A categorical approach to loops, neardomains and nearfields

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Abstract

In this paper we study loops, neardomains and nearfields from a categorical point of view. By choosing the right kind of morphisms, we can show that the category of neardomains is equivalent to the category of sharply 2-transitive groups. The other categories are also shown to be equivalent with categories whose objects are sets of permutations with suitable extra properties.

Up to now the equivalence between neardomains and sharply 2-transitive groups was only known when both categories were equipped with the obvious isomorphisms as morphisms. We thank Hubert Kiechle for this observation [6].

1 Introduction

Loops and nearfields are structures in algebra which generalize groups and fields, respectively. The first examples of finite proper nearfields were constructed by L.E. Dickson in 1905. Thirty years later the finite nearfields were completely classified by H. Zassenhaus. In 1965 H. Karzel introduced neardomains (which are a weakening of nearfields) in such a way that there is a one-to-one correspondence with sharply 2-transitive groups (see [3]). At that time morphisms were not considered. The still unsolved problem is whether there exist proper neardomains, i.e. neardomains which are not nearfields.

The link between loops and regular (i.e. sharply 1-transitive) permutation sets is of a similar but simpler kind.

The present paper describes these links in a uniform way. By considering the right kind of morphisms we show that loops, resp. neardomains are categories equivalent to the categories of regular permutation sets resp. sharply 2-transitive groups. The latter equivalence nicely restricts to an equivalence between the category of nearfields and the category of sharply 2-transitive groups with the property that the translations form a subgroup.

In [3] the correspondence between sharply 2-transitive groups and neardomains is described. We define morphisms which turn this correspondence into an equivalence of categories. Kiechle [6] was also aware of an equivalence of categories but with a more restricted class of morphisms, namely isomorphisms.

2   Loops and regular permutation sets

2.1   Loops

A **loop** is a set L, together with a binary operation $(k,l) \mapsto kl$ with identity satisfying the *left and right loop property*. This means that for every $a,b \in L$ there exist unique elements $x,y \in L$ such that $ax = b$ and $ya = b$.

A **morphism of loops** $(L,.) \to (L',*)$ is a map $f\colon L \to L'$ preserving the operations. This means that $f(a.b) = f(a) * f(b)$ for all $a,b \in L$. It follows that f maps the identity of L onto the identity of L'.

We denote by $\mathscr{Loop}$ the category of all loops together with all morphisms of loops.

2.2   Regular permutation sets

Let Ω be a set and let $\mathrm{Sym}(\Omega)$ denote the set of all permutations of Ω. A **regular permutation set (r.p.s.)** on Ω is a subset M of $\mathrm{Sym}(\Omega)$ such that the identity permutation 1_Ω is in M and M acts regularly on Ω, i.e. $\forall \alpha, \beta \in \Omega\colon \exists! \; m \in M\colon m(\alpha) = \beta$.

We construct a category $\mathscr{rps}$ whose objects are triples (M, Ω, ω), where M is an r.p.s. on Ω and $\omega \in \Omega$ is a **base point**.

A **morphism of r.p.s.** $(M, \Omega, \omega) \to (N, \Sigma, \sigma)$ is a pair (f, Φ) such that $f\colon M \to N$ and $\Phi\colon \Omega \to \Sigma$ are maps satisfying $\Phi(\omega) = \sigma$ and $\forall m \in M$, $\forall \alpha \in \Omega\colon \Phi(m(\alpha)) = (f(m))(\Phi(\alpha))$. The latter property can be summarized by the following commutative diagram in which the horizontal maps are the (left) actions of M (resp. N) on the set Ω (resp. Σ).

$$
\begin{array}{ccc}
M \times \Omega & \longrightarrow & \Omega \\
{\scriptstyle f \times \Phi}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\
N \times \Sigma & \longrightarrow & \Sigma
\end{array}
$$

Composition of morphisms $(f, \Phi) \colon (M, \Omega, \omega) \to (N, \Sigma, \sigma)$ and $(g, \Psi) \colon (N, \Sigma, \sigma) \to (P, \Gamma, \tau)$ is defined by $(g \circ f, \Psi \circ \Phi)$. The identity $1_{(M,\Omega,\omega)}$ is defined as $(1_M, 1_\Omega)$.

One easily verifies that $\mathcal{rps}$ is a category.

Notice that, by regularity, a morphism $(f, \Phi) \colon (M, \Omega, \omega) \to (N, \Sigma, \sigma)$ satisfies $f(1_\Omega) = 1_\Sigma$.

2.3 Equivalence of the categories $\mathcal{Loop}$ and $\mathcal{rps}$

The one-to-one correspondence between loops and regular permutation sets is folklore (see for instance [2]). However we think it is useful to describe an explicit categorical equivalence.

Let (M, Ω, ω) be an object of $\mathcal{rps}$. By regularity, the map $\mu \colon M \to \Omega$: $m \mapsto m(\omega)$ is a bijection such that $\mu(1_\Omega) = \omega$. Now define an operation

$$\begin{aligned} \otimes_\omega \colon \quad M \times M \quad &\to \quad M \\ (m, n) \quad &\mapsto \quad m \otimes_\omega n := \mu^{-1}((m \circ n)(\omega)) \end{aligned}$$

Notice that $m \otimes_\omega n$ is equivalently defined by $(m \otimes_\omega n)(\omega) = (m \circ n)(\omega)$ (compare [2, p. 618]).

It is easy to check that

PROPERTY 2.1. *The pair $(M, \otimes_\omega)$ is a loop with identity 1_Ω.*

We can also construct a loop structure on the set Ω. Let (M, Ω, ω) be an object of $\mathcal{rps}$. We still have the bijection $\mu \colon M \to \Omega$. We define the operation

$$\begin{aligned} \cdot_\omega \colon \quad \Omega \times \Omega \quad &\to \quad \Omega \\ (\alpha, \beta) \quad &\mapsto \quad \alpha \cdot_\omega \beta := (\mu^{-1}(\alpha) \otimes_\omega \mu^{-1}(\beta))(\omega) = (\mu^{-1}(\alpha) \circ \mu^{-1}(\beta))(\omega) \end{aligned}$$

One easily verifies the following

PROPERTY 2.2. *The pair $(\Omega, \cdot_\omega)$ is a loop with identity ω, and $\mu \colon (M, \otimes_\omega) \to (\Omega, \cdot_\omega)$ is a loop isomorphism.*

The following property gives a useful characterization of morphisms of regular permutation sets.

PROPERTY 2.3. *Let (M, Ω, ω), (N, Σ, σ) be objects of $\mathcal{rps}$, $f \colon M \to N$ and $\Phi \colon \Omega \to \Sigma$ maps with $\Phi(\omega) = \sigma$. Then (f, Φ) is a morphism $(M, \Omega, \omega) \to (N, \Sigma, \sigma)$ if and only if the following conditions are both satisfied.*

1. $\forall m \in M \colon f(m)(\sigma) = \Phi(m(\omega))$

2. $f \colon (M, \otimes_\omega) \to (N, \otimes_\sigma)$ is a morphism of loops.

Proof. Let $(f, \Phi)\colon (M, \Omega, \omega) \to (N, \Sigma, \sigma)$ be a morphism, then (1) follows immediately, since $\forall m \in M\colon f(m)(\sigma) = f(m)(\Phi(\omega)) = \Phi(m(\omega))$.

In order to prove (2) it suffices, by the regular action of N on Σ, to establish that $\forall m_1, m_2 \in M\colon (f(m_1 \otimes_\omega m_2))(\sigma) = (f(m_1) \otimes_\sigma f(m_2))(\sigma)$. The left hand side equals

$$\Phi((m_1 \otimes_\omega m_2)(\omega)) = \Phi((m_1 \circ m_2)(\omega)) = \Phi(m_1(m_2(\omega))) =$$
$$(f(m_1))(\Phi(m_2(\omega))) = (f(m_1))((f(m_2))(\Phi(\omega))) =$$
$$(f(m_1) \circ f(m_2))(\sigma) = (f(m_1) \otimes_\sigma f(m_2))(\sigma)$$

Conversely (1) implies that the maps $f\colon M \to N$, $\Phi\colon \Omega \to \Sigma$ with $\Phi(\omega) = \sigma$ satisfy $f(m)(\Phi(\alpha)) = \Phi(m(\alpha))$ provided $m \in M$, and for $\alpha = \omega$. We have to show that this condition holds for all $\alpha \in \Omega$. For any such α there exists a unique $m' \in M$ such that $m'(\omega) = \alpha$ (by the regular action of M on Ω). Then it follows that for all $m \in M$ and all $\alpha \in \Omega$ we have

$$f(m)(\Phi(\alpha)) = f(m)(\Phi(m'(\omega))) \overset{(1)}{=} f(m)(f(m')(\sigma)) =$$
$$(f(m) \circ f(m'))(\sigma) = (f(m) \otimes_\sigma f(m'))(\sigma) \overset{(2)}{=} (f(m \otimes_\omega m'))(\sigma) \overset{(1)}{=}$$
$$\Phi((m \otimes_\omega m')(\omega)) = \Phi((m \circ m')(\omega)) = \Phi(m(\alpha)). \quad \blacksquare$$

COROLLARY 2.4. *Let* $(f, \Phi) \in rps((M, \Omega, \omega), (N, \Sigma, \sigma))$. *Then* $\Phi\colon (\Omega, \cdot_\omega) \to (\Sigma, \cdot_\sigma)$ *is a loop homomorphism.*

The proof easily follows from the definition of the operations $\cdot_\omega$ and $\cdot_\sigma$ and Properties 2.2 and 2.3.

Now let $(\Omega, \cdot)$ be a loop. For $\alpha \in \Omega$ we define $\lambda_\alpha\colon \Omega \to \Omega\colon \gamma \mapsto \alpha \cdot \gamma$.

PROPERTY 2.5. *For a loop* $(\Omega, \cdot)$ *with identity* ω *we write* $L = \{\lambda_\alpha \mid \alpha \in \Omega\}$, *the set of left translations of* Ω. *The triple* (L, Ω, ω) *is an object of r.p.s..*

Proof. Since $(\Omega, \cdot)$ is a (left) loop, L is a subset of $\mathrm{Sym}(\Omega)$. Moreover L acts regularly on Ω by the (right) loop property of Ω. Also $1_\Omega = \lambda_\omega \in L$. Hence (L, Ω, ω) is an object of r.p.s.. $\quad \blacksquare$

We now have a correspondence between the objects of rps and $Loop$ which can be extended to an equivalence of categories.

THEOREM 2.6.

$$
\begin{array}{ccc}
F\colon & rps & \longrightarrow & Loop \\
& (M, \Omega, \omega) & \longmapsto & (\Omega, \cdot_\omega) \\
& | & & | \\
& (f, \Phi) & \mapsto & \Phi \\
& \downarrow & & \downarrow \\
& (N, \Sigma, \sigma) & \longmapsto & (\Sigma, \cdot_\sigma)
\end{array}
$$

is an equivalence of categories.

Proof. The functorial properties of F are easily checked. We use [1, prop. 3.4.3(4)] to show that F is an equivalence of categories.

F is faithful when for every pair of objects (M, Ω, ω), (N, Σ, σ) of rps the induced map

$F_{(M,\Omega,\omega),(N,\Sigma,\sigma)} \colon \text{rps}((M, \Omega, \omega), (N, \Sigma, \sigma)) \to \mathcal{L}oop((\Omega, \cdot_\omega), (\Sigma, \cdot_\sigma))$ is injective. Let $(f, \Phi), (f', \Phi') \in \text{rps}((M, \Omega, \omega), (N, \Sigma, \sigma))$ be such that $F(f, \Phi) = F(f', \Phi')$. Then $\Phi = \Phi'$ follows immediately.

Hence, for all $(m, \alpha) \in M \times \Omega$, we have that $(f(m))(\Phi(\alpha)) = \Phi(m(\alpha)) = \Phi'(m(\alpha)) = (f'(m))(\Phi'(\alpha))$. In particular, for $\alpha = \omega$, it follows that $\nu(f(m)) = (f(m))(\sigma) = (f(m))(\Phi(\omega)) = (f'(m))(\Phi'(\omega)) = (f'(m))(\sigma) = \nu(f'(m))$ (where $\nu \colon N \to \Sigma \colon n \mapsto n(\sigma)$ is the bijection analogous to μ). Hence $f(m) = f'(m)$ for all $m \in M$ and thus $f = f'$.

F is full when all induced maps $F_{(M,\Omega,\omega),(N,\Sigma,\sigma)}$ (as above) are surjective. Let $\Phi \in \mathcal{L}oop((\Omega, \cdot_\omega), (\Sigma, \cdot_\sigma))$, then $\Phi(\omega) = \sigma$ (since ω and σ are identities in the respective loops). Define $f_\Phi \colon M \to N \colon m \mapsto (\nu^{-1} \circ \Phi \circ \mu)(m)$ (with μ and ν as above). One uses Property 2.3 and the regularity of the action of N on Σ to show that $(f_\Phi, \Phi) \in \text{rps}((M, \Omega, \omega), (N, \Sigma, \sigma))$. Clearly $F(f_\Phi, \Phi) = \Phi$.

We now take a loop $(\Omega, \cdot)$ with identity ω with its left translations as defined above. We then know (Prop. 2.5) that (L, Ω, ω) is an object of r.p.s.. For $\alpha, \beta \in \Omega$ we have that $(\lambda_\alpha \otimes_\omega \lambda_\beta)(\omega) = (\lambda_\alpha \circ \lambda_\beta)(\omega) = \alpha \cdot (\beta \cdot \omega) = \alpha \cdot \beta = (\alpha \cdot \beta) \cdot \omega = \lambda_{\alpha \cdot \beta}(\omega)$. Again by regularity, it follows that $\lambda_\alpha \otimes_\omega \lambda_\beta = \lambda_{\alpha \cdot \beta}$. Hence $F(L, \Omega, \omega) = (\Omega, \cdot_\omega)$ where $\alpha \cdot_\omega \beta = (\mu^{-1}(\alpha) \circ \mu^{-1}(\beta))(\omega) = (\lambda_\alpha \circ \lambda_\beta)(\omega) = \alpha \cdot \beta$, which shows that $(\Omega, \cdot) = F(L, \Omega, \omega)$. So F turns out to be even strictly surjective on objects. ∎

3 Neardomains, nearfields and sharply 2-transitive groups

3.1 Neardomains and nearfields

A triple $(F, +, .)$ is said to be a **neardomain** if

1. $(F, +)$ is a loop with neutral element 0;

2. $\forall a, b \in F \colon a + b = 0 \Rightarrow b + a = 0$;

3. $(F \setminus \{0\}, .)$ is a group (with neutral element 1);

4. $\forall a \in F \colon 0.a = 0$;

5. $\forall a, b, c \in F \colon a.(b + c) = a.b + a.c$;

6. $\forall a, b \in F \colon \exists d_{a,b} \in F \setminus \{0\} \colon (\forall x \in F \colon a + (b + x) = (a + b) + d_{a,b}.x)$.

Notice that $(F, +, .)$ is a nearfield if and only if all $d_{a,b}$ are 1. In that case $(F, +)$ is a group. Also notice that 1 and 5 imply that $\forall a \in F \colon a.0 = 0$.

THEOREM 3.1 (see [3], [4] or [5]). *A finite neardomain is a nearfield.*

It is still an open problem whether there exists a (necessarily infinite) neardomain which is not a nearfield.

We define the category $n\text{-}\mathcal{D}om$ of neardomains where objects are neardomains and morphisms are maps preserving both operations.

PROPERTY 3.2. *Neardomain morphisms are injective.*

Proof. Let $f\colon (F,+,.) \longrightarrow (F',+',.')$ be a morphism of neardomains. Suppose $f(x) = f(y)$ for some $x, y \in F$. Since $(F,+,.)$ is a neardomain we have by conditions 1, 2 a unique additive inverse $-y$ of y. Then,

$$f(x + (-y)) = f(x) +' f(-y) = f(y) +' f(-y) = f(y + (-y)) = f(0) = 0'.$$

If $x + (-y) \neq 0$ this element has an inverse z in $(F \setminus \{0\},.)$. Hence we get

$$1' = f(1) = f((x + (-y)).z) = f(x + (-y)).' f(z) = 0'.' f(z) = 0',$$

a contradiction. ∎

THEOREM 3.3. *Let $(F,+,.)$ be a neardomain and let*

$$T_2(F) = \{\tau_{a,b}\colon F \to F\colon x \mapsto a + bx \mid a \in F, b \in F \setminus \{0\}\}$$

Then $(T_2(F),\circ)$ is a group whose action on F is sharply 2-transitive, i.e. for any two ordered pairs (α_1, α_2), (β_1, β_2) of points of F with $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$ there exists a unique element $\tau_{a,b} \in T_2(F)$ such that $\tau_{a,b}(\alpha_1) = \beta_1$ and $\tau_{a,b}(\alpha_2) = \beta_2$.

Proof. See [3, (5.1)], [4, (6.1)] or [5, (7.8)]. ∎

3.2 Sharply 2-transitive groups and involutions

Let G be a sharply 2-transitive permutation group on a set Ω with $\#\Omega \geq 2$.

We denote by J the set of involutions in G, i.e.

$$J = \{g \in G \mid g^2 = 1_\Omega \neq g\}$$

One can quickly see that J is never empty. Indeed, as $\#\Omega \geq 2$, we can take $\alpha \neq \beta$ in Ω and find a (unique) $g \in G$ with $g(\alpha) = \beta$ and $g(\beta) = \alpha$. Such g must have order 2 (since g^2 fixes both α and β, implying $g^2 = 1_\Omega$, by sharp 2-transitivity).

PROPERTY 3.4. *J satisfies exactly one of the following properties:*

1. every $j \in J$ has a unique fixpoint;

2. all $j \in J$ are fixpoint-free.

In the latter case we say that G has characteristic 2 and write $\operatorname{char} G = 2$. In the first case we put $\operatorname{char} G \neq 2$.

Proof. For a detailed proof, we refer to [4, p.12], where the case char $G = 2$, resp. char $G \neq 2$ is referred to as G of type 0, resp. of type 1. (The type refers to the number of fixpoints of an involution in G.) The main idea behind the proof is that sharp 2-transitivity implies that a nontrivial element of G cannot have 2 (or more) fixed points. Moreover all elements of J are conjugate in G, hence they have the same number of fixpoints. ∎

The reason for this notation and for using the word *characteristic* will be clarified when we establish the correspondence between sharply 2-transitive groups and neardomains (see Property 3.6).

In the case char $G \neq 2$ we write ν for the unique involution fixing an arbitrarily chosen base point $\omega_0 \in \Omega$.

The following subset of G plays an important role. We define $A \subseteq G$ as follows

$$A = \begin{cases} J \circ \nu & \text{if char } G \neq 2 \\ J \cup \{1_\Omega\} & \text{if char } G = 2 \end{cases} \tag{1}$$

PROPERTY 3.5. *The triple (A, Ω, ω_0) is a regular permutation set with basepoint on Ω.*

Proof. See [3, (5.3)] or [4, (3.3)]. ∎

The category of sharply 2-transitive groups will be denoted by $s2t\text{-}\mathcal{G}n$. Its **objects** are quadruples $(G, \Omega, \omega_0, \omega_1)$ where G is a permutation group which operates sharply 2-transitively on the set Ω (with $\#\Omega \geq 2$), with two different base points ω_0 and ω_1 of Ω. Morphisms will be defined after Property 3.6.

Let $(G, \Omega, \omega_0, \omega_1)$ be an object in $s2t\text{-}\mathcal{G}n$. On Ω we define an **addition** and a **multiplication** as follows. For $\alpha, \beta \in \Omega$ we define $\alpha +_0 \beta$ to be $a(\beta)$, where $a \in A$ is unique such that $a(\omega_0) = \alpha$.

Since the stabilizer G_{ω_0} is regular on $\Omega \setminus \{\omega_0\}$ we can also define $\alpha \cdot_1 \beta$ as $g(\beta)$ where $g \in G_{\omega_0}$ is unique such that $g(\omega_1) = \alpha$. We also put $\alpha \cdot_1 \beta = \omega_0$ when $\alpha = \omega_0$ or $\beta = \omega_0$.

PROPERTY 3.6. *1. The triple $F = (\Omega, +_0, \cdot_1)$ is a neardomain.*

2. char $G = 2 \Leftrightarrow$ char $F = 2$, i.e. $1 + 1 = 0$ in F (1 denotes the multiplicative identity of the neardomain F).

Proof. 1. See [3, (5.2)] or [4, (6.2)].

2. See [5, (7.10)]. ∎

We now define the **morphisms** in $s2t\text{-}\mathcal{G}n$ to be pairs of maps (f, Φ): $(G, \Omega, \omega_0, \omega_1) \to (H, \Sigma, \sigma_0, \sigma_1)$ where either both G and H have characteristic 2 or both have characteristic different from 2 and where $f: G \to H$ is a group homomorphism, $\Phi: \Omega \to \Sigma$ an **injective** map with $\Phi(\omega_0) = \sigma_0$ and $\Phi(\omega_1) = \sigma_1$,

and such that the following diagram commutes.

$$
\begin{array}{ccc}
G \times \Omega & \longrightarrow & \Omega \\
{\scriptstyle f \times \Phi} \downarrow & & \downarrow {\scriptstyle \Phi} \\
H \times \Sigma & \longrightarrow & \Sigma
\end{array}
\qquad (2)
$$

In this diagram the horizontal maps λ and μ are the left action maps. (Do not confuse with the map $\mu : M \to \Omega$ introduced in 2.3.)

In the case either char $G = 2$ and char $H \neq 2$ or char $G \neq 2$ and char $H = 2$, we put $\mathit{s2t}\text{-}\mathcal{G}\!n\!\mathit{\rho}((G, \Omega, \omega_0, \omega_1), (H, \Sigma, \sigma_0, \sigma_1)) = \varnothing$ (see the second remark just below).

Remark 3.7.   1. The injectivity of Φ implies the injectivity of f since $1_\Sigma = f(g)$ for some $g \in G$ implies that $\forall \omega \in \Omega$ we have $\Phi(g(\omega)) = f(g)(\Phi(\omega)) = \Phi(\omega)$. Hence $\forall \omega \in \Omega$: $g(\omega) = \omega$, proving $g = 1_\Omega$.

2. The reason for not allowing any morphisms in the case where G and H have different characteristics is the following. If (f, Φ) is a morphism from $(G, \Omega, \omega_0, \omega_1)$ to $(H, \Sigma, \sigma_0, \sigma_1)$ with either char $G = 2$ and char $H \neq 2$ or char $G \neq 2$ and char $H = 2$, the map Φ between the associated neardomains $(\Omega, +_0, \cdot_1)$ and $(\Sigma, +_{0'}, \cdot_{1'})$ cannot be a morphism of neardomains. Indeed, the first case char $G = 2 \neq$ char H implies (by Property 3.6.2.) that the multiplicative identities 1 of $(\Omega, +_0, \cdot_1)$ and $1'$ of $(\Sigma, +_{0'}, \cdot_{1'})$ satisfy $1 + 1 = 0$ and $1' + 1' \neq 0'$, which conflicts with $\Phi : (\Omega, +_0, \cdot_1) \to (\Sigma, +_{0'}, \cdot_{1'})$ being a morphism of neardomains, as can be seen by $0' \neq 1' + 1' = \Phi(1) +' \Phi(1) = \Phi(1 + 1) = \Phi(0) = 0'$. Similarly the second case char $G \neq 2 =$ char H leads to the contradiction $0' = 1' +' 1' = \Phi(1) +' \Phi(1) = \Phi(1 + 1) \neq \Phi(0) = 0'$, where we used the injectivity of Φ.

   Our elimination of "bad" morphisms in $\mathit{s2t}\text{-}\mathcal{G}\!n\!\mathit{\rho}$ enables us to show (in Property 3.9) that the corresponding Φ is a morphism of neardomains.

LEMMA 3.8. *For a morphism $(f, \Phi) : (G, \Omega, \omega_0, \omega_1) \to (H, \Sigma, \sigma_0, \sigma_1)$ the following hold.*

1. *$f(J) \subseteq K$ (where K denotes the set of involutions in H);*

2. *$f(A) \subseteq B$ where $A \subseteq G$ is defined in (1) above Property 3.5 and $B \subseteq H$ is defined by*

$$
B = \begin{cases} K \circ f(\nu) & \text{if char } H \neq 2 \\ K \cup \{1_\Sigma\} & \text{if char } H = 2 \end{cases}
$$

   ($f(\nu)$ is the unique element of K fixing the base point $\sigma_0 = \Phi(\omega_0)$)

Proof.   1. Let $g \in J$, then $f(g)$ satisfies $(f(g))^2 = f(g^2) = f(1_\Omega) = 1_\Sigma \neq f(g)$, since $f(1_\Omega) = f(g)$ would imply (using the injectivity of f) that $g = 1_\Omega$, contradicting the fact that $g \in J$. Hence $f(J) \subseteq K$.

2. By the previous remark 2., the existence of the morphism (f, Φ) implies that either char $G =$ char $H \neq 2$ or char $G =$ char $H = 2$. In the first case it follows that $f(A) = f(J \circ \nu) = f(J) \circ f(\nu) \subseteq K \circ f(\nu) = B$. In the second case we have $f(A) = f(J \cup \{1_\Omega\}) = f(J) \cup \{f(1_\Omega)\} \subseteq K \cup \{1_\Sigma\} = B$.   ∎

PROPERTY 3.9. *For a morphism* $(f, \Phi) \colon (G, \Omega, \omega_0, \omega_1) \to (H, \Sigma, \sigma_0, \sigma_1)$ *the map* $\Phi \colon (\Omega, +_0, \cdot_1) \to (\Sigma, +_{0'}, \cdot_{1'})$ *is a morphism of neardomains.*

Proof. Consider the diagram (2) above involving the morphism (f, Φ) from $(G, \Omega, \omega_0, \omega_1)$ to $(H, \Sigma, \sigma_0, \sigma_1)$. For $\alpha, \beta \in \Omega$, $\alpha +_0 \beta = a(\beta) \in \Omega$ where $a \in A \subseteq G$ is unique such that $a(\omega_0) = \alpha$. Now $\Phi(\alpha +_0 \beta) = \Phi(a(\beta)) = \Phi(\lambda(a, \beta)) = (\mu \circ (f \times \Phi))(a, \beta) = \mu(f(a), \Phi(\beta)) = f(a)(\Phi(\beta))$. On the other hand, $\Phi(\alpha) +_{0'} \Phi(\beta) = b(\Phi(\beta))$ where $b \in B \subseteq H$ is unique such that $b(\sigma_0) = \Phi(\alpha)$.

Since $f(A) \subseteq B$ we also have $f(a) \in B$ and moreover $(f(a))(\sigma_0) = \mu(f(a), \Phi(\omega_0)) = \Phi(\lambda(a, \omega_0)) = \Phi(a(\omega_0)) = \Phi(\alpha)$. Since B acts regularly on Σ it follows that $f(a) = b$. This implies that $\Phi(\alpha +_0 \beta) = f(a)(\Phi(\beta)) = b(\Phi(\beta)) = \Phi(\alpha) +_{0'} \Phi(\beta)$.

We recall that, for $\alpha, \beta \in \Omega$,

$$\alpha \cdot_1 \beta = \begin{cases} g(\beta) & \text{when } \alpha \in \Omega \setminus \{\omega_0\} \text{ with } g \in G_{\omega_0} \text{ unique such that } g(\omega_1) = \alpha \\ \omega_0 & \text{when } \alpha = \omega_0 \text{ or } \beta = \omega_0 \end{cases}$$

When $\alpha = \omega_0$ we have immediately

$$\Phi(\alpha) \cdot_{1'} \Phi(\beta) = \sigma_0 \cdot_{1'} \Phi(\beta) = \sigma_0 = \Phi(\omega_0) = \Phi(\omega_0 \cdot_1 \beta).$$

When $\alpha \neq \omega_0$, we have

$$\Phi(\alpha \cdot_1 \beta) = \Phi(g(\beta)) = \Phi(\lambda(g, \beta)) = (\mu \circ (f \times \Phi))(g, \beta) = \mu(f(g), \Phi(\beta)) =$$
$$f(g)(\Phi(\beta)) \qquad \text{(with } g \in G_{\omega_0} \text{ unique such that } g(\omega_1) = \alpha\text{)}.$$

On the other hand, $\Phi(\alpha) \cdot_{1'} \Phi(\beta) = h(\Phi(\beta))$ with $h \in H_{\sigma_0}$ unique such that $h(\sigma_1) = \Phi(\alpha)$. Note that, by the injectivity of Φ we have $\Phi(\alpha) \neq \Phi(\omega_0) = \sigma_0$. We must have $h = f(g)$ since

$$f(g)(\sigma_1) = \mu(f(g), \Phi(\omega_1)) = (\mu \circ (f \times \Phi))(g, \omega_1) =$$
$$(\Phi \circ \lambda)(g, \omega_1) = \Phi(g(\omega_1)) = \Phi(\alpha) = h(\sigma_1) \text{ and } f(g), h \in H_{\sigma_0}.$$

Hence

$$f(g)(\Phi(\beta)) = h(\Phi(\beta))$$

and Φ is a neardomain homomorphism. ∎

For any neardomain $(F, +, \cdot)$ we can construct the object $(T_2(F), F, 0, 1)$ in $\mathcal{A}ct\text{-}\mathcal{G}p$. It is clear that the stabilizer of 0 in $T_2(F)$ consists of the elements $\tau_{0,\delta}$ with $\delta \in F \setminus \{0\}$.

LEMMA 3.10. *When* $G = T_2(F)$ *we have* $A = \{\tau_{\gamma,1} \mid \gamma \in F\}$.

Proof. See [4, (6.5)]. ∎

4 Equivalence of the categories $s2t\text{-}\mathcal{Gp}$ and $n\text{-}\mathcal{Dom}$

THEOREM 4.1. *The functor*

$$
\begin{array}{rccc}
K: & s2t\text{-}\mathcal{Gp} & \longrightarrow & n\text{-}\mathcal{Dom} \\
& (G, \Omega, \omega_0, \omega_1) & \longmapsto & (\Omega, +_0, \cdot_1) \\
& | & & | \\
& (f, \Phi) & \mapsto & \Phi \\
& \downarrow & & \downarrow \\
& (H, \Sigma, \sigma_0, \sigma_1) & \longmapsto & (\Sigma, +_0, \cdot_1)
\end{array}
$$

is an equivalence of categories. Here $f: G \to H$, $\Phi: \Omega \to \Sigma$ with $\Phi(\omega_0) = \sigma_0$ and $\Phi(\omega_1) = \sigma_1$ are defined as in section 3.2. The neardomain operations $+_0: \Omega \times \Omega \to \Omega$ and $\cdot_1: \Omega \setminus \{\omega_0\} \times \Omega \setminus \{\omega_0\} \to \Omega \setminus \{\omega_0\}$ are defined as in the previous section.

Proof. The shortest proof should consist in proving that K is full and faithful and essentially surjective on objects ([1, prop.3.4.3 (4)]). However we prefer to show that there exists a functor $L: n\text{-}\mathcal{Dom} \to s2t\text{-}\mathcal{Gp}$, which is of interest in its own right, and two natural isomorphisms $1_{n\text{-}\mathcal{Dom}} \cong K \circ L$ and $L \circ K \cong 1_{s2t\text{-}\mathcal{Gp}}$ ([1, prop. 3.4.3 (3)]). The proof of functoriality of K is left as an exercise for the reader. The definition of $L: n\text{-}\mathcal{Dom} \to s2t\text{-}\mathcal{Gp}$ is as follows. A neardomain $(F, +, .)$ is sent to $L(F) = (T_2(F), F, 0, 1)$ where $(T_2(F), \circ)$, as defined in Theorem 3.3, is a sharply 2-transitive group acting on F and $0, 1$ are the identities of the loop $(F, +)$ and the group $(F \setminus \{0\}, .)$ respectively. A morphism $\Phi: (F, +, .) \to (F', +', .')$ of neardomains is sent to $L(\Phi) = (f_\Phi, \Phi): (T_2(F), F, 0, 1) \to (T_2(F'), F', 0', 1')$ where $f_\Phi: (T_2(F), \circ) \to (T_2(F'), \circ)$ maps $\tau_{a,b}$ to $\tau_{\Phi(a), \Phi(b)}$. Since $\Phi(1) = 1'$ it immediately follows, using Lemma 3.10, that $f_\Phi(A) = \{f_\Phi(\tau_{a,1}) \mid a \in F\} = \{\tau_{\Phi(a),1'} \mid a \in F\} \subseteq \{\tau_{a',1'} \mid a' \in F'\} = A'$. Moreover f_Φ is a group homomorphism since, on the one hand, for $a, k \in F$ and $b, l \in F \setminus \{0\}$ we have

$$
f_\Phi(\tau_{a,b} \circ \tau_{k,l}) = f_\Phi(\tau_{a+bk, d_{a,bk}bl}) = \tau_{\Phi(a+bk), \Phi(d_{a,bk}bl)} = \tau_{\Phi(a)+\Phi(b)\Phi(k), \Phi(d_{a,bk})\Phi(b)\Phi(l)}
$$

while, on the other hand,

$$
f_\Phi(\tau_{a,b}) \circ f_\Phi(\tau_{k,l}) = \tau_{\Phi(a), \Phi(b)} \circ \tau_{\Phi(k), \Phi(l)} = \tau_{\Phi(a)+\Phi(b)\Phi(k), d'_{\Phi(a), \Phi(b)\Phi(k)}\Phi(b)\Phi(l)}.
$$

Finally the missing link $\Phi(d_{a,bk}) = d'_{\Phi(a), \Phi(b)\Phi(k)}$ is provided by applying Φ to the identity $a + (bk + x) = (a + bk) + d_{a,bk}x$, which follows from axiom 6 for F, and expanding $\Phi(a) + (\Phi(b)\Phi(k) + \Phi(x))$, again using rule 6 for F'.

Then it follows that $L(\Phi)$ is a morphism in $s2t\text{-}\mathcal{Gp}$ by easily verifying that for all $a \in F$, $b \in F \setminus \{0\}$, $\omega \in F$ we have $\Phi(\tau_{a,b}(\omega)) = (f_\Phi(\tau_{a,b}))(\Phi(\omega))$.

The functoriality of L, i.e. $L(1_F) = 1_F$ for every neardomain F and $L(\Phi' \circ \Phi) = L(\Phi') \circ L(\Phi)$ for every composable pair $\Phi: F \to F'$, $\Phi': F' \to F''$ of morphisms of neardomains, is easily verified.

In order to prove $K \circ L = 1_{n\text{-}\mathcal{D}\mathit{om}}$ it suffices to check that $+_0$, $\cdot_1$ (resp. $+_{\{0'\}}$, $\cdot_{\{1'\}}$) coincide with $+$, $\cdot$ (resp. $+'$, $\cdot'$).

We first look at the multiplication. Take any α, β in F. By definition $\alpha \cdot_1 \beta$ equals $\tau_{0,\alpha}(\beta)$ since $T_2(F)_0 = \{\tau_{0,\delta} \mid \delta \in F \setminus \{0\}\}$ and $\tau_{0,\delta}(1) = \alpha \Leftrightarrow \delta = \alpha$. Hence we get $\alpha \cdot_1 \beta = \alpha \cdot \beta$.

For the addition we consider $\alpha, \beta \in F$. By Lemma 3.10, the unique element in A mapping 0 to α must be equal to $\tau_{\alpha,1}$. Therefore we get $\alpha +_0 \beta = \tau_{\alpha,1}(\beta) = \alpha + \beta$.

Secondly, we have to find a natural isomorphism $\gamma \colon L \circ K \Rightarrow 1_{s2t\text{-}\mathcal{G}r}$. We define each component $\gamma_{(G,\Omega,\omega_0,\omega_1)} \colon (L \circ K)(G,\Omega,\omega_0,\omega_1) = L(\Omega, +_0, \cdot_1) = (T_2(\Omega), \Omega, \omega_0, \omega_1) \to (G, \Omega, \omega_0, \omega_1)$ as follows.

For each $\tau_{\alpha,\beta} \in T_2(\Omega)$ there exists, by the sharp 2-transitivity of G on Ω a unique element $g \in G$, denoted by $g_{\alpha,\beta}$ such that $g_{\alpha,\beta}(\omega_0) = \alpha = \tau_{\alpha,\beta}(\omega_0)$ and $g_{\alpha,\beta}(\omega_1) = \alpha +_0 \beta \cdot_1 \omega_1 = \tau_{\alpha,\beta}(\omega_1)$. This correspondence defines a bijective map $k \colon T_2(\Omega) \to G \colon \tau_{\alpha,\beta} \mapsto g_{\alpha,\beta}$ and one verifies easily that $k \colon (T_2(\Omega), \circ) \to (G, \circ)$ is a group homomorphism (using also the sharp 2-transitivity of $T_2(\Omega)$ on Ω). Finally we define $\gamma_{(G,\Omega,\omega_0,\omega_1)} \colon (T_2(\Omega), \Omega, \omega_0, \omega_1) \to (G, \Omega, \omega_0, \omega_1)$ by $\gamma_{(G,\Omega,\omega_0,\omega_1)}(\tau_{\alpha,\beta}, \omega) = (k(\tau_{\alpha,\beta}), \omega) = (g_{\alpha,\beta}, \omega)$, which is an isomorphism in $s2t\text{-}\mathcal{G}r$. The naturality of γ amounts to the commutativity of each square

$$
\begin{array}{ccc}
(T_2(\Omega), \Omega, \omega_0, \omega_1) & \xrightarrow{\gamma_{(G,\Omega,\omega_0,\omega_1)}} & (G, \Omega, \omega_0, \omega_1) \\
{\scriptstyle (f_\Phi, \Phi)}\downarrow & & \downarrow{\scriptstyle (f, \Phi)} \\
(T_2(\Sigma), \Sigma, \sigma_0, \sigma_1) & \xrightarrow{\gamma_{(H,\Sigma,\sigma_0,\sigma_1)}} & (H, \Sigma, \sigma_0, \sigma_1)
\end{array}
$$

with $f \colon G \to H$ and $\Phi \colon \Omega \to \Sigma$ as in 3.2 and $f_\Phi \colon T_2(\Omega) \to T_2(\Sigma) \colon \tau_{\alpha,\beta} \mapsto \tau_{\Phi(\alpha),\Phi(\beta)}$.

On the one hand we have, for each $(\tau_{\alpha,\beta}, \omega) \in T_2(\Omega) \times \Omega$,

$$
\begin{aligned}
((f, \Phi) \circ \gamma_{(G,\Omega,\omega_0,\omega_1)})(\tau_{\alpha,\beta}, \omega) &= (f, \Phi)(k(\tau_{\alpha,\beta}), \omega) = \\
&(f, \Phi)(g_{\alpha,\beta}, \omega) = (f(g_{\alpha,\beta}), \Phi(\omega)) \in H \times \Sigma
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(\gamma_{(H,\Sigma,\sigma_0,\sigma_1)} \circ (f_\Phi, \Phi))(\tau_{\alpha,\beta}, \omega) &= \gamma_{(H,\Sigma,\sigma_0,\sigma_1)}(f_\Phi(\tau_{\alpha,\beta}), \Phi(\omega)) = \\
\gamma_{(H,\Sigma,\sigma_0,\sigma_1)}(\tau_{\Phi(\alpha),\Phi(\beta)}, \Phi(\omega)) &= (h_{\Phi(\alpha),\Phi(\beta)}, \Phi(\omega)) \in H \times \Sigma.
\end{aligned}
$$

Finally $f(g_{\alpha,\beta}) = h_{\Phi(\alpha),\Phi(\beta)}$ since $(f(g_{\alpha,\beta}))(\sigma_0) = (f(g_{\alpha,\beta}))(\Phi(\omega_0)) = \Phi(\alpha) = h_{\Phi(\alpha),\Phi(\beta)}(\sigma_0)$ and, similarly,

$$
\begin{aligned}
(f(g_{\alpha,\beta}))(\sigma_1) &= (\mu \circ (f \times \Phi))(g_{\alpha,\beta}, \omega_1) = \Phi(g_{\alpha,\beta}(\omega_1)) = \\
\Phi(\alpha +_0 \beta \cdot_1 \omega_1) &= \Phi(\alpha +_0 \beta) = \Phi(\alpha) +_{0'} \Phi(\beta) = \\
\Phi(\alpha) +_{0'} \Phi(\beta) \cdot_{1'} \sigma_1 &= \tau_{\Phi(\alpha),\Phi(\beta)}(\sigma_1) = h_{\Phi(\alpha),\Phi(\beta)}(\sigma_1).
\end{aligned}
$$

Hence, again by sharp 2-transitivity, $f(g_{\alpha,\beta}) = h_{\Phi(\alpha),\Phi(\beta)}$.

Hence $\gamma \colon L \circ K \Rightarrow 1_{s2t\text{-}\mathcal{G}r}$ is a natural transformation. ∎

Since a full and faithful functor reflects isomorphisms we immediately get

PROPERTY 4.2. *Let* (G, F) *and* (G', F') *be sharply* 2-*transitive permutation groups. Then* (G, F) *and* (G', F') *are isomorphic as permutation groups if and only if the associated neardomains* $(F, +, .)$ *and* $(F', +', .')$ *are isomorphic in* $n\text{-}\mathscr{D}om$.

(For a noncategorical proof, see [4, (6.3)].)

The equivalence obtained in Theorem 4.1 can be restricted to interesting subcategories of $s2t\text{-}\mathscr{G}p$ and $n\text{-}\mathscr{D}om$ respectively, which sheds new light on the possible difference between neardomains and nearfields.

THEOREM 4.3. *Let* $s2t\text{-}\mathscr{G}p_A$ *be the full subcategory of* $s2t\text{-}\mathscr{G}p$ *on objects* $(G, \Omega, \omega_0, \omega_1)$ *in which the subset* A *(as defined in (1)) is a* subgroup *of* G.

Then the functor K restricts (and corestricts) to an equivalence

$$K_A \colon s2t\text{-}\mathscr{G}p_A \longrightarrow n\text{-}\mathscr{F}ld,$$

where $n\text{-}\mathscr{F}ld$ *denotes the full subcategory of* $n\text{-}\mathscr{D}om$ *with nearfields as objects.*

Proof. In Theorem (7.1) of [3] it is shown that, for a sharply 2-transitive group on a set Ω, the associated neardomain is a nearfield if and only if $J^2 = \{gh \mid g, h \in J\}$ (with J the set of involutions in G) is a subgroup of G. In connection with Theorem (3.7) of [3] it follows that J^2 is a subgroup of G if and only if A is a subgroup of G.

Thus $K_A \colon s2t\text{-}\mathscr{G}p_A \longrightarrow n\text{-}\mathscr{F}ld$ is a functor into the category of nearfields.

On the other hand, the functor $L \colon n\text{-}\mathscr{D}om \to s2t\text{-}\mathscr{G}p$ restricts (and corestricts) to a functor $L_A \colon n\text{-}\mathscr{F}ld \to s2t\text{-}\mathscr{G}p_A$. It suffices to notice that when $(F, +, .)$ is a nearfield, $(F, +)$ and $(F \setminus \{0\}, .)$ are groups. Hence $(A, \circ) = (\{\tau_{\gamma,1} \mid \gamma \in F\}, \circ)$ is a group since, for each $\alpha, \beta, x \in F$ we have $(\tau_{\alpha,1} \circ \tau_{\beta,1})(x) = \tau_{\alpha,1}(\beta + x) = \alpha + (\beta + x) = (\alpha + \beta) + x = \tau_{\alpha+\beta,1}(x)$, which shows that $\tau_{\alpha,1} \circ \tau_{\beta,1} = \tau_{\alpha+\beta,1}$. Similarly $\tau_{\alpha,1}^{-1} = \tau_{-\alpha,1}$.

Finally one verifies that $K_A \circ L_A = 1_{n\text{-}\mathscr{F}ld}$ (as in the proof of Theorem 4.1) and that the natural transformation $\gamma_A \colon L_A \circ K_A \Rightarrow 1_{s2t\text{-}\mathscr{G}p_A}$ with components $\gamma_{(G,\Omega,\omega_0,\omega_1)}$ as defined in the proof of Theorem 4.1, is a natural isomorphism. ∎

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