Calibrated Toposes

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Abstract

We study a particular structure on a topos $\mathcal{E}$, related to the notion of a 'class of étale maps' due to Joyal and Moerdijk [10] and to Bénabou's notion of 'calibration' [2], which corresponds to giving for each object A of $\mathcal{E}$ a 'natural' comparison between the slice category $\mathcal{E}/A$ and a smaller 'petit topos' associated with A. We show that there are many naturally-arising examples of such structures; but rather few of them satisfy the condition that the relation between the 'gros' and 'petit' toposes of every object is expressed by a local geometric morphism.

1 Calibrations

There are various aspects to the question 'What features of a topos $\mathcal{E}$ qualify it for being considered as a 'gros topos' whose objects are spaces in their own right, rather than a 'petit topos' representing a single (generalized) space?'. One of these, relating to the relationship between $\mathcal{E}$ and its full subcategory $\mathcal{S}$ of 'discrete spaces', has been investigated in papers such as [12] and [8]. Here we wish to focus on a different (though related) aspect of the question: namely, how to describe the relationship between the slice categories $\mathcal{E}/A$, $A \in \text{ob}\ \mathcal{E}$, and the corresponding 'petit toposes' of 'sheaves on A'. I am indebted to my student Nick Duncan for directing my attention to this problem: the original idea which led to the notion of 'calibration' presented here was his, though almost all the mathematical development is my own.

To begin with, let us recall the original 'gros topos' of J. Giraud, described in [1], IV 2.5.

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Example 1.1. Let $\mathcal{C}$ be a small full subcategory of the category $\mathbf{Sp}$ of topological spaces, which is closed under passage to open subspaces, and let J be the coverage (= Grothendieck topology) on $\mathcal{C}$ in which a sieve covers iff it contains a jointly-surjective family of open inclusions. For any space X (whether or not it belongs to $\mathcal{C}$), the restriction to $\mathcal{C}^{\mathrm{op}}$ of the functor $\mathbf{Sp}\,(-, X) \colon \mathbf{Sp}^{\mathrm{op}} \to \mathbf{Set}$ is a J-sheaf $l(X)$; this defines a functor $l \colon \mathbf{Sp} \to \mathbf{Sh}(\mathcal{C}, J)$ which is faithful (at least provided $\mathcal{C}$ contains the one-point space), and full when restricted to $\mathcal{C}$. Moreover, at least for a space X in $\mathcal{C}$, the inclusion of the poset $\mathcal{O}(X)$ of open subsets of X in the slice category $\mathcal{C}/X$ is full, preserves finite limits, and preserves and reflects covers, from which it follows that it induces a local geometric morphism $q_X \colon \mathbf{Sh}(\mathcal{C}, J)/l(X) \to \mathbf{Sh}(X)$.

Similar examples may be constructed on replacing $\mathcal{C}$ by the category $\mathbf{Mf}$ of smooth manifolds (note that we do not need to 'cut down' to a small subcategory in this case, since the hypotheses of the Comparison Lemma apply to the inclusion $\mathbf{Mf}_0 \to \mathbf{Mf}$, where $\mathbf{Mf}_0$ is the full subcategory of connected manifolds, and the latter is essentially small), or by the category of affine schemes of finite type over an algebraically closed field K (the dual of the category of finitely-presented K-algebras). These are all instances of the following general construction (essentially due to E.J. Dubuc [4]).

Lemma 1.2. *Let $\mathcal{C}$ be a small category with a terminal object* 1*, and let $\mathcal{U}$ be a set of monomorphisms of $\mathcal{C}$ (to be thought of as 'open inclusions') with the properties*

(*a*) *$\mathcal{U}$ contains all isomorphisms, and is closed under composition.*

(*b*) *Morphisms in $\mathcal{U}$ may be pulled back along arbitrary morphisms of $\mathcal{C}$, and their pullbacks are in $\mathcal{U}$.*

For each object X of $\mathcal{C}$, let $|X|$ denote the space obtained by equipping the set $\mathcal{C}\,(1, X)$ of points of X with the topology whose basic open sets are the images of the mappings $\mathcal{C}\,(1, Y) \to \mathcal{C}\,(1, X)$ induced by morphisms $Y \to X$ in $\mathcal{U}$. Let J be the coverage on $\mathcal{C}$ consisting of those sieves which contain a family $(Y_i \to X \mid i \in I)$ of morphisms in $\mathcal{U}$ for which the induced mappings $(|Y_i| \to |X| \mid i \in I)$ are jointly surjective, and let $l \colon \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$ denote the composite of the Yoneda embedding with the associated sheaf functor. Then, for each X, we have a local geometric morphism $q_X \colon \mathbf{Sh}(\mathcal{C}, J)/l(X) \to \mathbf{Sh}(|X|)$, and for each morphism $\alpha \colon Y \to X$ in $\mathcal{C}$ we have a commutative square

$$\begin{tikzcd}
\mathbf{Sh}(\mathcal{C}, J)/l(Y) \arrow[r] \arrow[d, "q_Y"] & \mathbf{Sh}(\mathcal{C}, J)/l(X) \arrow[d, "q_Y"] \\
\mathbf{Sh}(|Y|) \arrow[r] & \mathbf{Sh}(|X|)
\end{tikzcd}$$

where the horizontal morphisms are induced by α.

Proof. See [9], 1.10. ∎

The proof in [9] does not in fact require the hypothesis that the members of $\mathcal{U}$ are monomorphisms, but we have imposed it since we shall require it for subsequent developments — it is satisfied in all the examples of interest. Note, incidentally, that this assumption implies a partial converse to (a): if (α, β) is a composable pair of morphisms of $\mathcal{C}$ such that $\alpha\beta$ and α are both in $\mathcal{U}$, then $\beta \in \mathcal{U}$, since it is (isomorphic to) the pullback of $\alpha\beta$ along α. In particular, this implies that if $\alpha\colon Y \to X$ is a morphism in $\mathcal{U}$, then the induced mapping $|Y| \to |X|$ is an open embedding, and the square in the statement of 1.2 is actually a pullback in the 2-category $\mathsf{BTop}/\mathbf{Set}$ of Grothendieck toposes (cf. 1.4(g) below). Note also that there is no loss of generality, in the situation of 1.2, in assuming that every object of $\mathcal{C}$ admits a point (that is, a morphism from 1); for any object which fails to do so will be J-covered by the empty sieve. It follows that $\mathbf{Sh}(\mathcal{C}, J)$ is hyperconnected over $\mathbf{Set}$, as well as being local (the latter since $|1|$ is the one-point space); and it will be punctually locally connected provided the spaces $|X|$ are all locally connected (cf. [8], 1.4).

However, if we are to regard *every* object of our gros topos $\mathcal{E}$ as having the characteristics of a 'space' of some kind, then we should expect to find some analogue of the geometric morphism $q_X\colon \mathbf{Sh}(\mathcal{C}, J)/l(X) \to \mathbf{Sh}(|X|)$ when $l(X)$ is replaced by an arbitrary object of $\mathcal{E}$. We should like this morphism to have the same good properties as q_X; but, for the moment, all we shall assume is that it is connected (i.e. that its inverse image is full and faithful). This means that we can identify the 'petit topos' associated with an object A as a full subcategory of $\mathcal{E}/A$, whose objects we may think of as the 'fibrewise discrete' morphisms with codomain A. In attempting to axiomatize this notion, we are led to the following definition. (The name 'calibration' was originally introduced by J. Bénabou [2] for a very similar concept.)

Definition 1.3. Let $\mathcal{E}$ be a topos. By a *calibration* in $\mathcal{E}$, we mean a class $\mathcal{D}$ of morphisms of $\mathcal{E}$, satisfying the following conditions:

(a) $\mathcal{D}$ contains all isomorphisms of $\mathcal{E}$, and in any commutative triangle

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[dr, "h"'] & B \arrow[d, "f"] \\ & A \end{tikzcd}$$

with $f \in \mathcal{D}$, we have $g \in \mathcal{D}$ iff $h \in \mathcal{D}$.

(b) $\mathcal{D}$ is stable under pullback: that is, given a pullback square

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "k"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & A \end{tikzcd}$$

with $f \in \mathcal{D}$, we also have $k \in \mathcal{D}$.

(c) $\mathcal{D}$ descends along epimorphisms; i.e., in a pullback square as in (b), if $k \in \mathcal{D}$ and g is epic then $f \in \mathcal{D}$.

(d) For every $f : B \to A$ in $\mathcal{E}$, there exists $g : C \to B$ such that $fg \in \mathcal{D}$, and universal among such morphisms (i.e. such that if $fh \in \mathcal{D}$ then h factors uniquely through g).

Before proceeding to the examples, we should note some immediate consequences of this definition.

Remarks 1.4. (e) By (a), we may regard $\mathcal{D}$ as a subcategory of $\mathcal{E}$; moreover, for any object A the slice category $\mathcal{D}/A$ is a full subcategory of $\mathcal{E}/A$. Condition (d) says that $\mathcal{D}/A$ is coreflective in $\mathcal{E}/A$, and it follows easily from (a) and (b) that it is closed under pullbacks and contains the terminal object. So, by a standard result ([7], A4.2.1), it is a topos, and the inclusion $\mathcal{D}/A \to \mathcal{E}/A$ is the inverse image of a connected geometric morphism p_A.

(f) For any morphism $f : B \to A$ in $\mathcal{E}$, there is a commutative square

$$\begin{tikzcd} \mathcal{E}/B \arrow[r, "f"] \arrow[d, "p_A"] & \mathcal{E}/A \arrow[d, "p_B"] \\ \mathcal{D}/B \arrow[r] & \mathcal{D}/A \end{tikzcd}$$

of geometric morphisms, where the top edge is the morphism whose inverse image is the pullback functor $f^* : \mathcal{E}/A \to \mathcal{E}/B$. For condition ($b$) implies that f^* restricts to a finite-limit-preserving functor $\mathcal{D}/A \to \mathcal{D}/B$, and we may obtain a right adjoint for it by applying Π_f to objects of $\mathcal{D}/B$ and then coreflecting into $\mathcal{D}/A$. Moreover, the assignment $A \mapsto \mathcal{D}/A$ is easily seen to be a (pseudo)functor $\mathcal{E} \to \mathsf{Top}$, and $(A \mapsto p_A)$ is a (pseudo)natural transformation from $(A \mapsto \mathcal{E}/A)$ to this functor.

(g) If f itself is in $\mathcal{D}$, then the square in (f) is a pullback in the 2-category Top of toposes and geometric morphisms, since (a) allows us to identify $\mathcal{D}/B$ with $(\mathcal{D}/A)/f$.

(h) The descent condition (c) is equivalent to saying that the square in (f) is a pushout in Top whenever f is an epimorphism in $\mathcal{E}$, since the corresponding diagram of inverse image functors is a pullback in CAT (cf. [7], section B3.4).

(i) We claim that *a morphism of $\mathcal{E}$ belongs to $\mathcal{D}$ iff both halves of its image factorization do so.* One direction is immediate from (a). Conversely, if $(f : B \to A) \in \mathcal{D}$, then the projections $B \times_A B \rightrightarrows B$ are in $\mathcal{D}$ by (b), and thus may be regarded as morphisms $f \times f \rightrightarrows f$ in $\mathcal{D}/A$; but by (e) $\mathcal{D}/A$ is closed under coequalizers in $\mathcal{E}/A$, and so the image $I \rightarrowtail A$ of f is in $\mathcal{D}$. Hence so is $B \twoheadrightarrow I$, by (a).

(j) Although we do not require that all monomorphisms of $\mathcal{E}$ belong to $\mathcal{D}$ (but see 3.1 below), we can show that *all complemented monomorphisms belong to $\mathcal{D}$.* To see this, suppose $m : B \rightarrowtail A$ is a monomorphism with complement $m' : B' \rightarrowtail A$. Since the coreflection (c, say) from $\mathcal{E}/A$ to $\mathcal{D}/A$ is the direct image of a connected geometric morphism, it preserves coproducts by [7], C3.4.14; so we have

$c(m) \amalg c(m') \cong c(1_A) \cong 1_A$. But this says that $c(B)$ and $c(B')$ are complementary subobjects of A, and they satisfy $c(B) \leq B$ and $c(B') \leq B'$; so both of the latter inequalities must be isomorphisms.

(k) We may now deduce that *a morphism $B \to A_1 \amalg A_2$ belongs to $\mathcal{D}$ iff its pullbacks along both the coprojections do so*. One direction is immediate from (b); conversely, if the pullbacks $B_i \to A_i$ are both in $\mathcal{D}$, then so are the composites $B_i \to A_i \to A_1 \amalg A_2$, by ($a$) and ($j$), and hence $B \to A_1 \amalg A_2$ is in $\mathcal{D}$ since $\mathcal{D}/(A_1 \amalg A_2)$ is closed under coproducts in $\mathcal{E}/(A_1 \amalg A_2)$, by ($e$). Hence the functor $\mathcal{E} \to \mathsf{Top}$ sending A to $\mathcal{D}/A$ preserves finite coproducts. If $\mathcal{E}$ is cocomplete (e.g. if $\mathcal{E}$ is a Grothendieck topos), then similar remarks apply to infinite coproducts.

(l) Following on from (k), we may strengthen the descent condition (c) to say that if the pullback of f along each member of a finite epimorphic family is in $\mathcal{D}$, then $f \in \mathcal{D}$ — and in a Grothendieck topos we may strengthen it to a similar condition on arbitrary (set-indexed) epimorphic families.

Remark 1.5. We should also comment on the relationship between our definition and the notion of a 'class of étale maps' introduced by Joyal and Moerdijk in [10]. Their definition comes in two versions, an elementary one (interpretable in an arbitrary pretopos), and a 'set-based' one interpretable in Grothendieck toposes; it is the latter version which concerns us here. In this version, a *class of étale maps* $\mathcal{D}$ in a Grothendieck topos $\mathcal{E}$ satisfies

(a) $\mathcal{D}$ contains all isomorphisms of $\mathcal{E}$, and is closed under composition.

(b) $\mathcal{D}$ is stable under pullback.

(c) $\mathcal{D}$ descends along epimorphisms.

(d) If $(f_i \colon B_i \to A_i) \in \mathcal{D}$ for each $i \in I$, then $\coprod_{i \in I} f_i \colon \coprod_{i \in I} B_i \to \coprod_{i \in I} A_i$ is in $\mathcal{D}$.

(e) For any object A and any set I, the codiagonal map $\coprod_{i \in I} A \to A$ is in $\mathcal{D}$. (Note that, by (b), it would be sufficient to demand this condition for $A = 1$. By convention, we interpret the condition as including the case $I = \emptyset$, in which case it asserts that the unique morphism $0 \to A$ is in $\mathcal{D}$.)

(f) Given $f \colon B \to A$ and $g \colon C \to B$, if g is epic and a member of $\mathcal{D}$ and $fg \in \mathcal{D}$, then $f \in \mathcal{D}$.

(g) If $(f \colon B \to A) \in \mathcal{D}$, then the diagonal map $B \rightarrowtail B \times_A B$ is also in $\mathcal{D}$.

It is easy to see that all these conditions follow from those of 1.3: (a–c) are contained in (a–c) of 1.3, (d) was verified in (k) of 1.4, and (e) follows easily from closure of $\mathcal{D}/A$ under coproducts in $\mathcal{E}/A$. Similarly, (f) follows from the closure of $\mathcal{D}/A$ under coequalizers in $\mathcal{E}/A$ (since g is the coequalizer of its kernel-pair, and the projections $C \times_B C \rightrightarrows C$ are pullbacks of g and hence in $\mathcal{D}$), and (g) follows from (a) of 1.3 and the fact that the projections $B \times_A B \rightrightarrows B$ are (pullbacks of f, and hence) in $\mathcal{D}$.

In the converse direction, the Joyal–Moerdijk axioms imply the stronger form of (a) that we have adopted; for if we are given $f\colon B \to A$ and $g\colon C \to B$ with fg and f both in $\mathcal{D}$, then we can factor g as

$$C \xrightarrow{(1,g)} C \times_A B \xrightarrow{\pi_2} B$$

where the first factor is a pullback of the diagonal $B \rightarrowtail B \times_A B$, and hence in $\mathcal{D}$ by (g) and (b), and the second is a pullback of fg and so also in $\mathcal{D}$. As a full subcategory of $\mathcal{E}/A$, $\mathcal{D}/A$ is closed under arbitrary coproducts, by (d) and (e). Finally, it is closed under coequalizers of equivalence relations; for if we are given a coequalizer

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & B \arrow[r, two heads, "q"] \arrow[d] & Q \arrow[dl] \\ & A & \end{tikzcd}$$

where $R \rightrightarrows B$ is an equivalence relation, and $R \to A$ and $B \to A$ are in $\mathcal{D}$, then q is epic and its pullback along itself is in $\mathcal{D}$, so it is in $\mathcal{D}$ by (c), and hence $Q \to A$ is in $\mathcal{D}$ by (f).

It follows from the arguments above that, if $\mathcal{D}$ is a class of étale maps in a Grothendieck topos $\mathcal{E}$, then $\mathcal{D}/A$ is closed under arbitrary colimits in $\mathcal{E}/A$, for any A; so we might expect to be able to apply the Adjoint Functor Theorem to conclude that it is coreflective, and thus that $\mathcal{D}$ is a calibration in our sense. Unfortunately, there does not seem to be any reason why the necessary solution-set condition should hold in general. On the other hand, we shall see below that there are many examples of interest where it does hold, and we have not been able to find any example where it fails; so it remains a possibility that our definition is equivalent to that of Joyal and Moerdijk.

If $\mathcal{D}$ is a calibration in a Grothendieck topos $\mathcal{E}$, then the categories $\mathcal{D}/A$, being coreflective in $\mathcal{E}/A$, are (co)complete and locally small; but, once again, there seems to be no *a priori* reason why they should possess generating sets (i.e. why they should be Grothendieck toposes). However, this condition is also satisfied in all the examples which we shall discuss below. It seems reasonable to call $\mathcal{D}$ a *Grothendieck calibration* if this condition holds.

2 Examples of Calibrations

We recall from [8] that another desideratum for a 'gros topos' $\mathcal{E}$ is that it should be punctually locally connected over a topos $\mathcal{S}$ of 'discrete spaces' — that is, that there should exist a geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$ which is locally connected, hyperconnected and local. In simple cases, one might suppose that knowledge of p alone (that is, of which spaces are discrete) could be enough to determine which morphisms are fibrewise discrete, as in the following construction. In fact we need only two of the three properties just mentioned in order to obtain a calibration:

Lemma 2.1. *Let $p\colon \mathcal{E} \to \mathcal{S}$ be a geometric morphism, and let $\mathcal{D}_p$ denote the class of morphisms $f\colon B \to A$ in $\mathcal{E}$ for which the naturality square*

$$\begin{tikzcd} p^*p_*B \arrow[r, "\epsilon_B"] \arrow[d, "p^*p_*f"] & B \arrow[d, "f"] \\ p^*p_*A \arrow[r, "\epsilon_A"] & A \end{tikzcd}$$

(where ϵ is the counit of $(p^* \dashv p_*)$) is a pullback. If p is local and hyperconnected, then $\mathcal{D}_p$ is a calibration.

Proof. Using familiar properties of pullbacks, and the fact that p^* and p_* both preserve them, it is easy to see that $\mathcal{D}_p$ satisfies conditions (*a*) and (*b*) of 1.3. Condition (*c*) follows from the fact that p_* preserves epimorphisms (cf. [8], 3.3); for if the pullback of f along an epimorphism g is in $\mathcal{D}_p$, then the canonical map from p^*p_*f to the pullback of f along ϵ_A becomes an isomorphism when pulled back along p^*p_*g. For condition (*d*), we proceed as follows: first recall from [7], A4.6.6(iv) that ϵ is monic if p is hyperconnected. Given $f\colon B \to A$, let $u\colon E \rightarrowtail B$ denote the pullback of ϵ_A along f, and let $v\colon B' \rightarrowtail B$ be the Heyting implication $(u \Rightarrow \epsilon_B)$ in $\mathrm{Sub}(B)$. We claim that the composite $fv\colon B' \to A$ is the coreflection of f in $\mathcal{D}_p/A$: first note that if $g\colon C \to B$ is such that $fg \in \mathcal{D}_p$, then $g^*(u) \cong \epsilon_C \leq g^*(\epsilon_B)$ in $\mathrm{Sub}(C)$, so that g factors uniquely through v. But we also have $v \cap u \cong v \cap \epsilon_B \cong v\epsilon_{B'}$ in $\mathrm{Sub}(B)$ (the second isomorphism because $v \in \mathcal{D}_p$), so $fv \in \mathcal{D}_p$. ∎

Remarks 2.2. (*a*) Since the only part of the above argument which uses hyperconnectedness is the verification of (*d*), it is tempting to conjecture that one might be able to extend 2.1 to the case where p is merely local, replacing the Heyting implication $(u \Rightarrow \epsilon_B)$ in $\mathrm{Sub}(B)$ by the object $v = \Pi_u w$ of $\mathcal{E}/B$, where $w\colon p^*p_*B \to E$ is the factorization of ϵ_B through u. However, it does not seem possible to prove directly that the composite fv belongs to $\mathcal{D}_p$. In 2.7(*d*) below, we shall show by a different method that $\mathcal{D}_p$ is a calibration when p is any local geometric morphism, provided $\mathcal{E}$ is a Grothendieck topos.

(*b*) On the other hand, the requirement that p should be local (which we used in verifying the descent condition (*c*)) cannot be relaxed. Consider the (hyperconnected) morphism $p\colon [G, \mathbf{Set}] \to \mathbf{Set}$, where G is a nontrivial group: it is easy to verify that the class $\mathcal{D}_p$ consists of morphisms $f\colon A \to B$ which 'reflect G-fixed points', i.e., if $a \in A$ and $g.f(a) = f(a)$ for all $g \in G$, then $g.a = a$ for all g. Hence, if A is a nonempty G-set without G-fixed points, the projections $A \times A \rightrightarrows A$ are in $\mathcal{D}_p$ but the unique morphism $A \to 1$ is not, so the descent condition fails.

(*c*) Note that if A is of the form p^*C for some $C \in \mathrm{ob}\ \mathcal{S}$, then $\mathcal{D}_p/A$ is equivalent to $\mathcal{S}/C$. Note also that $\mathcal{D}_p$ is the unique largest calibration $\mathcal{D}$ in $\mathcal{E}$ for which $p_1\colon \mathcal{E} \to \mathcal{D}/1$ is equivalent to p; for in any such calibration we have $\mathcal{D}/p^*C \simeq \mathcal{S}/C$ for all $C \in \mathrm{ob}\ \mathcal{S}$, by 1.4(*g*), and so if $f\colon B \to A$ is in $\mathcal{D}$ then its pullback along ϵ_A must be (in the image of p^*, and hence) isomorphic to $p^*p_*(f)$. In general, however, a calibration $\mathcal{D}$ is not uniquely determined by $\mathcal{D}/1$.

(*d*) It is tempting to suppose that, given a suitable (in particular, hyperconnected) geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$, we might obtain a calibration in $\mathcal{E}$ by defining $\mathcal{E}/A \to \mathcal{D}/A \to \mathcal{S}$ to be the hyperconnected–localic factorization of the composite $\mathcal{E}/A \to \mathcal{E} \to \mathcal{S}$ — equivalently, by taking $\mathcal{D}/A$ to be the topos of $\mathcal{S}$-valued sheaves on the internal locale $p_*(\Omega^A)$. However, this definition does not in general satisfy the descent condition (*c*) of 1.3. For a counterexample, take $\mathcal{S} = \mathbf{Set}$ and $\mathcal{E}$ to be the topos $[\mathbf{\Delta}_1^{\mathrm{op}}, \mathbf{Set}]$ of reflexive graphs (here $\mathbf{\Delta}_1$ denotes the full subcategory of the simplicial category on the objects 0 and 1). $\mathcal{E}$ is punctually locally connected over $\mathbf{Set}$ by [8], 1.4, and it is easy to verify that the calibration $\mathcal{D}_p$ of 2.1 consists of those morphisms $f\colon B \to A$ which reflect identity arrows, i.e. such that, if x is an arrow of B such that $f(x)$ is an identity arrow in A, then x itself is an identity. On the other hand, $\mathcal{E}/A$ may be identified with $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, where $\mathcal{A}$ is the total category of the discrete fibration over $\mathbf{\Delta}_1$ corresponding to A, and its localic reflection may thus be identified with $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ where $\mathbb{A}$ is the preorder reflection of $\mathcal{A}$ (cf. [7], A4.6.9). It is easy to see that this topos is equivalent to $\mathcal{D}_p/A$ if A has no non-identity endo-arrows (that is, arrows with the same source and target), but otherwise it is strictly smaller. Thus, if we take B to be the directed graph with two vertices and one non-identity arrow between them, and A to be the quotient of B obtained by identifying the two vertices, then the quotient map $B \twoheadrightarrow A$ is not in $\mathcal{D}/A$, but its pullback along itself is in $\mathcal{D}/B$.

There is one particular (and fundamental) case in which we may verify by elementary means that $\mathcal{D}_p$ (defined as in 2.1) is a calibration without p being hyperconnected. This is the case when $\mathcal{E}$ is the *Sierpiński topos* $[\mathbf{2}, \mathcal{F}]$ over some topos $\mathcal{F}$, that is the category whose objects are the morphisms of $\mathcal{F}$ and whose morphisms are commutative squares in $\mathcal{F}$.

Example 2.3. Consider the local geometric morphism $p\colon [\mathbf{2}, \mathcal{F}] \to \mathcal{F}$ for an arbitrary $\mathcal{F}$ defined by $p_*(A_0 \to A_1) = A_0$ and $p^*(A) = (1_A\colon A \to A)$. It is easy to see that $\mathcal{D}_p$ consists precisely of those morphisms in $[\mathbf{2}, \mathcal{F}]$ which are pullback squares in $\mathcal{F}$; in other words, it is the *canonical* class of étale maps in $[\mathbf{2}, \mathcal{F}]$, in the terminology of Joyal and Moerdijk [10]. To verify that it is a calibration, the only nontrivial condition is (*d*), for which we argue as follows. Given an object $(A_0 \to A_1)$ of $[\mathbf{2}, \mathcal{F}]$, the slice category $[\mathbf{2}, \mathcal{F}]/(A_0 \to A_1)$ may be identified with the diagram category $[\mathbb{A}, \mathcal{F}]$, where $\mathbb{A}$ is an internal poset (the domain of the discrete opfibration over $\mathbf{2}$ corresponding to the given object) whose object of objects is $A_0 \amalg A_1$. And $\mathcal{D}_p/(A_0 \to A_1)$ is equivalent to $\mathcal{F}/A_1 \cong [DA_1, \mathcal{F}]$, where DA_1 denotes the discrete internal category on A_1. It is easy to see that the inclusion $i\colon DA_1 \to \mathbb{A}$ has an internal left adjoint l, and that the equivalences above identify the inclusion $\mathcal{D}_p/(A_0 \to A_1) \to [\mathbf{2}, \mathcal{F}]/(A_0 \to A_1)$ with the functor l^*. So it is an inverse image functor. For future reference, we note that l in turn has a left adjoint (so that, by [7], C3.6.3(*b*), the geometric morphism $[\mathbf{2}, \mathcal{F}]/(A_0 \to A_1) \to \mathcal{D}_p/(A_0 \to A_1)$ is local) iff each connected component of $\mathbb{A}$ has an initial object, which is equivalent to saying that $A_0 \to A_1$ is a complemented monomorphism in $\mathcal{F}$.

If $\mathcal{D}$ is a calibration in a Grothendieck topos $\mathcal{E} = \mathbf{Sh}(\mathcal{C}, J)$, then it follows immediately from (*b*) of 1.3 and (*l*) of 1.4 that a morphism $f\colon B \to A$ belongs to

$\mathcal{D}$ iff its pullback along every morphism $l(X) \to A$, with $X \in \mathrm{ob}\,\mathcal{C}$, does so. This suggests the following result:

Proposition 2.4. *Let $(\mathcal{C}, J)$ be a small site, where J is subcanonical, and let l denote the Yoneda embedding $\mathcal{C} \to \mathcal{E} = \mathbf{Sh}(\mathcal{C}, J)$. Suppose we are given for each $X \in \mathrm{ob}\,\mathcal{C}$ a connected geometric morphism $q_X \colon \mathcal{E}/l(X) \to \mathcal{F}_X$ (where $\mathcal{F}_X$ is a Grothendieck topos), and for each morphism $\alpha \colon Y \to X$ in $\mathcal{C}$ a geometric morphism $\widehat{\alpha}$ making*

$$\begin{tikzcd}
\mathcal{E}/l(Y) \arrow[r, "l(\alpha)"] \arrow[d, "q"] & \mathcal{E}/l(X) \arrow[d, "q_X"] \\
\mathcal{F}_Y \arrow[r, "\widehat{\alpha}"] & \mathcal{F}_X
\end{tikzcd}$$

commute. Suppose further that the class $\mathcal{D}_0$ of morphisms $A \to l(X)$ in $\mathcal{E}$ which are isomorphic to objects in the image of q_X^* (for some X) satisfies the following conditions:

(i) *Given*

$$B \xrightarrow{g} A \xrightarrow{f} l(X)$$

where $f \in \mathcal{D}_0$, we have $fg \in \mathcal{D}_0$ iff the pullback of g along any $l(Y) \to A$ belongs to $\mathcal{D}_0$.

(ii) *Given $f \colon A \to l(X)$, if there exists a J-covering sieve R on X such that the pullback of f along $l(\alpha)$ is in $\mathcal{D}_0$ for each $\alpha \in R$, then $f \in \mathcal{D}_0$.*

Then we obtain a Grothendieck calibration $\mathcal{D}$ on $\mathcal{E}$ by specifying that $f \colon B \to A$ is in $\mathcal{D}$ iff its pullback along every morphism $l(X) \to A$ belongs to $\mathcal{D}_0$. Moreover, the morphisms in $\mathcal{D}$ with representable codomains are exactly the members of $\mathcal{D}_0$, so that $\mathcal{D}/l(X) \simeq \mathcal{F}_X$ for all X.

Proof. Condition (*a*) of 1.3 follows immediately from (i) (incidentally, $\mathcal{D}_0$ contains all isomorphisms with representable codomains, since q_X^* preserves the terminal object); (*b*) is immediate from the form of the definition, and the strong descent condition of 1.4(*l*) follows straightforwardly from (ii). Also, the 'naturality' condition on the q_X ensures that $\mathcal{D}_0$ is stable under pullback along morphisms $l(Y) \to l(X)$, and hence that it coincides with the restriction of $\mathcal{D}$ to representable codomains. So it remains to verify (*d*) for an arbitrary object A of $\mathcal{E}$. For this, we use the ideas of 1.4(*h*) and (*k*): we may find an epimorphism $B = \coprod_{i \in I} l(X_i) \twoheadrightarrow A$ in $\mathcal{E}$, and observe first that $\mathcal{D}/B$ is equivalent to $\prod_{i \in I} \mathcal{D}/l(X_i) \simeq \prod_{i \in I} \mathcal{F}_{X_i}$, and hence coreflective in $\mathcal{E}/B \simeq \prod_{i \in I} \mathcal{E}/l(X_i)$. Now if we form the pushout

$$\begin{tikzcd}
\mathcal{E}/B \arrow[r] \arrow[d] & \mathcal{E}/A \arrow[d] \\
\mathcal{D}/B \arrow[r] & \mathcal{P}
\end{tikzcd}$$

in BTop/**Set**, then $\mathcal{P}$ is equivalent to $\mathcal{D}/A$, since the latter is the pullback in CAT of the corresponding diagram of inverse image functors. So $\mathcal{D}/A$ is coreflective in $\mathcal{E}/A$ (and has a generating set). ■

It is easy to see that the original gros topos of 1.1 satisfies the hypotheses of 2.4, with q_X taken to be the local morphism $\mathcal{E}/l(X) \to \mathbf{Sh}(X)$; and so do its 'smooth' and 'algebraic' variants. More generally, for any site $(\mathcal{C}, J)$ as in 1.2, the hypotheses of 2.4 are satisfied. For in this case the functor $q_X^* \colon \mathbf{Sh}(|X|) \to \mathbf{Sh}(\mathcal{C}, J)/l(X)$ is determined by the fact that any basic open $|Y| \rightarrowtail |X|$ (corresponding to a morphism $Y \rightarrowtail X$ in $\mathcal{U}$ is mapped to the corresponding morphism $l(Y) \rightarrowtail l(X)$ in $\mathbf{Sh}(\mathcal{C}, J)$, and the fact that it preserves colimits: thus a morphism $A \to l(X)$ is in $\mathcal{D}_0$ iff there exists a coequalizer diagram of the form

$$\begin{tikzcd}
\coprod_j l(Z_j) \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \coprod_i l(Y_i) \arrow[r] \arrow[d] & A \arrow[dl] \\
& l(X) &
\end{tikzcd}$$

for some family of morphisms $(Y_i \to X \mid i \in I)$ in $\mathcal{U}$ and some family of commutative squares (not necessarily pullbacks)

$$\begin{tikzcd}
Z_j \arrow[r] \arrow[d] & Y_{f(j)} \arrow[d] \\
Y_{g(j)} \arrow[r] & X
\end{tikzcd}$$

with all edges in $\mathcal{U}$. Using this description, it is easy to verify that conditions (i) and (ii) of 2.4 are satisfied.

Here is an example of a rather different kind:

Example 2.5. Given a (finite or infinite) open interval $U = (a, b) \subseteq \mathbb{R}$, let U_+ denote the same set retopologized with the 'one-way' topology whose only open sets are the subintervals (c, b), $a \leq c \leq b$ (we interpret (b, b) as the empty set). Then the geometric morphism $q_U \colon \mathbf{Sh}(U) \to \mathbf{Sh}(U_+)$ induced by the identity mapping $U \to U_+$ is connected (cf. [6], p. 229). Taking the set of all open intervals, ordered by inclusion, as the underlying category of a site for the topos $\mathbf{Sh}(\mathbb{R})$, it is not hard to verify that the family of morphisms q_U satisfy the hypotheses of 2.4, and hence that we have a calibration $\mathcal{D}$ on $\mathbf{Sh}(\mathbb{R})$, for which $\mathcal{D}/1$ is equivalent to $\mathbf{Sh}(\mathbb{R}_+)$. In fact $\mathcal{D}$ is the class of maps with 'unique path-lifting towards the future', in the sense of [10], section 6.

Another important source of examples is given by the following result:

Proposition 2.6. *Let $f \colon \mathcal{F} \to \mathcal{E}$ be a geometric morphism between Grothendieck toposes, and let $\mathcal{D}$ be a Grothendieck calibration in $\mathcal{F}$. Then the class*

$$f_*(\mathcal{D}) = \{g \in \operatorname{mor} \mathcal{E} \mid f^*(g) \in \mathcal{D}\}$$

is a Grothendieck calibration in $\mathcal{E}$.

Proof. Conditions (*a*–*c*) of 1.3 all follow easily from the preservation properties of f^*. To verify (*d*), we argue as in the proof of 2.4: given an object A of $\mathcal{E}$, consider the pushout

$$\begin{tikzcd} \mathcal{F}/f^*A \arrow[r, "(f/A)"] \arrow[d] & \mathcal{E}/A \arrow[d] \\ \mathcal{D}/f^*A \arrow[r] & \mathcal{P} \end{tikzcd}$$

in $\mathsf{BTop}/\mathbf{Set}$ (where (f/A) denotes the geometric morphism whose inverse image is f^* applied to objects and morphisms of $\mathcal{E}/A$). Once again, the corresponding diagram of inverse image functors is a pullback in CAT, which enables us to identify $\mathcal{P}$ with $f_*(\mathcal{D})/A$, by an equivalence identifying the inverse image functor $\mathcal{P} \to \mathcal{E}/A$ with the inclusion. ∎

In [10], Joyal and Moerdijk showed that any class of étale maps in a Grothendieck topos $\mathcal{E}$ satisfying their 'collection axiom' may be expressed as $f_*(\mathcal{D})$ for some geometric morphism $f\colon [\mathbf{2}, \mathcal{F}] \to \mathcal{E}$, where $\mathcal{D}$ is the canonical class of 2.3. So it follows from 2.3 and 2.6 that all such classes are calibrations. There are many particular examples of interest:

Examples 2.7. (*a*) Given a small category $\mathcal{C}$ and a particular morphism $\alpha\colon U \to V$ of $\mathcal{C}$, there is a calibration $\mathcal{D}_\alpha$ in $[\mathcal{C}, \mathbf{Set}]$ consisting of those natural transformations such that the naturality square for α is a pullback: for this is exactly $f_{\alpha *}(\mathcal{D})$, where $\mathcal{D}$ is the canonical calibration in $[\mathbf{2}, \mathbf{Set}]$, and f_α is the geometric morphism induced by the functor $\mathbf{2} \to \mathcal{C}$ whose image contains the morphism α.

(*b*) For an elementary topos $\mathcal{E}$, there seems to be no reason to suppose that an intersection of calibrations in $\mathcal{E}$ is a calibration, since an intersection of coreflective subcategories need not be coreflective. (The other conditions of 1.3 are of course trivial to verify.) However, for a Grothendieck topos $\mathcal{E}$, we may use 2.6 to show that *if $\{\mathcal{D}_i \mid i \in I\}$ is any set-indexed family of Grothendieck calibrations in $\mathcal{E}$, then $\bigcap_{i \in I} \mathcal{D}_i$ is a calibration.* For we may combine the $\mathcal{D}_i$ into a single calibration $\widetilde{\mathcal{D}}$ in $\mathcal{E}/p^*I$ (where $p\colon \mathcal{E} \to \mathbf{Set}$ is the unique geometric morphism), by defining a morphism of $\mathcal{E}/p^*I$ to be in $\widetilde{\mathcal{D}}$ iff its pullback along each $i\colon 1 \to p^*I$ is in $\mathcal{D}_i$. Clearly, $\widetilde{\mathcal{D}}/(f\colon A \to p^*I)$ is the coproduct in $\mathsf{BTop}/\mathbf{Set}$ (that is, the product in CAT) of the toposes $\mathcal{D}_i/A_i$ (where A_i denotes the pullback of $A \to p^*I$ along i), and it is coreflective in $\mathcal{E}/A \simeq \coprod_{i \in I} \mathcal{E}/A_i$. Now $\bigcap_{i \in I} \mathcal{D}_i$ is simply $f_{I*}(\widetilde{\mathcal{D}})$, where f_I denotes the geometric morphism $\mathcal{E}/p^*I \to \mathcal{E}$ whose inverse image is $(p^*I)^*$.

(*c*) In particular, if $\mathcal{C}$ is a small category and $S = \{\alpha_i \mid i \in I\}$ is any set of morphisms of $\mathcal{C}$, the class $\mathcal{D}_S = \bigcap_{i \in I} \mathcal{D}_{\alpha_i}$ is a calibration in $[\mathcal{C}, \mathbf{Set}]$. As a particular case of some interest, we mention the class of idempotent-reflecting morphisms; we say a morphism $f\colon B \to A$ in $[\mathcal{C}, \mathbf{Set}]$ is *idempotent-reflecting* if, whenever $x \in B(X)$ satisfies $A(e)(f(x)) = f(x)$ for some idempotent $e\colon X \to X$ in $\mathcal{C}$, we also have $B(e)(x) = x$. For if idempotents split in $\mathcal{C}$ (which we may assume without loss of generality), this class is precisely $\mathcal{D}_S$ where S is the set of split monomorphisms in $\mathcal{C}$. Recalling that the slice category $[\mathcal{C}, \mathbf{Set}]/A$ may be identified with $[\mathcal{A}, \mathbf{Set}]$, where $\mathcal{A} \to \mathcal{C}$ is the discrete opfibration corresponding to the functor A, it is not hard to show that $\mathcal{D}_S/A$ may be identified with the topos $[\overline{\mathcal{A}}, \mathbf{Set}]$, where $\overline{\mathcal{A}}$ is the quotient of $\mathcal{A}$ by the smallest congruence which identifies all idempotents in $\mathcal{A}$ with identity morphisms — and the geometric morphism p_A may be identified with that induced by the quotient map $\mathcal{A} \to \overline{\mathcal{A}}$. In the topos $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ of simplicial sets, it would be more natural to call idempotent-reflecting morphisms *degeneracy-reflecting*: they are the morphisms such that, given a simplex $x \in B_n$ such that $f(x)$ is in the image of some degeneracy map $s_i^{n-1} \colon A_{n-1} \to A_n$, then x is in the image of $s_i^{n-1} \colon B_{n-1} \to B_n$.

(*d*) Let $\theta \colon h \to k$ be a geometric transformation between geometric morphisms $h, k \colon \mathcal{F} \rightrightarrows \mathcal{E}$ (that is, a natural transformation between their inverse image functors), where $\mathcal{E}$ and $\mathcal{F}$ are Grothendieck toposes. Then the class $\mathcal{D}_\theta$ of morphisms $f \colon A \to B$ in $\mathcal{E}$ for which the naturality square

$$\begin{tikzcd}
h^*(A) \arrow[r, "\theta_A"] \arrow[d, "h^*(f)"] & k^*(A) \arrow[d, "k^*(f)"] \\
h^*(B) \arrow[r, "\theta_B"] & k^*(B)
\end{tikzcd}$$

is a pullback is a calibration: for we may identify it as $\widehat{\theta}_*(\mathcal{D})$, where $\mathcal{D}$ is the canonical calibration in $[\mathbf{2}, \mathcal{F}]$ and $\widehat{\theta} \colon [\mathbf{2}, \mathcal{F}] \to \mathcal{E}$ is the geometric morphism corresponding to the 2-cell θ (recall that $[\mathbf{2}, \mathcal{F}]$ is the tensor of $\mathcal{F}$ with $\mathbf{2}$ in the 2-category Top).

(*e*) In particular, if $p \colon \mathcal{E} \to \mathcal{S}$ is a local geometric morphism between Grothendieck toposes, then both p^* and p_* are inverse image functors, so we may apply (*d*) to the counit $\epsilon \colon p^* p_* \to 1_{\mathcal{E}}$. We may thus conclude, as promised after 2.1, that the class $\mathcal{D}_p$ is a calibration for any local morphism p.

(*f*) Similarly, let $g \colon T \to U$ be a morphism between tiny objects of a Grothendieck topos $\mathcal{E}$ (recall that an object T is said to be *tiny* if $(-)^T$ has a right adjoint). Then since both $(-)^T$ and $(-)^U$ are inverse image functors $\mathcal{E} \to \mathcal{E}$ and $(-)^g$ is a natural transformation between them, we see that the class of morphisms $f \colon B \to A$ which are *internally right orthogonal* to g, in the sense that the square

$$\begin{tikzcd}
B^U \arrow[r, "B^g"] \arrow[d, "f^U"] & B^T \arrow[d, "f^T"'] \\
A^U \arrow[r, "A^g"] & A^T
\end{tikzcd}$$

is a pullback, is a calibration. And using the ideas of (*b*) above, we may replace the single morphism g by a set-indexed family $\{g_i \colon T_i \to U_i \mid i \in I\}$ of morphisms between tiny objects. For example, in any model of synthetic differential geometry in which the spectra of Weil algebras are tiny, the class of *formally étale maps* in the sense of [11], section I 17 (that is, the morphisms internally right orthogonal to $0\colon 1 \to D_W$ for all Weil algebras W) is a calibration.

3 Properties of Calibrations

Given a property P of geometric morphisms (e.g. local connectedness, hyperconnectedness), we shall say that a calibration $\mathcal{D}$ has the property P if all the geometric morphisms p_A do so. For example, since the counits $B' \to B$ of the coreflections are monomorphisms, the calibration of 2.1 is hyperconnected. More generally, we have:

Lemma 3.1. *Let $\mathcal{D}$ be a calibration in $\mathcal{E}$. The following are equivalent:*

(i) *$\mathcal{D}$ is hyperconnected.*

(ii) *$\mathcal{D}$ contains all monomorphisms of $\mathcal{E}$.*

(iii) *Given a commutative triangle*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[dr, "h"'] & B \arrow[d, "f"] \\ & A \end{tikzcd}$$

with h in $\mathcal{D}$ and g epic, we have $f \in \mathcal{D}$.

(iv) *$\mathcal{D}$ is a class of open maps in the sense of* [10].

Proof. (ii) and (iii) respectively say that, for any A, $\mathcal{D}/A$ is closed under arbitrary subobjects (resp. arbitrary quotients) in $\mathcal{E}/A$. But these are exactly conditions (ii) and (iii) of [7], A4.6.6, for hyperconnectedness of p_A. Finally, given that (as we saw in 1.5) any calibration is a class of étale maps in the sense of [10], condition (iii) is the only part of the Joyal–Moerdijk characterization of classes of open maps which it lacks in general. ∎

We remark in passing that if a class of étale maps in the sense of [10] is also a class of open maps, then it is easy to verify the solution-set condition for the existence of a right adjoint to the inclusion $\mathcal{D}/A \to \mathcal{E}/A$: the coreflection of $B \to A$ is its largest subobject $B' \rightarrowtail B \to A$ which belongs to $\mathcal{D}$ (equivalently, the union of all such subobjects). So all such classes are (hyperconnected) calibrations.

The calibration of idempotent-reflecting maps considered in 2.7(c) is clearly hyperconnected; and the calibration $\mathcal{D}_\theta$ of 2.7(d) is hyperconnected iff θ is pointwise monic, by (ii) above and [7], A1.6.9. Note also that an intersection of hyperconnected calibrations is hyperconnected, and more generally that a calibration of the form $f_*(\mathcal{D})$ is hyperconnected provided $\mathcal{D}$ is; both of these follow easily from 3.1(ii). However, not all calibrations of interest are hyperconnected: for example, that induced by a site $(\mathcal{C}, J)$ with a distinguished class $\mathcal{U}$ as in 1.2 can only be hyperconnected if all morphisms $1 \to X$ in $\mathcal{C}$ belong to $\mathcal{U}$, i.e. if all the spaces $|X|$ are discrete. Again, in the canonical calibration of 2.3, and the calibration of 2.5, the morphisms $p_A\colon \mathcal{E}/A \to \mathcal{D}/A$ are all localic.

Regarding local connectedness, we have

Lemma 3.2. *Let $\mathcal{D}$ be a calibration in $\mathcal{E}$. The following are equivalent:*

(i) *$\mathcal{D}$ is locally connected*

(ii) *$p_A^*\colon \mathcal{D}/A \to \mathcal{E}/A$ has a left adjoint $p_{A!}$ for each object A of $\mathcal{E}$.*

(iii) *$\mathcal{D}$ is closed under fibrewise exponentiation, i.e. if $f\colon B \to A$ and $g\colon C \to A$ are in $\mathcal{D}$, then so is the exponential f^g in $\mathcal{E}/A$.*

(iv) *(if $\mathcal{E}$ and the $\mathcal{D}/A$ are all bounded over some topos $\mathcal{S}$) $\mathcal{D}/A$ is closed in $\mathcal{E}/A$ under $\mathcal{S}$-indexed products, for all A.*

Proof. (i)⇒(ii) is trivial. For (ii)⇒(i), we have to show that the left adjoints form a $\mathcal{D}/A$-indexed functor, for each A. But the observation in 1.4(g) that the 'naturality square' for $A \mapsto p_A$ is a pullback whenever $(f\colon B \to A) \in \mathcal{D}$ easily yields the Beck–Chevalley condition $p_A^*\Pi_f \cong \Pi_f p_B^*\colon \mathcal{D}/B \to \mathcal{E}/A$; taking left adjoints, we obtain $p_{B!}f^* \cong f^* p_{A!}$.

(i)⇔(iii) follows from the characterization of locally connected morphisms in [7], C3.3.1, again using the fact that the naturality squares are pullbacks when $f \in \mathcal{D}$.

(ii)⇔(iv) follows from the $\mathcal{S}$-indexed adjoint functor theorem ([7], B2.4.6). ∎

We note that condition (iv) holds for the calibration of 2.1 if p is locally connected, since then p^* and p_* both preserve arbitrary $\mathcal{S}$-indexed limits. Even if p is bounded, it does not seem to be automatically the case that the toposes $\mathcal{D}_p/A$ are all bounded over $\mathcal{D}_p/1 \simeq \mathcal{S}$; but when $\mathcal{S} = \mathbf{Set}$ we may deduce this by the argument of 2.7(e).

Once again, it follows from 3.2(iii) that an intersection of locally connected calibrations is locally connected. And the same holds for stably locally connected calibrations, which we may easily characterize by combining 3.2(iii) with a result from [8]:

Corollary 3.3. *A calibration $\mathcal{D}$ is stably locally connected iff $\mathcal{D}/A$ is an exponential ideal in $\mathcal{E}/A$ for each object A.*

Proof. The condition is necessary, by [8], 2.6(ii); but it is sufficient by 3.2(iii). ∎

The canonical calibration of 2.3 is in fact totally connected in the sense of [3], since the adjunction $(l \dashv i)$ in $\mathsf{Cat}(\mathcal{E})$ gives rise to an adjunction $(i^* \dashv l^*)$. So the geometric morphism induced by i is right adjoint to that induced by l in Top. Combined with the remark before 2.7, this makes it easy to give examples showing that local connectedness (unlike hyperconnectedness) is not preserved under the passage from $\mathcal{D}$ to $f_*(\mathcal{D})$ as in 2.6.

For the property of being local, things are much more delicate. We have the following characterization of local calibrations:

Lemma 3.4. *For a Grothendieck calibration $\mathcal{D}$ in $\mathcal{E}$, the following are equivalent:*

(i) *$\mathcal{D}$ is local.*

(ii) *p_{A*} preserves epimorphisms for every A.*

(iii) *If $f\colon B \twoheadrightarrow A$ is an epimorphism in $\mathcal{E}$, then $p_{A*}(f)$ is an epimorphism.*

(iv) *Given an epimorphism $f\colon B \twoheadrightarrow A$ in $\mathcal{E}$, there exists an epimorphism $g\colon C \twoheadrightarrow A$ which factors through f and belongs to $\mathcal{D}$.*

Proof. We recall from [8] that a connected morphism between Grothendieck toposes is local iff its direct image preserves epimorphisms, which yields (i) $\Leftrightarrow$ (ii). Now (ii) $\Rightarrow$ (iii) by considering f as an epimorphism $f \to 1_A$ in $\mathcal{E}/A$; and (iii) $\Leftrightarrow$ (iv) is trivial. For (iii) $\Rightarrow$ (ii), suppose given $C \twoheadrightarrow B \to A$ in $\mathcal{E}$. Form the diagram

$$\begin{tikzcd}
D' \arrow[r] \arrow[dr] & D \arrow[r] \arrow[d, two heads] & C \arrow[d, two heads] \\
& B' \arrow[r] \arrow[dr] & B \arrow[d] \\
& & A
\end{tikzcd}$$

where $B' \to A$ is the coreflection of $B \to A$ in $\mathcal{D}/A$, the square is a pullback and $D' \to B'$ is the coreflection of $D \to B'$ in $\mathcal{D}/B'$. Then by 1.3(a) the composite $D' \to B' \to A$ is in $\mathcal{D}$, and it is easily verified that it is the coreflection of $C \to A$ in $\mathcal{D}/A$. But $D' \to B'$ is epic by (iii). ∎

It follows easily from 3.4(iv) that a local calibration satisfies the 'collection axiom' of [10]. However, actual examples of local calibrations seem rather hard to come by.

Examples 3.5. (a) Let $\mathcal{E} = [\mathbf{\Delta}_1^{\mathrm{op}}, \mathbf{Set}]$ be the category of reflexive graphs. In 2.2(d) we identified the calibration $\mathcal{D}_p$ in $\mathcal{E}$ as the class of morphisms $f\colon B \to A$ which reflect identities. The coreflection of an arbitrary f may be obtained simply by deleting those non-identity arrows of B which map to identities in A; and it is clear that if f is surjective then so is its coreflection, since the image of any of the deleted arrows is also the image of an identity arrow of B. So $\mathcal{D}_p$ is a local calibration; indeed, it is punctually locally connected.

We note, however, that the simple description of the coreflection, and consequently the proof that it preserves epimorphisms, depends on the law of excluded middle in $\mathbf{Set}$: the same argument will not work for the topos of internal reflexive graphs in a non-Boolean topos. Moreover, the argument does not extend to the category $[\mathbf{\Delta}_2^{\mathrm{op}}, \mathbf{Set}]$ of 2-dimensional simplicial sets (let alone to $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$). For let B be the 2-dimensional simplicial set represented by the object 2 of $\mathbf{\Delta}_2$, and let A be the quotient of B by the relation which identifies two vertices of the 2-simplex, and identifies the 1-simplex joining them with the degenerate 1-simplex on that vertex. In order to obtain the coreflection of the quotient map $q\colon B \twoheadrightarrow A$ in $\mathcal{D}_p/A$, we have to delete the non-degenerate 2-simplex of B as well as one of its 1-dimensional faces; so the result is not surjective.

(*b*) The canonical calibration of 2.3 is not local, since there is no epimorphism in $\mathcal{D}$ factoring through the obvious epimorphism $(2 \to 2) \twoheadrightarrow (2 \to 1)$ in $[\mathbf{2}, \mathcal{E}]$. However, it does satisfy the collection axiom, as was shown in [10]. And, as we observed already in 2.3, we may characterize those objects $\mathbf{A} = (A_0 \to A_1)$ of $[\mathbf{2}, \mathcal{E}]$ for which $p_{\mathbf{A}}$ is local: they are exactly the complemented monomorphisms in $\mathcal{E}$.

(*c*) The calibration of degeneracy-reflecting maps in $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$, mentioned in 2.7(*c*), is not local, but we may again characterize those objects A of $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ for which $p_A\colon [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]/A \to \mathcal{D}/A$ is local: *p_A is local iff all faces of non-degenerate simplices of A are non-degenerate*. For if A contains a non-degenerate simplex (of dimension n, say) one of whose faces is degenerate, then the epic part of the image factorization of the corresponding morphism $\Delta^n \to A$ is not preserved by the coreflection $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]/A \to \mathcal{D}/A$ (cf. (*a*) above). But if A has the given property, then the coreflection has a simpler description, similar to that in (*a*): the coreflection of $(B \to A)$ is the sub-simplicial set B' of B generated by those (necessarily non-degenerate) simplices which map to non-degenerate simplices of A. From this description, it is easily seen that it does preserve epimorphisms. We may give an explicit description of the right adjoint $p_A^\#$ of p_{A*} in this case: given $f\colon B \to A$ in $\mathcal{D}/A$, an n-simplex of $p_A^\# B$ is given by a pair (g, h), where $g\colon \Delta^n \to A$ is an arbitrary n-simplex of A and $h\colon p_{A*}g \to f$ in $\mathcal{D}/A$. In fact we also have a simpler description of $\mathcal{D}/A$ in this case: it may be identified with the functor category $[\mathcal{A}_n^{\mathrm{op}}, \mathbf{Set}]$, where $\mathcal{A} \to \mathbf{\Delta}$ denotes the discrete fibration corresponding to A, and $\mathcal{A}_n$ is the full subcategory of $\mathcal{A}$ on the non-degenerate simplices. The condition on A is equivalent to saying that the inclusion $i\colon \mathcal{A}_n \to \mathcal{A}$ has a left adjoint l, which sends an arbitrary simplex of A to the unique non-degenerate simplex of which it is a degeneration (equivalently, its largest non-degenerate face); and the geometric morphism $p_A\colon [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]/A \simeq [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{D}/A$ may be identified with that induced by l (and its left adjoint in Top is the morphism induced by i).

In contrast to the other properties we have considered, there does not seem to be any reason why an intersection of local calibrations should be local.

For calibrations obtained by the method of 2.4, we have the following criterion for localness:

Lemma 3.6. *With the notation of* 2.4, *suppose that the morphisms q_X are all local, and additionally that the commutative squares in the statement of* 2.4 *satisfy the Beck–Chevalley condition that*

$$\begin{tikzcd}
\mathcal{E}/l(X) \arrow[r, "l(\alpha)^*"] \arrow[d, "q_{X*}"] & \mathcal{E}/l(Y) \arrow[d, "q_{Y*}"] \\
\mathcal{F}_X \arrow[r, "\widehat{\alpha}^*"] & \mathcal{F}_Y
\end{tikzcd}$$

commutes up to isomorphism. Then the induced calibration in $\mathcal{E}$ is local.

Proof. From the proof of 2.4, it is clear that p_A is local whenever A is a coproduct of representables, and that the Beck–Chevalley condition holds for morphisms $\coprod_{j\in J} l(Y_j) \to \coprod_{i\in I} l(X_i)$ which are induced by J-indexed families of the form $(l(Y_j) \to l(X_{f(j)}) \to \coprod_{i\in I} l(X_i) \mid j \in J)$. Now, given any object A of $\mathcal{E}$, we may find a coequalizer diagram

$$\begin{tikzcd} C \arrow[r, shift left] \arrow[r, shift right] & B \arrow[r] & A \end{tikzcd}$$

where B and C are coproducts of representables, and the morphisms between them are of the form described. Consider the diagram

$$\begin{tikzcd} \mathcal{E}/C \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left, "p_C"] & \mathcal{E}/B \arrow[r] \arrow[d, shift left, "p_B"] & \mathcal{E}/A \arrow[d, "p_A"] \\ \mathcal{D}/C \arrow[r, shift left] \arrow[r, shift right] \arrow[u, shift left, "c_C"] & \mathcal{D}/B \arrow[r] \arrow[u, shift left, "c_B"] & \mathcal{D}/A \end{tikzcd}$$

where c_B and c_C denote the centres of p_B and p_C (i.e. their left adjoints in Top). Since epimorphisms in a topos are descent morphisms, the top row is a coequalizer in BTop/**Set**; and since the right-hand square is a pushout, the bottom row is also a coequalizer. But the Beck–Chevalley condition says that the two 'upward' squares on the left commute up to isomorphism; so we may obtain a left adjoint c_A for p_A by factoring $\mathcal{D}/B \to \mathcal{E}/B \to \mathcal{E}/A$ through $\mathcal{D}/B \to \mathcal{D}/A$. ∎

It seems likely that the Beck–Chevalley condition of 3.6 is necessary as well as sufficient for the corresponding calibration to be local, but we have not been able to prove this. (It is worth noting that in the local calibration of 3.5(a), the condition holds for the squares corresponding to arbitrary morphisms of $[\mathbf{\Delta}_1^{\mathrm{op}}, \mathbf{Set}]$, as may easily be verified from the explicit description of the coreflection which we gave.) Unfortunately, however, this condition seems to be satisfied in very few other cases.

Lemma 3.7. *Let $(\mathcal{C}, J)$ be a site as in Lemma* 1.2, *and suppose that the spaces $|X|$, $X \in$* ob $\mathcal{C}$, *are not all discrete. Then*

(i) *The Beck–Chevalley condition of* 3.6 *fails for some morphisms of $\mathcal{C}$.*

(ii) *The calibration in $\mathbf{Sh}(\mathcal{C}, J)$ defined in* 2.4 *is not local.*

Proof. (i) Let $\alpha\colon 1 \to X$ be a non-open point of $|X|$ for some X. Then the Beck–Chevalley condition fails for the square induced by α; for if we consider $l(\alpha)\colon 1 \to l(X)$ as an object of $\mathcal{E}/l(X)$ and chase it around the top and right edges of the square we obtain the terminal object 1 of $\mathbf{Sh}(1) \cong \mathbf{Set}$, but if we chase it around the left and bottom edges we obtain 0.

(ii) Let α be as in (i), and let $p_1, p_2\colon l(X) \rightrightarrows A$ be the cokernel-pair of $l(\alpha)$ in $\mathbf{Sh}(\mathcal{C}, J)$. We have an epimorphism $l(X) \amalg l(X) \twoheadrightarrow A$ in $\mathbf{Sh}(\mathcal{C}, J)$; we shall show that its coreflection ($B \to A$, say) in $\mathcal{D}/A$ is not epimorphic, by showing that the point $x = p_1 l(\alpha) = p_2 l(\alpha)$ of A does not factor through it. (Note that 1 is an indecomposable projective in $\mathbf{Sh}(\mathcal{C}, J)$, since — as we remarked after 1.2 — the topos is hyperconnected and local over **Set**.) Clearly x has just two factorizations through $l(X) \amalg l(X) \twoheadrightarrow A$, so we need to show that neither of these factors through the counit $B \to l(X) \amalg l(X)$. Now if we form the pullback squares

$$\begin{tikzcd}
B_1 \arrow[r] \arrow[d] & B \arrow[d] \\
l(X) \amalg 1 \arrow[r, "1 \amalg l(\alpha)"] \arrow[d, "(1, l(\alpha))"] & l(X) \amalg l(X) \arrow[d] \\
l(X) \arrow[r, "p_1"] & A
\end{tikzcd}$$

the left vertical composite is in $\mathcal{D}_0$, so $B_1 \to l(X) \amalg 1$ must factor through the coreflection of its codomain in $\mathcal{D}_0/l(X)$ — i.e., by the argument of (i), through the coprojection $l(X) \rightarrowtail l(X) \amalg 1$. this shows that the second of the two points of $l(X) \amalg l(X)$ cannot factor through B; and a similar argument using pullback along p_2 shows that the first one also cannot do so. ■

Thus, at present, our only nontrivial example of a local calibration is provided by the identity-reflecting maps of reflexive graphs, as in 3.5(*a*). Since one advantage of local morphisms is that, being adjoint pairs of geometric morphisms, they are 'homotopy equivalences' in Top — in particular, they induce isomorpisms between the cohomology of the gros topos $\mathcal{E}/A$ and that of the petit topos $\mathcal{D}/A$ — this is perhaps a disappointment. However, a more positive conclusion to draw would be that, just as we do not expect *all* the objects in a model of synthetic differential geometry to be 'synthetic manifolds' (for example, not all objects are infinitesimally linear), so we should not expect all the objects of a gros topos to be 'sufficiently spacelike' to be represented up to homotopy equivalence by their petit toposes. Rather, we should perhaps be aiming to characterize those which do have this property (as we have done in a couple of cases above) and to investigate the closure properties of this class of 'spacelike' objects. (For example, it seems reasonable to hope that the 'spacelike' objects might be closed under finite limits, or at least under finite products, and under coproducts; but the proof of 3.7(ii) shows that we cannot expect them to be closed under coequalizers.) We leave the detailed consideration of such questions to a subsequent paper.

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