Topological monomorphisms between free paratopological groups

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Abstract

Suppose that X is a subspace of a Tychonoff space Y. Then the embedding mapping $e_{X,Y} : X \to Y$ can be extended to a continuous monomorphism $\hat{e}_{X,Y} : AP(X) \to AP(Y)$, where $AP(X)$ and $AP(Y)$ are the free Abelian paratopological groups over X and Y, respectively. In this paper, we mainly discuss when $\hat{e}_{X,Y}$ is a topological monomorphism, that is, when $\hat{e}_{X,Y}$ is a topological embedding of $AP(X)$ to $AP(Y)$.

1 Introduction

In 1941, free topological groups were introduced by A.A. Markov in [9] with the clear idea of extending the well-known construction of a free group from group theory to topological groups. Now, free topological groups have become a powerful tool of study in the theory of topological groups and serve as a source of various examples and as an instrument for proving new theorems, see [1, 5, 11].

In [5], M.I. Graev extended continuous pseudometrics on a space X to invariant continuous pseudometrics on $F(X)$ (or $A(X)$). Apparently, the description of a local base at the neutral element of the free Abelian topological group $A(X)$ in terms of continuous pseudometric on X was known to M.I. Graev, but appeared explicitly in [10] and [11]. When working with free topological groups, it is also

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very important to know under which conditions on a subspace X of a Tychonoff space Y, the subgroup $F(X,Y)$ of $F(Y)$ generated by X is topologically isomorphic to the group $F(X)$, under the natural isomorphism extending the identity embedding of X to Y. V.G. Pestov and E.C. Nummela gave some answers (see e.g. Theorem 3.1) in [13] and [12], respectively. In the Abelian case, M.G. Tkackenko gave an answer in [22], see Theorem 3.2.

It is well known that paratopological groups are good generalizations of topological groups, see e.g. [1]. The Sorgenfrey line ([3, Example 1.2.2]) with the usual addition is a first-countable paratopological group but not a topological group. The absence of continuity of inversion, the typical situation in paratopological groups, makes the study in this area very different from that in topological groups. Paratopological groups attract a growing attention of many mathematicians and articles in recent years. As in free topological groups, S. Romaguera, M. Sanchis and M.G. Tkackenko in [18] define free paratopological groups. Recently, N.M. Pyrch has investigated some properties of free paratopological groups, see [14, 15, 16]. In this paper, we will discuss the topological monomorphisms between free paratopological groups, and extend several results valid for free (abelian) topological groups to free (abelian) paratopological groups.

2 Preliminaries

Firstly, we introduce some notions and terminology.

Recall that a *topological group* G is a group G with a (Hausdorff) topology such that the product mapping of $G \times G$ into G is jointly continuous and the inverse mapping of G onto itself associating x^{-1} with an arbitrary $x \in G$ is continuous. A *paratopological group* G is a group G with a topology such that the product mapping of $G \times G$ into G is jointly continuous.

Definition 2.1. [9] Let X be a subspace of a topological group G. Assume that

1. The set X generates G algebraically, that is $< X >= G$;

2. Each continuous mapping $f : X \to H$ to a topological group H extends to a continuous homomorphism $\hat{f} : G \to H$.

Then G is called the *Markov free topological group on* X and is denoted by $F(X)$.

Definition 2.2. [18] Let X be a subspace of a paratopological group G. Assume that

1. The set X generates G algebraically, that is $< X >= G$;

2. Each continuous mapping $f : X \to H$ to a paratopological group H extends to a continuous homomorphism $\hat{f} : G \to H$.

Then G is called the *Markov free paratopological group on* X and is denoted by $FP(X)$.

Again, if all the groups in the above definitions are Abelian, then we get the definitions of the *Markov free Abelian topological group* and the *Markov free Abelian paratopological group on* X which will be denoted by $A(X)$ and $AP(X)$ respectively.

By a *quasi-uniform space* $(X, \mathscr{U})$ we mean the natural analog of a *uniform space* obtained by dropping the symmetry axiom. For each quasi-uniformity $\mathscr{U}$ the filter $\mathscr{U}^{-1}$ consisting of the inverse relations $U^{-1} = \{(y, x) : (x, y) \in U\}$ where $U \in \mathscr{U}$ is called the *conjugate quasi-uniformity* of $\mathscr{U}$. We recall that the standard base of the *left quasi-uniformity* $\mathscr{G}_G$ on a paratopological group G consists of the sets

$$W_U^l = \{(x, y) \in G \times G : x^{-1}y \in U\},$$

where U is an arbitrary open neighborhood of the neutral element in G. If X is a subspace of G, then the base of the left induced quasi-uniformity $\mathscr{G}_X = \mathscr{G}_G \mid X$ on X consists of the sets

$$W_U^l \cap (X \times X) = \{(x, y) \in X \times X : x^{-1}y \in U\}.$$

Similarly, we can define the *right induced quasi-uniformity* on X.

We also recall that the *universal quasi-uniformity* $\mathscr{U}_X$ of a space X is the finest quasi-uniformity on X that induces on X its original topology. Throughout this paper, if $\mathscr{U}$ is a quasi-uniformity of a space X then $\mathscr{U}^*$ denotes the smallest uniformity on X that contains $\mathscr{U}$, and $\tau(\mathscr{U})$ denotes the topology of X generated by $\mathscr{U}$. A quasi-uniform space $(X, \mathscr{U})$ is called *bicomplete* if $(X, \mathscr{U}^*)$ is complete.

Definition 2.3. A function $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ is called *quasi-uniformly continuous* if for each $V \in \mathscr{V}$ there exists an $U \in \mathscr{U}$ such that $(f(x), f(y)) \in V$ whenever $(x, y) \in U$, where $\mathscr{U}$ and $\mathscr{V}$ are quasi-uniformities for X and Y respectively.

Definition 2.4. A *quasi-pseudometric* d on a set X is a function from $X \times X$ into the set of non-negative real numbers such that for $x, y, z \in X$: (a) $d(x, x) = 0$ and (b) $d(x, y) \leq d(x, z) + d(z, y)$. If d satisfies the additional condition (c) $d(x, y) = 0 \Leftrightarrow x = y$, then d is called a *quasi-metric* on X.

Every quasi-pseudometric d on X generates a topology $\mathscr{F}(d)$ on X which has as a base the family of d-balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

A topological space $(X, \mathscr{F})$ is called *quasi-(pseudo)metrizable* if there is a quasi-(pseudo)metric d on X compatible with $\mathscr{F}$, where d is compatible with $\mathscr{F}$ provided $\mathscr{F} = \mathscr{F}(d)$.

Denote by $\mathscr{U}^\star$ the upper quasi-uniformity on $\mathbb{R}$ the standard base of which consists of the sets

$$U_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y < x + r\},$$

where r is an arbitrary positive real number.

Definition 2.5. Given a group G with the neutral element e, a function $N : G \to [0, \infty)$ is called a *quasi-prenorm* on G if the following conditions are satisfied:

1. $N(e) = 0$; and
2. $N(gh) \leq N(g) + N(h)$ for all $g, h \in G$.

Definition 2.6. Let X be a subspace of a Tychonoff space Y.

1. The subspace X is *P-embedded* in Y if each continuous pseudometric on X admits a continuous extension over Y;

2. The subspace X is *P^*-embedded* in Y if each bounded continuous pseudometric on X admits a continuous extension over Y;

3. The subspace X is *quasi-P-embedded* in Y if each continuous quasi-pseudometric from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$ admits a continuous extension from $(Y \times Y, \mathscr{U}_Y^{-1} \times \mathscr{U}_Y)$ to $(\mathbb{R}, \mathscr{U}^\star)$;

4. The subspace X is *quasi-P^*-embedded* in Y if each bounded continuous quasi-pseudometric from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$ admits a continuous extension from $(Y \times Y, \mathscr{U}_Y^{-1} \times \mathscr{U}_Y)$ to $(\mathbb{R}, \mathscr{U}^\star)$.

Throughout this paper, we use $G(X)$ to denote the topological groups $F(X)$ or $A(X)$, and $PG(X)$ to denote the paratopological groups $FP(X)$ or $AP(X)$. For a subset Y of a space X, we use $G(Y, X)$ and $PG(Y, X)$ to denote the subgroups of $G(X)$ and $PG(X)$ generated by Y respectively. Moreover, we denote the abstract groups of $F(X), FP(X)$ by $F_a(X)$ and of $A(X)$ and $AP(X)$ by $A_a(X)$, respectively.

Since X generates the free group $F_a(X)$, each element $g \in F_a(X)$ has the form $g = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $x_1, \cdots, x_{n1} \in X$ and $\varepsilon_1, \cdots, \varepsilon_n = \pm 1$. This word for g is called *reduced* if it contains no pair of consecutive symbols of the form xx^{-1} or $x^{-1}x$. It follow that if the word g is reduced and non-empty, then it is different from the neutral element of $F_a(X)$. In particular, each element $g \in F_a(X)$ distinct from the neutral element can be uniquely written in the form $g = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$, where $n \geq 1$, $r_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$, and $x_i \neq x_{i+1}$ for each $i = 1, \cdots, n-1$. Such a word is called the *normal form* of g. Similar assertions are valid for $A_a(X)$.

We denote by $\mathbb{N}$ the set of all natural numbers. The letter e denotes the neutral element of a group. Readers may consult [1, 3, 7] for notations and terminology not explicitly given here.

3 Backgrounds

If X is an arbitrary subspace of a Tychonoff space Y, then let $e_{X,Y}$ be the natural embedding mapping from X to Y. The following two theorems are well known in the theory of free topological groups.

Theorem 3.1. *[12, 13, Nummela-Pestov] Let X be a dense subspace of a Tychonoff space Y. Then the embedding mapping $e_{X,Y}$ can be extended to a topological monomorphism $\hat{e}_{X,Y} : F(X) \to F(Y)$ if and only if X is P-embedded in Y.*

Theorem 3.2. *[22, M.G. Tkackenko] Let X be an arbitrary subspace of a Tychonoff space Y. Then the embedding mapping $e_{X,Y}$ can be extended to a topological monomorphism $\hat{e}_{X,Y} : A(X) \to A(Y)$ if and only if X is P^*-embedded in Y.*

Obviously, if X is a subspace of a Tychonoff space Y, then the embedding mapping $e_{X,Y} : X \to Y$ can be extended to a continuous monomorphism $\hat{e}_{X,Y} : PG(X) \to PG(Y)$. However, by Theorems 3.1 and 3.2, it is natural to ask the following two questions:

Question 3.3. *Let X be a dense subspace of a Tychonoff space Y. Is it true that the embedding mapping $e_{X,Y}$ can be extended to a topological monomorphism $\hat{e}_{X,Y} : FP(X) \to FP(Y)$ if and only if X is quasi-P-embedded in Y?*

Question 3.4. *Let X be a subspace of a Tychonoff space Y. Is it true that the embedding mapping $e_{X,Y}$ can be extended to a topological monomorphism $\hat{e}_{X,Y} : AP(X) \to AP(Y)$ if and only if X is quasi-P^*-embedded in Y?*

In this paper, we shall give an affirmative answer to Question 3.4. Moreover, we shall give a partial answer to Question 3.3, and prove that for a Tychonoff space Y if X is a dense subspace of the smallest uniformity containing $\mathscr{U}_Y$ induces on $\tilde{Y}$ of the bicompletion of $(Y, \mathscr{U}_Y)$ and the natural mapping $\hat{e}_{X,Y} : FP(X) \to FP(Y)$ is a topological monomorphism then X is quasi-P-embedded in Y.

4 Quasi-pseudometrics on free paratopological groups

In this section, we shall give some lemmas and theorems in order to prove our main results in Section 4.

We now outline some of the ideas of [18] in a form suitable for our applications.

Suppose that e is the neutral element of the abstract free group $F_a(X)$ on X, and suppose that ρ is a fixed quasi-pseudometric on X which is bounded by 1. Extend ρ from X to a quasi-pseudometric ρ_e on $X \cup \{e\}$ by putting

$$\rho_e(x,y) = \begin{cases} 0, & \text{if } x = y, \\ \rho(x,y), & \text{if } x, y \in X, \\ 1, & \text{otherwise} \end{cases}$$

for arbitrary $x, y \in X \cup \{e\}$. By [18], we extend ρ_e to a quasi-pseudometric ρ^* on $\tilde{X} = X \cup \{e\} \cup X^{-1}$ defined by

$$\rho^*(x,y) = \begin{cases} 0, & \text{if } x = y, \\ \rho_e(x,y), & \text{if } x, y \in X \cup \{e\}, \\ \rho_e(y^{-1}, x^{-1}), & \text{if } x, y \in X^{-1} \cup \{e\}, \\ 2, & \text{otherwise} \end{cases}$$

for arbitrary $x, y \in \tilde{X}$.

Let A be a subset of $\mathbb{N}$ such that $|A| = 2n$ for some $n \geq 1$. A *scheme* on A is a partition of A to pairs $\{a_i, b_i\}$ with $a_i < b_i$ such that each two intervals $[a_i, b_i]$ and $[a_j, b_j]$ in $\mathbb{N}$ are either disjoint or one contains the other.

If $\mathcal{X}$ is a word in the alphabet $\tilde{X}$, then we denote the reduced form and the length of $\mathcal{X}$ by $[\mathcal{X}]$ and $\ell(\tilde{X})$ respectively.

For each $n \in \mathbb{N}$, let $\mathcal{S}_n$ be the family of all schemes φ on $\{1, 2, \cdots, 2n\}$. As in [18], define

$$\Gamma_\rho(\mathcal{X}, \varphi) = \frac{1}{2} \sum_{i=1}^{2n} \rho^*(x_i^{-1}, x_{\varphi(i)}).$$

Then we define a quasi-prenorm $N_\rho : F_a(X) \to [0, +\infty)$ by setting $N_\rho(g) = 0$ if $g = e$ and

$$N_\rho(g) = \inf\{\Gamma_\rho(\mathcal{X}, \varphi) : [\mathcal{X}] = g, \ell(\tilde{X}) = 2n, \varphi \in \mathcal{S}_n, n \in \mathbb{N}\}$$

if $g \in F_a(X) \setminus \{e\}$. It follows from Claim 3 in [18] that N_ρ is an invariant quasi-prenorm on $F_a(X)$. Put $\hat{\rho}(g, h) = N_\rho(g^{-1}h)$ for all $g, h \in F_a(X)$. We refer to $\hat{\rho}$ as the Graev extension of ρ to $F_a(X)$.

Given a word $\mathcal{X}$ in the alphabet $\tilde{X}$, we say that $\mathcal{X}$ is *almost irreducible* if $\tilde{X}$ does not contain two consecutive symbols of the form u, u^{-1} or u^{-1}, u (but $\mathcal{X}$ may contain several letters equal to e), see [18].

The following two lemmas are essentially Claims in the proof of Theorem 3.2 in [18].

Lemma 4.1. *[18] Let ϱ be a quasi-pseudometric on X bounded by 1. If g is a reduced word in $F_a(X)$ distinct from e, then there exists an almost irreducible word $\mathcal{X}_g = x_1 x_2 \cdots x_{2n}$ of length $2n \geq 2$ in the alphabet $\tilde{X}$ and a scheme $\varphi_g \in \mathcal{S}_n$ that satisfy the following conditions:*

1. *for $i = 1, 2, \cdots, 2n$, either x_i is e or x_i is a letter in g;*
2. *$[\mathcal{X}_g] = g$ and $n \leq \ell(g)$; and*
3. *$N_\rho(g) = \Gamma_\rho(\mathcal{X}_g, \varphi_g)$.*

Lemma 4.2. *[18] The family $\mathcal{N} = \{U_\rho(\varepsilon) : \varepsilon > 0\}$ is a base at the neutral element e for a paratopological group topology $\mathscr{F}_\rho$ on $F_a(X)$, where $U_\rho(\varepsilon) = \{g \in F_a(X) : N_\rho(g) < \varepsilon\}$. The restriction of $\mathscr{F}_\rho$ to X coincides with the topology of the space X generated by ρ.*

Lemma 4.3. *[4] For every sequence $V_0, V_1, \cdots$, of elements of a quasi-uniformity $\mathscr{U}$ on a set X, if*

$$V_0 = X \times X \text{ and } V_{i+1} \circ V_{i+1} \circ V_{i+1} \subset V_i, \text{ for } i \in \mathbb{N},$$

where '$\circ$' denotes the composition of entourages in the quasi-uniform space $(X, \mathscr{U})$, then there exists a quasi-pseudometric ρ on the set X such that, for each $i \in \mathbb{N}$,

$$V_i \subset \{(x, y) : \rho(x, y) \leq \frac{1}{2^i}\} \subset V_{i-1}.$$

Lemma 4.4. *For every quasi-uniformity $\mathscr{V}$ on a set X and each $V \in \mathscr{V}$ there exists a quasi-pseudometric ρ bounded by 1 on X which is quasi-uniform with respect to $\mathscr{V}$ and satisfies the condition*

$$\{(x, y) : \rho(x, y) < 1\} \subset V.$$

Proof. By the definition of a quasi-uniformity, we can find a sequence $V_0, V_1, \cdots, V_n, \cdots$ of members of $\mathscr{V}$ such that $V_1 = V$ and $V_{i+1} \circ V_{i+1} \circ V_{i+1} \subset V_i$, for each $i \in \mathbb{N}$. Let $\rho = \min\{1, 4\rho_0\}$, where ρ_0 is a quasi-pseudometric as in Lemma 4.3. Then ρ is a quasi-pseudometric which has the required property. ∎

Given a finite subset B of $\mathbb{N}$ on B with $|B| = 2n \geq 2$, we say that a bijection $\varphi : B \to B$ is an *Abelian scheme* on B if φ is an involution without fixed points, that is, $\varphi(i) = j$ always implies $j \neq i$ and $\varphi(j) = i$.

Lemma 4.5. *Suppose that ρ is a quasi-pseudometric on a set X, and suppose that $m_1x_1 + \cdots + m_nx_n$ is the normal form of an element $h \in A_a(X) \setminus \{e\}$ of the length $l = \sum_{i=1}^{n} |m_i|$. Then there is a representation*

$$h = (-u_1 + v_1) + \cdots + (-u_k + v_k) \qquad (1)$$

where $2k = l$ if l is even and $2k = l + 1$ if l is odd, $u_1, v_1, \cdots, u_k, v_k \in \{\pm x_1, \cdots, \pm x_n\}$ (but $v_k = e$ if l is odd), and such that

$$\hat{\rho}_A(e, h) = \sum_{i=1}^{k} \rho^*(u_i, v_i). \qquad (2)$$

In addition, if $\hat{\rho}_A(e, h) < 1$, then $l = 2k$, and one can choose $y_1, z_1, \cdots, y_k, z_k \in \{x_1, \cdots, x_n\}$ such that

$$h = (-y_1 + z_1) + \cdots + (-y_k + z_k) \qquad (3)$$

and

$$\hat{\rho}_A(e, h) = \sum_{i=1}^{k} \rho^*(y_i, z_i). \qquad (4).$$

Proof. Obviously, we have $h = h_1 + \cdots + h_l$, where $h_i \in \{\pm x_1, \cdots, \pm x_n\}$ for each $1 \leq i \leq l$. Obviously, there exists an integer k such that $2k - 1 \leq l \leq 2k$. Without loss of generality, we may assume that l is even. In fact, if $l = 2k - 1$, then one can additionally put $h_{2k} = e$. It follows from the proof of Lemma 4.1 (see [18]) that we have a similar assertion is valid for the case of $A_a(X)$. Then there exists an Abelian scheme φ on $\{1, 2, \cdots, 2k\}$ such that

$$\hat{\rho}_A(e, h) = \frac{1}{2} \sum_{i=1}^{2k} \rho^*(-h_i, h_{\varphi(i)}). \qquad (5)$$

Since the group $A_a(X)$ is Abelian, we may assume that $\varphi(2i - 1) = 2i$ for each $1 \leq i \leq k$. Obviously, $\varphi(2i) = 2i - 1$ for each $1 \leq i \leq k$. Hence, we have

$$h = (h_1 + h_2) \cdots + (h_{2k-1} + h_{2k}). \qquad (6)$$

For each $1 \leq i \leq k$, put $u_i = -h_{2i-1}$ and $v_i = h_{2i}$. Then it follows from (5) and (6) that (1) and (2) are true.

Finally, suppose that $\hat{\rho}_A(e, h) < 1$. Since $\rho(x, e) = 1$ and $\rho(e, x) = 1$, we have $\rho^*(x, e) = 1$, $\rho^*(e, x) = 1$, $\rho^*(-x, y) = 2$ and $\rho^*(x, -y) = 2$ for all $x, y \in X$. However, it follows from (5) that $\rho^*(-h_{2i-1}, h_{2i}) < 1$ for each $1 \leq i \leq k$, and therefore, one of the elements h_{2i-1}, h_{2i} in X while the other is in $-X$. Thus, for each $1 \leq i \leq k$, we have $h_{2i-1} + h_{2i} = -y_i + z_i$, where $y_i, z_i \in X$. Obviously, $y_i, z_i \in \{x_1, \cdots, x_n\}$ for each $1 \leq i \leq k$. Next, we only need to replace h_{2i-1} and h_{2i} by the corresponding elements $\pm y_i$ and $\pm z_i$ in (5) and (6), respectively. Hence we obtain (3) and (4). ∎

Lemma 4.6. *If d is a quasi-pseudometric on a set X quasi-uniform such that it is quasi-uniform with respect to $\mathscr{U}_X$, then d is continuous as a mapping from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$.*

Proof. Take an arbitrary point $(x_0, y_0) \in X \times X$. It is sufficient to show that d is continuous at the point (x_0, y_0). For each $\varepsilon > 0$, since d is quasi-uniform with respect to $\mathscr{U}_X$, there exists an $U \in \mathscr{U}_X$ such that $d(x, y) < \frac{\varepsilon}{2}$ for each $(x, y) \in U$.

Let $U_1 = \{x \in X : (x, x_0) \in U\}$ and $U_2 = \{y \in X : d(y_0, y) < \frac{\varepsilon}{2}\}$. Then U_1, U_2 are neighborhoods of the points x_0 and y_0 in the spaces $(X, \mathscr{U}_X^{-1})$ and $(X, \mathscr{U}_X)$ respectively. Put $V = U_1 \times U_2$. Then V is a neighborhood of the point (x_0, y_0) in $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$. For each $(x, y) \in V$, we have

$$\begin{aligned} d(x,y) - d(x_0,y_0) &\leq d(x,x_0) + d(x_0,y_0) + d(y_0,y) - d(x_0,y_0) \\ &= d(x,x_0) + d(y_0,y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, the quasi-pseudometric d is continuous at the point (x_0, y_0). ∎

Lemma 4.7. *[11] Let $\{V_i : i \in \mathbb{N}\}$ be a sequence of subsets of a group G with identity e such that $e \in V_i$ and $V_{i+1}^3 \subset V_i$ for each $i \in \mathbb{N}$. If $k_1, \cdots, k_n, r \in \mathbb{N}$ and $\sum_{i=1}^{n} 2^{-k_i} \leq 2^{-r}$, then we have $V_{k_1} \cdots V_{k_n} \subset V_r$.*

In the next theorem we prove that the family of quasi-pseudometrics $\{\hat{\rho}_A : \rho \in \mathscr{P}_X\}$, where $\mathscr{P}_X$ is the family of all continuous quasi-pseudometrics from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$, generates the topology of the free Abelian paratopological group $AP(X)$.

Theorem 4.8. *Let X be a Tychonoff space, and let $\mathscr{P}_X$ be the family of all continuous quasi-pseudometrics from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$ which are bounded by 1. Then the sets*

$$V_\rho = \{g \in AP(X) : \hat{\rho}_A(e, g) < 1\}$$

with $\rho \in \mathscr{P}_X$ form a local base at the neutral element e of $AP(X)$.

Proof. Let V be an open neighborhood of e in $AP(X)$. Since $AP(X)$ is a paratopological group, there exists a sequence $\{V_n : n \in \mathbb{N}\}$ of open neighborhoods of e in $AP(X)$ such that $V_1 \subset V$ and $V_{i+1} + V_{i+1} + V_{i+1} \subset V_i$ for every $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, put

$$U_n = \{(x, y) \in X \times X : -x + y \in V_n\}.$$

Then each U_n is an element of the universal quasi-uniformity $\mathscr{U}_X$ on the space X and $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset U_n$. Hence, it follows from Lemmas 4.3 and 4.6 that there is a continuous quasi-pseudometric ρ_1 on X such that, for each $n \in \mathbb{N}$,

$$\{(x, y) \in X \times X : \rho_1(x, y) < 2^{-n}\} \subset U_n.$$

Let $\rho = \min\{1, 4\rho_1\}$. Then $\rho \in \mathscr{P}_X$.

Claim: We have $V_\rho \subset V$.

Indeed, let $h \in V_\rho$. It follows from Lemma 4.5 that the element h can be written in the form

$$h = (-x_1 + y_1) + \cdots + (-x_m + y_m), \text{ where } x_i, y_i \in X \text{ for each } 1 \leq i \leq m,$$

such that

$$\hat{\rho}_A(e, h) = \rho(x_1, y_1) + \cdots + \rho(x_m, y_m) < 1.$$

It follows from the definition of ρ and ρ_1 that $\hat{\rho} = 4\hat{\rho}_1$. Therefore, we have

$$\hat{\rho}_1(e, h) = \rho_1(x_1, y_1) + \cdots + \rho_1(x_m, y_m) < \frac{1}{4}.$$

If $1 \leq i \leq m$ and $\rho_1(x_i, y_i) > 0$ then we choose a $k_i \in \mathbb{N}$ such that

$$2^{-k_i-1} \leq \rho_1(x_i, y_i) < 2^{-k_i}.$$

And then, if $1 \leq i \leq m$ and $\rho_1(x_i, y_i) = 0$ then we choose a sufficiently large $k_i \in \mathbb{N}$ such that $\sum_{i=1}^{m} 2^{-k_i} < \frac{1}{2}$. For every $1 \leq i \leq m$, since $-x_i + y_i \in V_{k_i}$, it follows from Lemma 4.7 that

$$h = (-x_1 + y_1) + \cdots + (-x_m + y_m) \in V_{k_1} + \cdots + V_{k_m} \subset V_1 \subset V.$$

Therefore, we have $V_\rho \subset V$. ■

We don't know whether a similar assertion is valid for the free paratopological group $FP(X)$.

However, we have the following Theorem 4.10. Our argument will be based on the following combinatorial lemma (Readers can consult the proof in [1, Lemma 7.2.8].).

Lemma 4.9. *Let $g = x_1 \cdots x_{2n}$ be a reduced element of $F_a(X)$, where $x_1, \cdots, x_{2n} \in X \cup X^{-1}$, and let φ be a scheme on $\{1, 2, \cdots, 2n\}$. Then there are natural numbers $1 \leq i_1 < \cdots < i_n \leq 2n$ and elements $h_1, \cdots, h_n \in F_a(X)$ satisfying the following two conditions:*
i) $\{i_1, \cdots, i_n\} \cup \{i_{\varphi(1)}, \cdots, i_{\varphi(n)}\} = \{1, 2, \cdots, 2n\}$;
ii) $g = (h_1 x_{i_1} x_{\varphi(i_1)} h_1^{-1}) \cdots (h_n x_{i_n} x_{\varphi(i_n)} h_n^{-1})$.

A paratopological group G has an *invariant basis* if there exists a family $\mathscr{L}$ of continuous and invariant quasi-pseudometric on G such that the family $\{U_\rho : \rho \in \mathscr{L}\}$ as a base at the neutral element e in G, where each $U_\rho = \{g \in G : \rho(e, g) < 1\}$.

Theorem 4.10. *For each Tychonoff space X, if the abstract group $F_a(X)$ admits the maximal paratopological group topology $\mathscr{F}_{inv}$ with invariant basis such that every continuous mapping $f : X \to H$ to a paratopological group H with invariant basis can be extended to a continuous homomorphism $\tilde{f} : (F_a(X), \mathscr{F}_{inv}) \to H$, then the family of all sets of the form*

$$U_\rho = \{g \in F_a(X) : \hat{\rho}(e, g) < 1\},$$

where ρ is a continuous quasi-pseudometric from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$, with $\rho \leq 1$ constitutes a base of the topology $\mathscr{F}_{inv}$ at the neutral element e of $F_a(X)$.

Proof. For each $\rho \in \mathscr{P}_X$, put

$$U_\rho = \{g \in F_a(X) : N_\rho(g) < 1\}, \text{ and } \mathscr{N} = \{U_\rho : \rho \in \mathscr{P}_X\},$$

where $N_\rho(g)$ is the invariant quasi-prenorm on $F_a(X)$ defined by $\hat{\rho}(g, h) = N_\rho(g^{-1}h)$. By Lemma 4.2 and Proposition 3.8 in [15], it is easy to see that $\mathscr{N}$ as a base at the neutral element e of $F_a(X)$ for a Hausdorff paratopological group topology. We denote this topology by $\mathscr{F}_{\text{inv}}$. Since $\hat{\rho}$ is invariant on $F_a(X)$, the paratopological group $FP_{\text{inv}}(X) = (FP(X), \mathscr{F}_{\text{inv}})$ has an invariant basis, and hence $\hat{\rho}$ is continuous on $FP_{\text{inv}}(X)$.

Let $f : X \to H$ be a continuous mapping of X to a paratopological group H with invariant basis. Let $\tilde{f}$ be the extension of f to a homomorphism of $F_a(X)$ to H.

Claim: The map $\tilde{f} : FP_{\text{inv}}(X) \to H$ is a continuous homomorphism.

Let V be an open neighborhood of the neutral element of H. Then there exists an invariant quasi-prenorm N on H such that $W = \{h \in H : N(h) < 1\} \subset V$ by Lemma 4.4. Therefore, we can define a quasi-pseudometric ρ on X by $\rho(x, y) = N(f^{-1}(x)f(y))$ for all $x, y \in X$. Next, we shall show that $\tilde{f}(U_\rho) \subset W$. Indeed, take an arbitrary reduced element $g \in U_\rho$ distinct from the neutral element of $F_a(X)$. Obviously, we have $\hat{\rho}(e, g) < 1$. Moreover, it is easy to see that g has even length, say $g = x_1 \cdots x_{2n}$, where $x_i \in X \cup X^{-1}$ for each $1 \leq i \leq 2n$. It follows from $\hat{\rho}(e, g) < 1$ that there is a scheme φ on $\{1, 2, \cdots, 2n\}$ such that

$$\hat{\rho}(e, g) = \frac{1}{2} \sum_{i=1}^{2n} \rho^*(x_i^{-1}, x_{\varphi(i)}) < 1.$$

By Lemma 4.9, we can find a partition $\{1, 2, \cdots, 2n\} = \{i_1, \cdots, i_n\} \cup \{i_{\varphi(1)}, \cdots, i_{\varphi(n)}\}$ and a representation of g as a product $g = g_1 \cdots g_n$ such that $g_k = h_k x_{i_k} x_{\varphi(i_k)} h_k^{-1}$ for each $k \leq n$, where $h_k \in F_a(X)$. Since N is invariant, we have

$$\begin{aligned} N(\tilde{f}(g)) &\leq \sum_{i=1}^{n} N(\tilde{f}(g_k)) = \sum_{i=1}^{n} N(\tilde{f}(x_k)\tilde{f}(x_{\varphi(k)})) \\ &= \rho^*(x_1^{-1}, x_{\varphi(1)}) + \cdots + \rho^*(x_n^{-1}, x_{\varphi(n)}) \\ &< 1. \end{aligned}$$

Therefore, we have $\tilde{f}(g) \in W$, and it follows that $\tilde{f}(U_\rho) \subset W \subset V$. Hence $\tilde{f}$ is a continuous homomorphism. ∎

5 Topological monomorphisms between free paratopological groups

In order to prove one of our main theorems, we also need the following lemma.

Lemma 5.1. *Let $(X, \mathscr{U}_X)$ be a quasi-uniform subspace of a Tychonoff space $(Y, \mathscr{U}_Y)$. Then X is quasi-P^*-embedded in Y.*

Proof. Let d be a bounded, continuous quasi-pseudometric from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$. One can assume that d is bounded by $\frac{1}{2}$. For each $i \in \mathbb{N}$, take a $V_i \in \mathscr{U}_Y$ satisfying $V_i \cap (X \times X) \subset \{(x, y) \in X \times X : d(x, y) < \frac{1}{2^i}\}$, and then by [20, Chap. 3, Proposition 2.4 and Theorem 2.5], take a continuous quasi-pseudometric d_i from $(Y \times Y, \mathscr{U}_Y^{-1} \times \mathscr{U}_Y)$ to $(\mathbb{R}, \mathscr{U}^\star)$ such that d_i is bounded by 1, quasi-uniform with respect to $\mathscr{U}_Y$ and $\{(x, y) \in Y \times Y : d_i(x, y) < \frac{1}{4}\} \subset V_i$. Put

$$\rho(x, y) = 8 \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x, y).$$

One can easily prove that ρ is a continuous quasi-pseudometric from $(Y \times Y, \mathscr{U}_Y^{-1} \times \mathscr{U}_Y)$ to $(\mathbb{R}, \mathscr{U}^\star)$. Moreover, it is easy to see that ρ is quasi-uniform with respect to $\mathscr{U}_Y$ and satisfies $d(x,y) \leq \rho(x,y)$ for all $x, y \in X$. Put

$$\rho'(x,y) = \inf\{\rho(x,a) + d(a,b) + \rho(b,y) : a, b \in X\}, \text{ where } x, y \in Y.$$

Let

$$\tilde{d} = \min\{\rho(x,y), \rho'(x,y)\}.$$

Obviously, $\tilde{d}$ is quasi-uniform with respect to $\mathscr{U}_Y$. It follows from Lemma 4.6 that $\tilde{d}$ is a continuous quasi-pseudometric from $(Y \times Y, \mathscr{U}_Y^{-1} \times \mathscr{U}_Y)$ to $(\mathbb{R}, \mathscr{U}^\star)$. Moreover, we have $\tilde{d}|X \times X = d$. Therefore, X is quasi-P*-embedded in Y. ∎

Now, we shall prove our main theorem, which gives an affirmative answer to Question 3.4.

Theorem 5.2. *Let X be an arbitrary subspace of a Tychonoff space Y. Then the natural mapping $\hat{e}_{X,Y} : AP(X) \to AP(Y)$ is a topological monomorphism if and only if X is quasi-P*-embedded in Y.*

Proof. Necessity. Let d be an arbitrary bounded continuous quasi-pseudometric from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$, where $\mathscr{U}_X$ is the universal quasi-uniformity on X. Then $U_d = \{(x,y) \in X \times X : d(x,y) < 1\} \in \mathscr{U}_X$. Put $V_d = \{g \in AP(X) : \hat{d}(e,g) < 1\}$. Then V_d is a neighborhood of the neutral element of $AP(X)$. Since $AP(X) \subset AP(Y)$, it follows from Theorem 4.8 that there is some continuous quasi-pseudometric ρ from $(Y \times Y, \mathscr{U}_Y^{-1} \times \mathscr{U}_Y)$ to $(\mathbb{R}, \mathscr{U}^\star)$ such that $V_\rho \cap AP(X) \subset V_d$, where $\mathscr{U}_Y$ is the universal quasi-uniformity on Y and $V_\rho = \{g \in AP(Y) : \hat{\rho}(e,g) < 1\}$. Note that $U_\rho = \{(x,y) \in Y \times Y : \rho(x,y) < 1\} \in \mathscr{U}_Y$ and $U_\rho \cap (X \times X) \subset U_d$. Moreover, one can see that $\hat{\rho}(e, x^{-1}y) = \rho(x,y)$ and $\hat{d}(e, x^{-1}y) = d(x,y)$ for all x, y. Therefore, $(X, \mathscr{U}_X)$ is a quasi-uniform subspace of $(Y, \mathscr{U}_Y)$. Hence X is quasi-P*-embedded in Y by Lemma 5.1.

Sufficiency. Let X be quasi-P*-embedded in Y. Denote by $e_{X,Y}$ the identity embedding of X in Y. Obviously, the monomorphism $\hat{e}_{X,Y}$ is continuous. Next, we need to show that the isomorphism $\hat{e}_{X,Y}^{-1} : AP(X,Y) \to AP(X)$ is continuous. Assume that U is a neighborhood of the neutral element e_X in $AP(X)$. It follows from Theorem 4.8 that there is a continuous quasi-pseudometric ρ from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$ such that $V_\rho = \{g \in AP(X) : \hat{\rho}_A(e_X, g) < 1\} \subset U$. Without loss of generality, we may assume that $\rho \leq 1$ (otherwise, replace ρ with $\rho' = \min\{\rho, 1\}$). Since X is quasi-P*-embedded in Y, the quasi-pseudometric ρ can be extended to a continuous quasi-pseudometric d from $(Y \times Y, \mathscr{U}_Y^{-1} \times \mathscr{U}_Y)$ to $(\mathbb{R}, \mathscr{U}^\star)$. Suppose that $\hat{d}_A$ is the Graev extension of d over $AP(Y)$. It follows from Theorem 4.8 again that $V_d = \{g \in AP(Y) : \hat{d}_A(e_Y, g) < 1\}$ is an open neighborhood of the neutral element e_Y in $AP(Y)$. Obviously, one can identify the abstract group $A_a(X)$ with the subgroup $\hat{e}_{X,Y}(A_a(X)) = A_a(X,Y)$ of $A_a(Y)$ generated by the subset X of $A_a(Y)$. Since $d \mid X = \rho$, it follows from Lemma 4.5 that, for each $h \in A_a(X,Y)$, $\hat{d}_A(e_Y, h) = \hat{\rho}_A(e_Y, h)$. Hence we have $A_a(X,Y) \cap V_d = V_\rho$, that is, $AP(X,Y) \cap V_d = \hat{e}_{X,Y}(V_\rho)$. Therefore, the isomorphism $\hat{e}_{X,Y}^{-1} : AP(X,Y) \to AP(X)$ is continuous. ∎

In order to give a partial answer to Question 3.3, we need to prove some lemmas.

Lemma 5.3. *Let X be a Tychonoff space. Then the restriction $\mathscr{G}_X = \mathscr{G}_{PG(X)} \mid X$ of the left uniformity $\mathscr{G}_{PG(X)}$ of the paratopological group $PG(X)$ to the subspace $X \subset PG(X)$ coincides with the universal quasi-uniformity $\mathscr{U}_X$ of X.*

Proof. Since the topology on X generated by the left uniformity $\mathscr{G}_{PG(X)}$ of $PG(X)$ coincides with the original topology of the space X, we have $\mathscr{G}_X \subset \mathscr{U}_X$. Next, we need to show that $\mathscr{U}_X \subset \mathscr{G}_X$. Take an arbitrary element $U \in \mathscr{U}_X$. It follows from Lemmas 4.4 and 4.6 that there exists a continuous quasi-pseudometric ρ from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$ such that $\{(x,y) \in X \times X : \rho(x,y) < 1\} \subset U$. By Theorem 3.2 in [18], the quasi-pseudometric ρ on set X extends to a left invariant quasi-pseudometric $\hat{\rho}$ on the abstract group $PG(X)$. One can see that $\hat{\rho}$ is continuous from $(PG(X) \times PG(X), \mathscr{U}_{PG(X)} \times \mathscr{U}_{PG(X)}^{-1})$ to $(\mathbb{R}, \mathscr{U}^\star)$. It follows from Theorem 4.8 that $V = \{g \in PG(X) : \hat{\rho}(e,g) < 1\}$ is an open neighborhood of the neutral element e in $PG(X)$. If $x, y \in X$ and $x^{-1}y \in V$, then

$$\rho(x,y) = \hat{\rho}(x,y) = \hat{\rho}(e, x^{-1}y) < 1,$$

which implies that the element $W_V^l = \{(g,h) \in G \times G : g^{-1}h \in V\}$ of $\mathscr{G}_{PG(X)}$ satisfies $W_V^l \cap (X \times X) \subset U$. Therefore, $\mathscr{U}_X \subset \mathscr{G}_X$. ∎

Lemma 5.4. *[17] The finest quasi-uniformity of each quasi-pseudometrizable topological space is bicomplete.*

Lemma 5.5. *Let X be a subspace of a Tychonoff space Y, and let X be $\tau(\tilde{\mathscr{U}_Y}^*)$-dense in $(\tilde{Y}, \tilde{\mathscr{U}_Y})$, where $\mathscr{U}_Y$ is the universal quasi-uniformity and $(\tilde{Y}, \tilde{\mathscr{U}_Y})$ is the bicompletion of $(Y, \mathscr{U}_Y)$. Then the following conditions are equivalent:*

1. *X is quasi-P^*-embedded in Y;*
2. *X is quasi-P-embedded in Y;*
3. *$\mathscr{U}_Y \mid X = \mathscr{U}_X$;*
4. *$X \subset Y \subset \tilde{X}$, where $(\tilde{X}, \tilde{\mathscr{U}_X})$ is the bicompletion of $(X, \mathscr{U}_X)$.*

Proof. Obviously, (2) $\Rightarrow$ (1). Hence it is suffices to show that (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2).

(1) $\Rightarrow$ (3). Assume that X is quasi-P*-embedded in Y. For each $U \in \mathscr{U}_X$, it follows from Lemmas 4.4 and 4.6 that there exists a bounded continuous quasi-pseudometric ρ_X from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$ such that

$$W_X = \{(x,x') \in X \times X : \rho_X(x,x') < 1\} \subset U.$$

Since X is quasi-P*-embedded in Y, let ρ_Y be an extension of ρ_X to a continuous quasi-pseudometric from $(Y \times Y, \mathscr{U}_Y^{-1} \times \mathscr{U}_Y)$ to $(\mathbb{R}, \mathscr{U}^\star)$. Put

$$W_Y = \{(y,y') \in Y \times Y : \rho_Y(y,y') < 1\}.$$

Then it is obvious that $W_Y \in \mathscr{U}_Y$ and $W_Y \cap (X \times X) = W_X \subset U$. Therefore, the quasi-uniformity $\mathscr{U}_Y \mid X$ is finer than $\mathscr{U}_X$. Moreover, it is clear that $\mathscr{U}_Y \mid X \subset \mathscr{U}_X$. Hence $\mathscr{U}_Y \mid X = \mathscr{U}_X$.

(3) $\Rightarrow$ (4). Assume that $\mathscr{U}_Y \mid X = \mathscr{U}_X$. Let $(\tilde{Y}, \tilde{\mathscr{U}}_Y)$ be the bicompletion of quasi-uniform space $(Y, \mathscr{U}_Y)$. Because $\tilde{\mathscr{U}}_Y \mid Y = \mathscr{U}_Y$, we have $\tilde{\mathscr{U}}_Y \mid X = \mathscr{U}_X$. Moreover, since X is $\tau(\tilde{\mathscr{U}}_Y{}^*)$-dense in $\tilde{Y}$ and $X \subset Y$, $(\tilde{Y}, \tilde{\mathscr{U}}_Y)$ is the bicompletion of the quasi-uniform space $(X, \mathscr{U}_X)$. Hence $X \subset Y \subset \tilde{X}$.

(4) $\Rightarrow$ (2). Assume that $Y \subset \tilde{X}$. Consider an arbitrary continuous quasi-pseudometric ρ from $(X \times X, \mathscr{U}_X^{-1} \times \mathscr{U}_X)$ to $(\mathbb{R}, \mathscr{U}^\star)$. Let $(\overline{X}, \overline{\rho})$ be the quasi-metric space obtained from (X, ρ) by identifying the points of X lying at zero distance one from another with respect to ρ. Let $\pi : X \to \overline{X}$ be the natural quotient mapping. Obviously, $\rho(x, y) = \overline{\rho}(\pi(x), \pi(y))$ for all $x, y \in X$. Suppose that $\mathscr{U}_{\overline{X}}$ is the universal quasi-uniformity on $\overline{X}$. Then π is a quasi-uniformly continuous map from $(X, \mathscr{U}_X)$ to $(\overline{X}, \mathscr{U}_{\overline{X}})$ by [2]. Moreover, by Lemma 5.4, $(\overline{X}, \mathscr{U}_{\overline{X}})$ is bicomplete. Therefore, it follows from Theorem 16 in [8] that π admits a quasi-uniformly continuous extension $\overline{\pi} : (\tilde{X}, \tilde{\mathscr{U}}_X) \to (\overline{X}, \mathscr{U}_{\overline{X}})$. Since $Y \subset \tilde{X}$, we can define a continuous mapping of d from $(Y \times Y, \mathscr{U}_Y^{-1} \times \mathscr{U}_Y)$ to $(\mathbb{R}, \mathscr{U}^\star)$ by $d(x, y) = \overline{\rho}(\overline{\pi}(x), \overline{\pi}(y))$ for all $x, y \in Y$. Clearly, the restriction of d to X coincides with ρ. Hence X is quasi-P-embedded in Y. ■

Theorem 5.6. *Let X be an arbitrary $\tau(\tilde{\mathscr{U}}_Y{}^*)$-dense subspace of a Tychonoff space Y. If the natural mapping $\hat{e}_{X,Y} : FP(X) \to FP(Y)$ is a topological monomorphism, then X is quasi-P-embedded in Y.*

Proof. Assume that the monomorphism $\hat{e}_{X,Y} : FP(X) \to FP(Y)$ extending the identity mapping $e_{X,Y} : X \to Y$ is a topological embedding. Therefore, it is easy to see that we can identify the group $FP(X)$ with the subgroup $FP(X, Y)$ of $FP(Y)$ generated by the set X. We denote by $\mathscr{G}_X$ and $\mathscr{G}_Y$ the left quasi-uniformities of the groups $FP(X)$ and $FP(Y)$, respectively. Since $FP(X)$ is a subgroup of $FP(Y)$, we obtain that $\mathscr{G}_Y \mid FP(X) = \mathscr{G}_X$. Moreover, it follows from Lemma 5.3 that $\mathscr{G}_X \mid X = \mathscr{U}_X$ and $\mathscr{G}_Y \mid Y = \mathscr{U}_Y$. Hence we have

$$\mathscr{G}_Y \mid X = \mathscr{G}_X \mid X = \mathscr{U}_X.$$

Therefore, it follows from Lemma 5.5 that X is quasi-P-embedded in Y. ■

Question 5.7. *Let X be an arbitrary $\tau(\tilde{\mathscr{U}}_Y{}^*)$-dense subspace of a Tychonoff space Y. If X is quasi-P-embedded in Y, is the natural mapping $\hat{e}_{X,Y} : FP(X) \to FP(Y)$ a topological monomorphism?*

In [21], O.V. Sipacheva has proved that if Y is a subspace of a Tychonoff X then the subgroup $F(Y, X)$ of $F(X)$ is topologically isomorphic to $F(Y)$ iff Y is P^*-embedded in X. In [22], M.G. Tkackenko has proved that if Y is a subspace of a Tychonoff X then the subgroup $A(Y, X)$ of $A(X)$ is topologically isomorphic to $A(Y)$ iff Y is P^*-embedded in X.Therefore, we have the following question:

Question 5.8. *Let X be an arbitrary subspace of a Tychonoff space Y. Is it true that the subgroup $PG(Y, X)$ of $PG(X)$ is topologically isomorphic to $PG(Y)$ iff Y is quasi-P^*-embedded in X?*

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