z)\,\zeta(z,y) = \delta(x,y), \tag{3}$$

and similarly with μ and ζ interchanged. By definition of μ, this equation holds when the sum is taken over all $z \in [x,y]_{\mathbf{A}}$. But $[x,y]_{\mathbf{A}} \subseteq [a,b]_{\mathbf{A}}$, and conversely if $z \in [a,b]_{\mathbf{A}}$ with $\mu(x,z)\zeta(z,y) \neq 0$ then $z \in [x,y]_{\mathbf{A}}$ (since $\mu \in k_{\mathrm{p}}\mathbf{A}$). This gives (3).

Conversely, suppose that for each $a, b \in \mathbf{A}$, the patch $[a,b]_{\mathbf{A}}$ has coarse Möbius inversion, with coarse Möbius function $\mu_{a,b}$. Define $\mu \in k_{\mathrm{p}}\mathbf{A}$ by

$$\mu(a,b) = \begin{cases} \mu_{a,b}(a,b) & \text{if } \mathbf{A}(a,b) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

We prove that μ is the coarse Möbius function of $\mathbf{A}$. Indeed, let $a, b \in \mathbf{A}$. Then

$$\sum_{c \in [a,b]_{\mathbf{A}}} \mu(a,c)\,\zeta(c,b) = \sum_{c \in [a,b]_{\mathbf{A}}} \mu_{a,c}(a,c)\,\zeta(c,b). \tag{4}$$

It is straightforward to show that $[a,c]_{([a,b]_{\mathbf{A}})} = [a,c]_{\mathbf{A}}$ whenever $c \in [a,b]_{\mathbf{A}}$, using composition. So by the first part of the proof (with $[a,b]_{\mathbf{A}}$ playing the role of $\mathbf{A}$), the coarse Möbius function of $[a,c]_{\mathbf{A}}$ is the restriction of that of $[a,b]_{\mathbf{A}}$. The right-hand side of (4) is therefore unchanged if we replace $\mu_{a,c}(a,c)$ by $\mu_{a,b}(a,c)$, and the result follows by definition of $\mu_{a,b}$. ■

Examples 3.8. i. Let $\mathbf{D}_{\mathrm{inj}}$ be the category whose objects are the natural numbers and whose maps $m \to n$ are the order-preserving injections $\{1,\ldots,m\} \to \{1,\ldots,n\}$. For $a, b \in \mathbb{N}$, the patch $[a,b]_{\mathbf{D}_{\mathrm{inj}}}$ is the full subcategory on $\{n \in \mathbb{N} \mid a \leq n \leq b\}$. This is always finite, so $\mathbf{D}_{\mathrm{inj}}$ is patch-finite. It has coarse Möbius inversion: $\zeta(m,n) = \binom{n}{m}$ and $\mu(m,n) = (-1)^{n-m}\binom{n}{m}$. (Compare Example 1.2(c) of [20].)

ii. The same is true with surjections in place of injections; now $\zeta(m,n) = \binom{m-1}{n-1}$ and $\mu(m,n) = (-1)^{m-n}\binom{m-1}{n-1}$.

We can now generalize Haigh's comparison theorem and Menni's corollary:

Theorem 3.9. *Theorem 1.6 holds when* $\mathbf{A}$ *is merely patch-finite.*

Corollary 3.10. *Corollary 1.7 holds when* $\mathbf{A}$ *and* $\mathbf{A}'$ *are merely patch-finite.*

Rota's seminal paper [31] on Möbius inversion contained two 'main theorems'. The first, Theorem 1, described the compatibility of Möbius functions across a Galois connection between posets. It was generalized in [20] to adjunctions between finite categories. We now generalize it further, to patch-finite categories.

Proposition 3.11. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be patch-finite categories with coarse Möbius inversion. Let* $\mathbf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{B}$ *be functors with finite fibres, with* F *left adjoint to* G*. Then for all* $a \in \mathbf{A}$ *and* $b \in \mathbf{B}$*,*

$$\sum_{a' : F(a') = b} \mu_{\mathbf{A}}(a, a') = \sum_{b' : G(b') = a} \mu_{\mathbf{B}}(b', b).$$

Proof. Exactly as for Proposition 4.4 of [20]. ∎

4 Möbius inversion for enriched categories

The theory of fine Möbius inversion does not seem to generalize to enriched categories in an obvious way, speaking as it does of *individual* morphisms. Coarse Möbius inversion, however, generalizes easily. All one needs is a notion of size for the objects of the enriching category. In fact, coarse Möbius inversion for enriched categories has already been used extensively in the case of metric spaces (Example 4.1(iii)).

We confine ourselves to enriched categories with a finite number of objects, although by imitating the previous section, the theory can also be set up for infinitely many objects.

Fix a monoidal category $\mathbf{V} = (\mathbf{V}, \otimes, I)$, a rig k, and a monoid homomorphism

$$|\cdot| : (\mathbf{V}_0/\cong, \otimes, I) \to (k, \cdot, 1)$$

where the domain is the monoid of isomorphism classes of objects of $\mathbf{V}$.

Let $\mathbf{A}$ be a $\mathbf{V}$-category with finitely many objects. The **coarse incidence algebra** $k_c\mathbf{A}$ is defined exactly as in the non-enriched case. The **coarse zeta function** $\zeta_{\mathbf{A}} \in k_c\mathbf{A}$ is given by

$$\zeta_{\mathbf{A}}(a, b) = |\mathbf{A}(a, b)| \in k$$

($a, b \in \mathbf{A}$). If $\zeta_{\mathbf{A}}$ has an inverse in $k_c\mathbf{A}$ then $\mathbf{A}$ has **coarse Möbius inversion** over k, and its **coarse Möbius function** is $\mu_{\mathbf{A}} = \zeta_{\mathbf{A}}^{-1} \in k_c\mathbf{A}$.

The assumption that $|\cdot|$ is a monoid homomorphism was not needed in order to make these definitions, but will be used in Proposition 4.2.

Examples 4.1.

i. Taking $\mathbf{V}$ to be the category of finite sets, with $|X| = n \cdot 1 \in k$ when X is an n-element set, we recover the definitions for non-enriched categories.

ii. Take $\mathbf{V}$ to be the category $\mathbf{2} = (0 \to 1)$ with min as tensor product. Take $k = \mathbb{Z}$, and put $|0| = 0$ and $|1| = 1$. Then a $\mathbf{V}$-category is a poset, and every finite $\mathbf{V}$-category has coarse Möbius inversion (Example 1.3).

iii. Let $\mathbf{V}$ be the poset $([0, \infty], \geq)$, with monoidal structure given by addition. As shown by Lawvere [17], a $\mathbf{V}$-category is a generalized metric space. Put $k = \mathbb{R}$ and $|x| = e^{-x}$ ($x \in [0, \infty]$). This gives a notion of Möbius inversion for metric spaces. Most metric spaces have Möbius inversion, in a sense made precise by Proposition 2.2.6(i) of [21]. For example, all finite subspaces of

Euclidean space do (Theorem 2.5.3 of [21]); more generally, so do all finite subspaces of $L^p[0,1]$ whenever $0 < p \leq 2$ (Theorem 3.6 of [25]).

The **magnitude** of a metric space, defined for finite spaces as $\sum_{a,b} \mu(a,b)$, is especially significant. The definition extends to a large class of compact metric spaces [21, 25], where its geometric meaning begins to emerge: to take the simplest example, the magnitude of a straight line segment is one plus half its length. Further connections with geometric measure are established in [21, 22, 25, 33, 34].

iv. Let $\mathbf{V}$ be the category of finite-dimensional vector spaces with its usual tensor product, let k be any rig, and put $|X| = (\dim X) \cdot 1 \in k$. Then we obtain a notion of coarse Möbius inversion for linear categories.

v. Let $\mathbf{V}$ be the category of finite categories with Euler characteristic [20], made monoidal by cartesian product. Let $k = \mathbb{Q}$, and put $|\mathbf{X}| = \chi(\mathbf{X}) \in \mathbb{Q}$. (This is a monoid homomorphism, by Proposition 2.6 of [20].) We obtain a notion of coarse Möbius inversion for (some) finite 2-categories.

vi. Let $\mathbf{V}$ be the category $\mathbf{FinSet}^{\mathbb{N}}$ of sequences of finite sets, with $(X \otimes Y)_n = \sum_{p+q=n} X_p \times Y_q$. A $\mathbf{V}$-category is a category in which each map f has a degree $\deg(f) \in \mathbb{N}$, such that for each a, b, there are only finitely many maps $a \to b$ of each degree, and $\deg(g \circ f) = \deg(f) + \deg(g)$. Let $k = \mathbb{Q}((t))$, the ring of formal Laurent series over $\mathbb{Q}$. Put $|X| = \sum_{n \in \mathbb{N}} \#X_n \cdot t^n$. We obtain a notion of coarse Möbius inversion for graded categories.

For example, let G be a finite directed graph. The free category FG on G need not be finite, but is naturally $\mathbf{V}$-enriched: a map in FG is a path in G, with degree defined as length. It has Möbius inversion, as follows. Write G_0 and G_1 for the sets of vertices and edges of G, and, for $a, b \in G$, write $G(a,b)$ for the set of edges from a to b. Define $\zeta_G \in k_c\mathbf{A}$ by $\zeta_G(a,b) = \#G(a,b)$. Then $\zeta_{FG} = \sum_{n \in \mathbb{N}} (\zeta_G \cdot t)^{*n}$ and $\mu_{FG} = \delta - \zeta_G \cdot t$. It follows that $\sum_{a,b} \mu_{FG}(a,b) = \#G_0 - \#G_1 \cdot t$. (For instance, if G has just one vertex and m edges then FG is the free monoid on m generators and $\sum_{a,b} \mu_{FG}(a,b) = 1 - mt$.) When $t = 1$, this is the Euler characteristic of G; compare Proposition 2.10 of [20].

Coarse Möbius inversion interacts well with tensor product of enriched categories. Assume now that $\mathbf{V}$ is symmetric, so that the tensor product of $\mathbf{V}$-categories is defined. The following result generalizes Lemma 1.13(b) of [20], and is proved using the multiplicative property of $|\cdot|$.

Proposition 4.2. *Let $\mathbf{A}$ and $\mathbf{B}$ be $\mathbf{V}$-categories with finite object-sets. If $\mathbf{A}$ and $\mathbf{B}$ have coarse Möbius inversion over k then so does $\mathbf{A} \otimes \mathbf{B}$, with*

$$\mu_{\mathbf{A}\otimes\mathbf{B}}((a,b),(a',b')) = \mu_{\mathbf{A}}(a,a')\mu_{\mathbf{B}}(b,b')$$

($a, a' \in \mathbf{A}$, $b, b' \in \mathbf{B}$).

There is a similar result on coproducts, generalizing Lemma 1.13(a) of [20]. (Compare also Proposition 1.4.4 of [21].) Our generalization of Rota's main theorem (Proposition 3.11) also extends easily to the enriched setting.

5 Functoriality revisited

The incidence algebra construction is functorial in two ways: covariant and contravariant. We have already used the covariant functoriality. Here we examine its contravariant counterpart. We then show that the two types of functoriality interact well enough that they can, in fact, be unified into a single functor.

A functor $F\colon \mathbf{A} \to \mathbf{B}$ has **unique lifting of factorizations**, or is **ULF**, if whenever f is a map in $\mathbf{A}$ and $Ff = g_2 \circ g_1$ in $\mathbf{B}$, there are unique maps f_1, f_2 in $\mathbf{A}$ such that $f_2 \circ f_1 = f$, $Ff_1 = g_1$ and $Ff_2 = g_2$:

$$
\begin{tikzcd}
 & \cdot \arrow[dr, dotted, "f_2"] & \\
a \arrow[ur, dotted, "f_1"] \arrow[rr, "f"'] & & b
\end{tikzcd}
\quad \overset{F}{\longmapsto} \quad
\begin{tikzcd}
 & \cdot \arrow[dr, "g_2"] & \\
Fa \arrow[ur, "g_1"] \arrow[rr, "Ff"'] & & Fb.
\end{tikzcd}
$$

This definition is implicit in Théorème 4.1 of [9], and is made explicit in Section 4 of [18]. Appendix B places the ULF concept into an abstract context.

Let $F\colon \mathbf{A} \to \mathbf{B}$ be a functor between finely finite categories. For each rig k, there is an induced k-linear map

$$F^*\colon k\mathbf{B} \to k\mathbf{A}$$

defined by

$$(F^*\beta)(f) = \beta(Ff)$$

where $\beta \in k\mathbf{B}$ and f is a map in $\mathbf{A}$. It is a fact that F is ULF if and only if F^* is an algebra homomorphism for all rings k: again, this is implicit in Théorème 4.1 of [9], and it is made explicit in Theorem 9.21 of [19]. Our Proposition 2.1 is a covariant companion of this fact.

For example, whenever X is an object of a category $\mathscr{C}$, the forgetful functor $X/\mathscr{C} \to \mathscr{C}$ is ULF. Lawvere (Section 4 of [18]) and Lawvere and Menni (Example 9.22 of [19]) point out the following. When $\mathscr{C}$ is the additive monoid of natural numbers, viewed as a one-object category $(\mathbb{N}, +, 0)$, this is the functor $(\mathbb{N}, \leq) \to (\mathbb{N}, +, 0)$ sending the map $m \to n$ in $(\mathbb{N}, \leq)$ to the map $n - m$ in $(\mathbb{N}, +, 0)$, whenever $m \leq n$. It induces an algebra homomorphism $k(\mathbb{N}, +, 0) \to k(\mathbb{N}, \leq)$, thus relating the monoid Möbius inversion of Cartier and Foata [7] to the poset Möbius inversion of Rota et al.

The class of ULF functors is closed under composition, so there is a category $\mathbf{Cat}^*$ of finely finite categories and ULF functors. There is then a functor $\mathbf{Cat}^{*\mathrm{op}} \to k\text{-}\mathbf{Alg}$ defined by $\mathbf{A} \mapsto k\mathbf{A}$ and $F \mapsto F^*$.

The covariant and contravariant constructions are linked by a result with a strong formal resemblance to the Beck–Chevalley theorem.

Theorem 5.1. *Let*

$$\begin{tikzcd} \mathbf{D} \arrow[r, "F'"] \arrow[d, "G'"'] & \mathbf{B} \arrow[d, "G"] \\ \mathbf{A} \arrow[r, "F"'] & \mathbf{C} \end{tikzcd}$$

be a pullback square in **Cat***. Suppose that all four categories are finely finite, F is ULF, and G is bijective on objects and has finite fibres. Then F' is ULF, G' is bijective on objects and has finite fibres, and the square*

$$\begin{tikzcd} k\mathbf{D} \arrow[r, "F'_!"] & k\mathbf{B} \\ k\mathbf{A} \arrow[u, "G'^*"] \arrow[r, "F_!"'] & k\mathbf{C} \arrow[u, "G^*"'] \end{tikzcd}$$

commutes for all rigs k.

Proof. That F' is ULF follows from the fact that the pullback of an ULF functor along an arbitrary functor is again ULF, which can be checked directly and also follows from Proposition B.4. That G' is bijective on objects and has finite fibres is straightforward. Now let $\alpha \in k\mathbf{A}$ and $g \in \mathbf{B}_1$. We have

$$(G^* F_! \alpha)(g) = (F_! \alpha)(Gg) = \sum_{f \in \mathbf{A}_1 \colon Ff = Gg} \alpha(f).$$

On the other hand,

$$(F'_! G'^* \alpha)(g) = \sum_{h \in \mathbf{D}_1 \colon F'h = g} \alpha(G'h) = \sum_{f \in \mathbf{A}_1 \colon Ff = Gg} \alpha(f)$$

since the square is a pullback. ∎

We can now unify the two types of functoriality for incidence algebras.

The bicategory of spans in **Cat** [4] has a sub-bicategory $\mathbf{Cat}_!^*$, defined as follows. The objects are the finely finite categories. The 1-cells from $\mathbf{A}$ to $\mathbf{B}$ are the spans

$$\mathbf{A} \xleftarrow[\text{ULF}]{F} \mathbf{C} \xrightarrow[\text{BO, FF}]{G} \mathbf{B} \tag{5}$$

in which F is ULF and G is bijective on objects and has finite fibres. The 2-cells are the isomorphisms. We may also view k-**Alg** as a bicategory, with only identity 2-cells.

Corollary 5.2. *There is a strict functor $\mathbf{Cat}_!^* \to k\text{-}\mathbf{Alg}$ defined on objects by $k \mapsto k\mathbf{A}$ and on 1-cells by sending* (5) *to the composite homomorphism $k\mathbf{A} \xrightarrow{F^*} k\mathbf{C} \xrightarrow{G_!} k\mathbf{B}$.*

Proof. Theorem 5.1 implies that composition is preserved, and the rest is trivial. ∎

A cruder version of the same result uses the *category* $\mathbf{Cat}_!^*$ whose maps are the *isomorphism classes* of spans (5). We still obtain a functor $\mathbf{Cat}_!^* \to k\text{-}\mathbf{Alg}$.

6 The Möbius categories of Leroux

In the work of Leroux et al. [9, 23], a central role is played by the 'Möbius categories'. (Beware that Haigh [13] uses the same term differently.) A category is **Möbius** if it is finely finite and satisfies the equivalent conditions of the following theorem.

Theorem 6.1 (Content–Lemay–Leroux)**.** *Let* $\mathbf{A}$ *be a finely finite category. The following conditions on* $\mathbf{A}$ *are equivalent:*

i. Every isomorphism or idempotent in $\mathbf{A}$ *is an identity.*

ii. Each map in $\mathbf{A}$ *can be expressed as a composite of a finite sequence of non-identity maps in only finitely many ways.*

iii. For all rings k*, an element* $\alpha \in k\mathbf{A}$ *is invertible if and only if* $\alpha(1_a) \in k$ *is invertible for all* $a \in \mathbf{A}$*.*

Proof. This is nearly Théorème 1.1 of [9], except that where we have condition (i), they have the conjunction of two conditions: (a) if $g \circ f = 1_a$ in $\mathbf{A}$ then $g = f = 1_a$, and (b) if h is an endomorphism in $\mathbf{A}$ with $h^m = h^n$ for some natural numbers $m \neq n$ then h is an identity.

Certainly (a) and (b) together imply (i). The converse does not seem to have been stated completely explicitly before, although essentially it goes back to [23] (and it is proved for finite categories in Proposition 3.5 of [19]). Suppose that (i) holds. For (a), if $g \circ f = 1_a$ in $\mathbf{A}$ then $f \circ g$ is idempotent, so $f \circ g$ is an identity, so f and g are isomorphisms and therefore identities. For (b), suppose that $h^n = h^{n+k}$ for some $n, k \geq 1$; then h^{nk} is idempotent, so $h^{nk} = 1$, which by (a) implies that h is an identity. ∎

Being Möbius is a strictly stronger condition than having fine Möbius inversion over all rings. It is stronger by (iii), and *strictly* stronger by the following example.

Example 6.2. Let $\mathbf{A}$ be the category freely generated by objects and maps $a \underset{i}{\overset{s}{\rightleftarrows}} b$ subject to $si = 1_b$. It is easily shown that $\mathbf{A}$ has fine Möbius inversion over all rings (with $\mu(1_a) = 1$, $\mu(1_b) = 2$, $\mu(s) = \mu(i) = -1$, and $\mu(is) = 0$). But $\mathbf{A}$ is not Möbius, since it contains the nontrivial idempotent is.

This example can be viewed as follows. By Theorem 6.1(ii), every subcategory (full or not) of a Möbius category is Möbius. In particular, every subcategory of a Möbius category has fine Möbius inversion over all rings. However, $\mathbf{A}$ contains the subcategory $\mathbf{B}$ consisting of the object a, the identity 1_a, and the idempotent $e = is \neq 1_a$, which does *not* have fine Möbius inversion over all rings: the Möbius function would have to satisfy $2\mu_{\mathbf{B}}(e) = -1$.

So, having a subcategory without fine Möbius inversion is an obstruction to being Möbius. The main result of this section is that it is the *only* obstruction.

Theorem 6.3. *Let* $\mathbf{A}$ *be a finely finite category. The following conditions on* $\mathbf{A}$ *are equivalent:*

i. $\mathbf{A}$ *is Möbius*

ii. every subcategory of $\mathbf{A}$ *has fine Möbius inversion over every ring*

iii. every subcategory of $\mathbf{A}$ *has fine Möbius inversion over* $\mathbb{Z}$.

Proof. We have just seen that (i) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (iii) trivially. Now suppose (iii). We prove condition (i) of Theorem 6.1.

Let $i\colon a \to b$ be an isomorphism in $\mathbf{A}$. Since $\mathbf{A}$ is finely finite, 1_a has only finitely many factorizations into two factors; write them as $g_1 \circ f_1, \ldots, g_n \circ f_n$. Then the distinct factorizations of i are $(ig_1) \circ f_1, \ldots, (ig_n) \circ f_n$. But $\mathbf{A}$ itself has fine Möbius inversion over $\mathbb{Z}$, and

$$\sum_{r=1}^{n} \mu_{\mathbf{A}}(f_r) = \delta(1_a), \qquad \sum_{r=1}^{n} \mu_{\mathbf{A}}(f_r) = \delta(i),$$

so $\delta(i) = \delta(1_a) = 1 \in \mathbb{Z}$. Hence i is an identity.

Now let $e\colon a \to a$ be an idempotent in $\mathbf{A}$. As above, the subcategory consisting of the object a and the maps 1_a and e can only have fine Möbius inversion over $\mathbb{Z}$ if $e = 1_a$. ∎

Further characterizations of Möbius categories can be found in [9, 19, 23].

A Zeros of the Möbius function

To extend the definition of coarse Möbius inversion to categories with infinitely many objects, we made essential use of Theorem 3.5, the proof of which depended in turn on a fact about matrices: Theorem A.4 below. Our task here is to prove this.

The same result was proved in the case $k = \mathbb{Q}$ as Theorem 4.1 of [20]. For arbitrary rigs, the proof is complicated by the need to avoid subtraction.

Fix a rig k. Write the (i, j)-entry of a matrix X as X_{ij}.

Definition A.1. An $n \times n$ matrix Z over k is **transitive** if for all $p \geq 0$ and $i_1, \ldots, i_p \in \{1, \ldots, n\}$,

$$Z_{i_0 i_p} = 0 \;\Rightarrow\; Z_{i_0 i_1} Z_{i_1 i_2} \cdots Z_{i_{p-1} i_p} = 0.$$

The case $p = 0$ states that $Z_{ii} = 0 \Rightarrow 1 = 0$; that is, if k is nontrivial then $Z_{ii} \neq 0$.

For an $n \times n$ matrix X over k, write

$$\det{}^{+} X = \sum_{\sigma \in A_n} \prod_{r=1}^{n} X_{r,\sigma(r)}, \qquad \det{}^{-} X = \sum_{\sigma \in S_n \setminus A_n} \prod_{r=1}^{n} X_{r,\sigma(r)}.$$

Thus, $\det X = \det^+ X - \det^- X$. Let $\operatorname{adj}^+ X$ and $\operatorname{adj}^- X$ be the $n \times n$ matrices with entries

$$\operatorname{adj}^+_{ij} X = \sum_{\sigma \in A_n \colon \sigma(j) = i} \prod_{r \neq j} X_{r,\sigma(r)}, \qquad \operatorname{adj}^-_{ij} X = \sum_{\sigma \in S_n \setminus A_n \colon \sigma(j) = i} \prod_{r \neq j} X_{r,\sigma(r)}.$$

Thus, $\operatorname{adj}^+ X - \operatorname{adj}^- X$ is the adjugate (classical adjoint) $\operatorname{adj} X$.

Lemma A.2. *The following identities hold, for $n \times n$ matrices X and Y over k.*

i. $\det^+ I = 1$ *and* $\det^- I = 0$.

ii. $(\det^+ X)(\det^+ Y) + (\det^- X)(\det^- Y) + \det^-(XY) =$ $(\det^+ X)(\det^- Y) + (\det^- X)(\det^+ Y) + \det^+(XY)$.

iii. $X(\operatorname{adj}^+ X) + (\det^- X)I = X(\operatorname{adj}^- X) + (\det^+ X)I$.

Proof. Part (i) is immediate. For (ii), first note that the general identity $\det(XY) = (\det X)(\det Y)$ can be regarded as an identity in the ring of polynomials over $\mathbb{Z}$ in $2n^2$ variables. Substituting $\det = \det^+ - \det^-$ gives the equation shown, which is again an identity in this polynomial ring. But all coefficients are nonnegative, so it is also an identity in the rig of polynomials over $\mathbb{N}$ in $2n^2$ variables. The result follows. Part (iii) is proved similarly, using the identity $X(\operatorname{adj} X) = (\det X)I$ and the fact that $\operatorname{adj} = \operatorname{adj}^+ - \operatorname{adj}^-$. ■

Lemma A.3. *Let Z be an invertible, transitive $n \times n$ matrix over k. Suppose that $Z_{1n} = 0$. Then:*

i. Both $(\det^+ Z)(Z^{-1})_{1n}$ and $(\det^- Z)(Z^{-1})_{1n}$ have additive inverses in k.

ii. $\operatorname{adj}^+_{1n} Z = \operatorname{adj}^-_{1n} Z = 0$.

Proof. First I claim that if $\sigma \in S_n$ with $\sigma(n) = 1$ then $\prod_{r=1}^{n-1} Z_{r,\sigma(r)} = 0$. To prove this, choose the least $p \geq 1$ such that $\sigma^p(1) = 1$. We have $\sigma^{p-1}(1) = n$, and the numbers $1, \sigma(1), \ldots, \sigma^{p-2}(1)$ are all distinct and less than n, so

$$Z_{1,\sigma(1)} Z_{\sigma(1),\sigma^2(1)} \cdots Z_{\sigma^{p-2}(1),n} \mid Z_{1,\sigma(1)} Z_{2,\sigma(2)} \cdots Z_{n-1,\sigma(n-1)}.$$

But by transitivity and the hypothesis $Z_{1n} = 0$, the left-hand side is 0, so the right-hand side is also 0, as claimed.

For (i), it is enough to prove that $(\prod_{r=1}^{n} Z_{r,\sigma(r)})(Z^{-1})_{1n}$ has an additive inverse for each $\sigma \in S_n$. When $\sigma(n) = 1$, this follows from the claim. Suppose, then, that $\sigma(n) \neq 1$. We have

$$\sum_{i=1}^{n} Z_{\sigma^{-1}(1),i} (Z^{-1})_{in} = I_{\sigma^{-1}(1),n} = 0,$$

so $Z_{\sigma^{-1}(1),1}(Z^{-1})_{1n}$ has an additive inverse, and the result follows.

Part (ii) follows immediately from the claim. ■

Theorem A.4. *Let Z be an invertible, transitive, $n \times n$ matrix over k. Let $i, j \in \{1, \ldots, n\}$. Then*

$$Z_{ij} = 0 \Rightarrow (Z^{-1})_{ij} = 0.$$

Proof. If $i = j$ then $Z_{ii} = 0$, so by transitivity, k is trivial and the result holds. So we may suppose without loss of generality that $i = 1$ and $j = n$.

Applying Lemma A.2(iii) to Z, then premultiplying by Z^{-1}, we have

$$\mathrm{adj}^+ Z + (\det^- Z)Z^{-1} = \mathrm{adj}^- Z + (\det^+ Z)Z^{-1}.$$

Now taking the $(1, n)$ entries on each side and using Lemma A.3(ii), we have

$$(\det^- Z)(Z^{-1})_{1n} = (\det^+ Z)(Z^{-1})_{1n}. \tag{6}$$

On the other hand, we may take $X = Z$ and $Y = Z^{-1}$ in Lemma A.2(ii), which, with the aid of Lemma A.2(i), gives

$$\begin{aligned}(\det^+ Z)(\det^+ Z^{-1}) + (\det^- Z)(\det^- Z^{-1}) &= \\ &(\det^+ Z)(\det^- Z^{-1}) + (\det^- Z)(\det^+ Z^{-1}) + 1. \end{aligned} \tag{7}$$

Multiply (7) by $(Z^{-1})_{1n}$ on each side. By (6), the result is an equation of the form $\lambda = \lambda + (Z^{-1})_{1n}$, where, by Lemma A.3(i), $\lambda \in k$ has an additive inverse. Hence $(Z^{-1})_{1n} = 0$. ■

B Pullback-homomorphisms

Here we place the notion of ULF functor into an abstract context. In doing so, we discover a new analogy between ULF functors and local homeomorphisms.

Definition B.1. Let $\mathbf{T} = (T, \eta, \mu)$ be a monad on a category $\mathscr{E}$. A homomorphism

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d] & TB \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd} \tag{8}$$

of $\mathbf{T}$-algebras is a **pullback-homomorphism** if the square (8) is a pullback.

Proposition B.2. *Let* $\mathbf{T}$ *be the free category monad on the category of directed graphs. Then the pullback-homomorphisms of* $\mathbf{T}$*-algebras are precisely the ULF functors.*

Proof. Let $F \colon \mathbf{A} \to \mathbf{B}$ be a functor between small categories, regarded as a homomorphism of $\mathbf{T}$-algebras. Write $\mathbf{A}_n$ for the set of paths $a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} a_n$ in $\mathbf{A}$, and similarly $\mathbf{B}_n$. Since limits in a presheaf category are computed pointwise, F is a pullback-homomorphism if and only if the squares

$$\begin{tikzcd} \mathbf{A}_0 \arrow[r] \arrow[d, "1"'] & \mathbf{B}_0 \arrow[d, "1"] \\ \mathbf{A}_0 \arrow[r] & \mathbf{B}_0 \end{tikzcd} \qquad \begin{tikzcd} \sum_{n\in\mathbb{N}} \mathbf{A}_n \arrow[r] \arrow[d, "\circ"'] & \sum_{n\in\mathbb{N}} \mathbf{B}_n \arrow[d, "\circ"] \\ \mathbf{A}_1 \arrow[r] & \mathbf{B}_1 \end{tikzcd}$$

are pullbacks in **Set**. (Here $\sum$ denotes coproduct.) The left-hand square certainly is, and the right-hand square is a pullback if and only if

$$\begin{tikzcd} \mathbf{A}_n \arrow[r] \arrow[d, "\circ"'] & \mathbf{B}_n \arrow[d, "\circ"] \\ \mathbf{A}_1 \arrow[r] & \mathbf{B}_1 \end{tikzcd}$$

is a pullback for each $n \in \mathbb{N}$. This reduces by induction to the cases $n = 0$ and $n = 2$. For the $n = 2$ square to be a pullback is precisely the ULF property. The $n = 0$ square is a pullback if and only if F reflects identities; but this is always true if F is ULF. ■

Pullback-homomorphisms have a three-for-two property: given homomorphisms $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$ with g a pullback-homomorphism, $g \circ f$ is a pullback-homomorphism if and only if f is. This follows from the elementary properties of pullbacks, and applies in particular to ULF functors.

Here are the pullback-homomorphisms for some other monads. Proofs are omitted.

Examples B.3. i. Fix a group G and consider the monad $G \times -$ on **Set**, whose algebras are G-sets. Then every map of G-sets is a pullback-homomorphism.

ii. At the other extreme, when $\mathbf{T}$ is the free group monad on **Set**, the only pullback-homomorphisms are the isomorphisms.

iii. Take the monad $1 + -$ on **Set**, adjoining to each set a new element. Its category of algebras is equivalent to the category of sets and partial functions. The pullback-homomorphisms are the *total* functions.

iv. Let $\mathscr{P}$ be the powerset monad on **Set**. Its algebras are the complete lattices; the homomorphisms are the maps preserving joins. Among them, the pullback-homomorphisms are the injections whose images are downwards closed.

v. Let $\mathbf{A}$ be a small category. The forgetful functor $\mathbf{Set}^{\mathbf{A}} \to \mathbf{Set}^{\mathbf{A}_0}$ is monadic. So, writing $\mathbf{T}$ for the induced monad, the homomorphisms of $\mathbf{T}$-algebras are the natural transformations between functors $\mathbf{A} \to \mathbf{Set}$. The pullback-homomorphisms are the cartesian natural transformations: those for which every naturality square is a pullback.

The unwirable maps of Bowler [6] provide further examples.

We have observed that the class of pullback-homomorphisms is closed under composition. For a general monad $\mathbf{T}$, it is not stable under pullback (Example B.6); that is, the pullback of a pullback-homomorphism along an arbitrary homomorphism need not be a pullback-homomorphism. However:

Proposition B.4. *Let $\mathscr{E}$ be a category with pullbacks and $\mathbf{T}$ a monad on $\mathscr{E}$ whose functor part preserves pullbacks. Then the class of pullback-homomorphisms of $\mathbf{T}$-algebras is stable under pullback along arbitrary homomorphisms.*

Proof. Elementary manipulation of pullbacks. ∎

Since the free category monad on directed graphs preserves pullbacks, the class of ULF functors is stable under pullback. Proposition B.4 also implies that the class of pullback-homomorphisms is stable under pullback in Examples B.3(i), (iii), (v). Furthermore, the same is true in Examples B.3(ii) and (iv), not by the proposition but by the explicit description of pullback-homomorphisms given there. This covers all of our examples so far.

It is now useful to extend the terminology.

Definition B.5. Let $\mathscr{E}$ be a category with pullbacks and $\mathbf{T}$ a monad on $\mathscr{E}$. A homomorphism $f\colon (A, \alpha) \to (B, \beta)$ of $\mathbf{T}$-algebras is a **stable pullback-homomorphism** if for every homomorphism $g\colon (C, \gamma) \to (B, \beta)$ of $\mathbf{T}$-algebras, the pullback of f along g is a pullback-homomorphism.

Thus, the class $\mathscr{S}$ of stable pullback-homomorphisms is the largest subclass of the pullback-homomorphisms that is stable under pullback along arbitrary homomorphisms. In all of our examples so far, every pullback-homomorphism is stably so.

We finish with a suggestive example in which pullback-homomorphisms are not stable under pullback. I thank Mike Shulman for pointing it out.

Example B.6. Let $\mathbf{T}$ be the ultrafilter monad on **Set**, whose algebras are the compact Hausdorff spaces. It is shown in [8] that not every pullback-homomorphism of $\mathbf{T}$-algebras is stably so. It is also shown that the stable pullback-homomorphisms are precisely the local homeomorphisms.

According to Lawvere and Menni, 'The definition of ULF-functor should be compared with that of local homeomorphism' ([19], p.230). We now have a general concept, stable pullback-homomorphism, of which both ULF functors and local homeomorphisms (between compact Hausdorff spaces) are special cases. A further possibility, suggested by Joachim Kock, is that there might also be a connection via the axiomatic notion of étale map [16].

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