On harmonic combination of univalent functions

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Abstract

Let $\mathcal{S}$ be the class of all functions f that are analytic and univalent in the unit disk $\mathbb{D}$ with the normalization $f(0) = f'(0) - 1 = 0$. Let $\mathcal{U}(\lambda)$ denote the set of all $f \in \mathcal{S}$ satisfying the condition

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| < \lambda \ \text{ for } z \in \mathbb{D},$$

and some $\lambda \in (0, 1]$. In this paper, among other things, we study a "harmonic mean" of two univalent analytic functions. More precisely, we discuss the properties of the class of functions F of the form

$$\frac{z}{F(z)} = \frac{1}{2} \left(\frac{z}{f(z)} + \frac{z}{g(z)} \right),$$

where $f, g \in \mathcal{S}$ or $f, g \in \mathcal{U}(1)$. In particular, we determine the radius of univalency of F, and propose two conjectures concerning the univalency of F.

1 Introduction and Main Results

For each $r > 0$, we denote by $\mathbb{D}_r$ the open disk $\{z \in \mathbb{C} : |z| < r\}$ and by $\mathbb{D}$ the unit disk $\mathbb{D}_1$. Let $\mathcal{A}$ be the class of all functions f that are analytic in $\mathbb{D}$ with the normalization $f(0) = f'(0) - 1 = 0$. Denote by $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{C}$, the subfamilies

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of $\mathcal{A}$ that are, respectively, univalent, starlike, convex, and close-to-convex in $\mathbb{D}$ (see [2, 3] for some detailed discussion on these classes). It is well-known that a function $f \in \mathcal{S}$ is starlike if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{for all } z \in \mathbb{D}.$$

Similarly, a function $f \in \mathcal{S}$ is close-to-convex if

$$\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0 \quad \text{for all } z \in \mathbb{D}$$

for some $g \in \mathcal{S}^*$. In [5], Mitrinović essentially investigated certain geometric properties of the functions f of the form

$$f(z) = \frac{z}{\phi(z)}, \quad \phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n. \tag{1}$$

In [11], Reade et al. derived coefficient conditions that guarantee the univalence, starlikeness, or convexity of rational functions of the form (1). These results have been improved and generalized by the authors in [7]. In connection with a problem due to [4], several authors (eg. [12]) discussed the univalency of functions in the set of convex linear combinations of the form

$$\mu f(z) + (1-\mu)g(z), \ \mu \in [0,1],$$

when f, g belonging to suitable subsets of $\mathcal{S}$. In this paper, we shall consider a similar problem for univalent functions f of the form (1).

Let $\mathcal{U}(\lambda)$ denote the set of all $f \in \mathcal{A}$ in $\mathbb{D}$ satisfying the condition ([6, 8])

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| < \lambda, \ \text{ for } z \in \mathbb{D} \tag{2}$$

and some $\lambda \in (0,1]$. Functions in $\mathcal{U}(1) =: \mathcal{U}$ is known to be univalent in $\mathbb{D}$, see [1, 10]. Clearly, $\mathcal{U}(\lambda) \subset \mathcal{U}$ for $\lambda \in (0,1]$ and so, functions in $\mathcal{U}(\lambda)$ are univalent in $\mathbb{D}$. Set

$$\mathcal{U}_2(\lambda) = \{f \in \mathcal{U}(\lambda) : \ f''(0) = 0\}.$$

For convenience, we may also let $\mathcal{U}_2 = \mathcal{U}_2(0)$. It is known (eg. [8]) that functions in $\mathcal{U}_2$ are included in the class $\mathcal{P}(1/2)$, where

$$\mathcal{P}(1/2) = \{f \in \mathcal{A} : \ \mathrm{Re}\,(f(z)/z) > 1/2 \ \text{ for } z \in \mathbb{D}\}.$$

We remark that $\mathcal{K} \subset \mathcal{P}(1/2)$ and there exist functions f in $\mathcal{S}$ such that $f \notin \mathcal{U}$.

It is convenient to say that f belongs to $\mathcal{U}(\lambda)$ in the disk $|z| < r$ if the inequality in (2) holds for $|z| < r$ instead of the whole unit disk $\mathbb{D}$. For instance, if $\lambda = 1$, this is equivalent to saying that g defined by $g(z) = r^{-1}f(rz)$ belongs to $\mathcal{U}$, whenever f belongs to $\mathcal{U}$ in the disk $|z| < r$. Similar terminology will be followed for other related classes of functions, eg., starlike functions in $|z| < r$.

Now, we state our main results.

Theorem 1. *Let $f, g \in \mathcal{S}$. Suppose that $\frac{f(z)+g(z)}{z} \neq 0$ for $z \in \mathbb{D}$ and consider the function F defined by*

$$F(z) = \frac{2f(z)g(z)}{f(z)+g(z)}. \tag{3}$$

Then G, defined by $G(z) = r^{-1}F(rz)$, belongs to $\mathcal{U}(\lambda)$ for $0 < r \leq \sqrt{\lambda/(1+\lambda)}$. In particular, F belongs to $\mathcal{U}$ in the disk $|z| < 1/\sqrt{2} \approx 0.707107$ (and hence, F is univalent in $\mathbb{D}_{1/\sqrt{2}}$). In addition, $r^{-1}F(rz)$ belongs $\mathcal{S}^*$ for

$$0 < r \leq r_0 = \sqrt{1 - (2/(4 - |b_1 + c_1|))},$$

where $b_1 + c_1 = -(f''(0) + g''(0))/2 = -F''(0)$.

If $b_1 + c_1 = 0$, then from Theorem 1 we obtain that G defined $G(z) = r^{-1}F(rz)$ belongs to $\mathcal{U} \cap \mathcal{S}^*$ whenever $0 < r \leq 1/\sqrt{2}$. Moreover, since $\mathcal{U} \subsetneq \mathcal{S}$, it is natural to prove an analog of Theorem 1 by replacing the assumption $f, g \in \mathcal{S}$ by $f, g \in \mathcal{U}$. Now, we are in a position to state our next result.

Theorem 2. *Let $f \in \mathcal{U}(\lambda_1)$, $g \in \mathcal{U}(\lambda_2)$ $(0 < \lambda_1, \lambda_2 \leq 1)$ and $\frac{f(z)+g(z)}{z} \neq 0$ for $z \in \mathbb{D}$. Define F by (3) and G by $G(z) = r^{-1}F(rz)$. Then G belongs to $\mathcal{U}(\lambda)$ whenever*

$$0 < r \leq \sqrt{\frac{-K^2 + K\sqrt{K^2+4}}{2}} \ \ with \ \ K = \sqrt{2\lambda^2/(\lambda_1+\lambda_2)}. \tag{4}$$

In particular, if $f, g \in \mathcal{U}$, then $G \in \mathcal{U}$ for $0 < r \leq \sqrt{\frac{\sqrt{5}-1}{2}}$; that is F is univalent in the disk $|z| < \sqrt{\frac{\sqrt{5}-1}{2}} \approx 0.78615$.

At this place, it is appropriate to present a two parameters family of analytic functions dealing with a number of issues concerning our investigation.

Example 1. For $0 \neq \alpha \in [-1, 1]$, we consider

$$f_\alpha(z) = \frac{z(1-\alpha z)}{1-z^2}.$$

By a computation, we obtain that

$$(1-z^2)f'_\alpha(z) = \frac{1+z^2-2\alpha z}{1-z^2}$$

and so,

$$\mathrm{Re}\left((1-z^2)f'_\alpha(z)\right) = \frac{(1-|z|^2)(1+|z|^2-2\alpha \mathrm{Re}\, z)}{|1-z^2|^2} > 0 \ \text{ for } z \in \mathbb{D}.$$

We conclude that for each α, the function f_α is close-to-convex in $\mathbb{D}$. Now, let $F(z) = F_{\alpha,\beta}(z)$ in the unit disk $\mathbb{D}$ be defined by

$$\frac{z}{F(z)} = \frac{1}{2}\left(\frac{z}{f_\alpha(z)} + \frac{z}{f_\beta(z)}\right)$$

where $\alpha, \beta \in [-1,1]\backslash\{0\}$. A computation gives

$$F(z) = \frac{z(1-\alpha z)(1-\beta z)}{(1-z^2)(1-((\alpha+\beta)/2)z)}$$

and

$$\frac{z}{F(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

where

$$b_1 = \frac{\alpha+\beta}{2} \text{ and } b_n = -\frac{1}{2}\left(\alpha^{n-2}(1-\alpha^2) + \beta^{n-2}(1-\beta^2)\right) \text{ for } n \geq 2.$$

First we wish to show that

$$S := \sum_{n=2}^{\infty} (n-1)|b_n|^2 \leq 1$$

which is a necessary condition for F to belong to $\mathcal{S}$ (see the well-known Area Theorem [3, Theorem 11 on p.193 of Vol. 2]). As

$$4|b_n|^2 = \alpha^{2(n-2)}(1-\alpha^2)^2 + \beta^{2(n-2)}(1-\beta^2)^2 + 2(1-\alpha^2)(1-\beta^2)(\alpha\beta)^{n-2}$$

for $n \geq 2$ and

$$\sum_{n=2}^{\infty} (n-1)x^{n-2} = \frac{1}{(1-x)^2} \text{ for } |x| < 1,$$

it follows easily that

$$S = \frac{1}{4}\left(1 + 1 + \frac{2(1-\alpha^2)(1-\beta^2)}{(1-\alpha\beta)^2}\right) \leq 1$$

and the equality holds if $\alpha = \beta$. Thus, F satisfies the necessary condition for F to belong to the class $\mathcal{S}$ whenever $\alpha, \beta \in [-1,1]\backslash\{0\}$. On the other hand for certain values of α, β these functions $F = F_{\alpha,\beta}$ belong neither to $\mathcal{U}$ nor to $\mathcal{S}^*$.

We note that $z/F(z) \neq 0$ in $\mathbb{D}$ and, by a lengthy computation, we obtain that

$$\left(\frac{z}{F(z)}\right)^2 F'(z) - 1 = z^2 \frac{(1-\alpha\beta)(1-\alpha z)(1-\beta z) - (1-z^2)((\alpha-\beta)^2/2)}{(1-\alpha z)^2(1-\beta z)^2}$$

Setting $\beta = -\alpha$, we see that for the function $F_\alpha(z) := F_{\alpha,-\alpha}(z)$, we have

$$\left(\frac{z}{F_\alpha(z)}\right)^2 F'_\alpha(z) - 1 = -(1-\alpha^2)\frac{z^2(1+\alpha^2 z^2)}{(1-\alpha^2 z^2)^2}.$$

Therefore, we have

$$\left|\left(\frac{z}{F_\alpha(z)}\right)^2 F'_\alpha(z) - 1\right| \leq (1-\alpha^2)\frac{|z|^2(1+\alpha^2|z|^2)}{(1-\alpha^2|z|^2)^2}$$

which is less than 1 whenever

$$(2\alpha^2-1)\alpha^2|z|^4-(1+\alpha^2)|z|^2+1>0.$$

Solving the last inequality gives the condition $|z|^2<r_{\mathcal{U}}^2$, where

$$r_{\mathcal{U}}=\sqrt{\frac{2}{1+\alpha^2+\sqrt{(1-\alpha^2)(7\alpha^2+1)}}}.$$

The above discussion shows that

$$\left|\left(\frac{z}{F_\alpha(z)}\right)^2F_\alpha'(z)-1\right|<1 \text{ for } |z|<r_{\mathcal{U}}.$$

Also, for $z=r$, where $r_{\mathcal{U}}\leq r<1$, we have that

$$\left|\left(\frac{z}{F_\alpha(z)}\right)^2F_\alpha'(z)-1\right|=(1-\alpha^2)\frac{r^2(1+\alpha^2r^2)}{(1-\alpha^2r^2)^2}\geq 1,$$

showing that the function F_α is in the class $\mathcal{U}$ in the disk $|z|<r_{\mathcal{U}}$ (so F_α is univalent in this disk) but not in any larger disk. That is, $r^{-1}F_\alpha(rz)$ belongs to $\mathcal{U}$ for $0<r\leq r_{\mathcal{U}}$, but not for a larger value of r.

Next, we show that for certain values of α,β, the functions $F=F_{\alpha,\beta}$ are not starlike in the unit disk $\mathbb{D}$. A straightforward computation shows that

$$F'(z)=\frac{M(z)}{(1-z^2)^2(1-((\alpha+\beta)/2)z)^2},$$

where

$$\begin{aligned} M(z) \;&=\; 1-2(\alpha+\beta)z+(1+3\alpha\beta+(\alpha+\beta)^2/2)z^2 \\ &\qquad -(\alpha+\beta)(1+\alpha\beta)z^3+((\alpha^2+\beta^2)/2)z^4. \end{aligned}$$

If $\beta=-\alpha$ with $|\alpha|>1/9$ then in this case $M(z)$ takes the form

$$M(z)=\alpha^2[z^2+A][z^2+\overline{A}],\quad A=\frac{1-3\alpha^2+i\sqrt{(9\alpha^2-1)(1-\alpha^2)}}{2\alpha^2}$$

and we see that $|A|\geq 1$ showing that $F_\alpha'(z)\neq 0$ in $\mathbb{D}$.

Also, if $|\alpha|\leq 1/9$, then we see that

$$M(z)=\alpha^2[z^2+B_+][z^2+B_-]$$

where

$$B_\pm=\frac{1-3\alpha^2\pm\sqrt{(1-9\alpha^2)(1-\alpha^2)}}{2\alpha^2}\geq 1.$$

Again, we see that $F_\alpha'(z)\neq 0$ in $\mathbb{D}$. Thus, F_α is locally univalent in $\mathbb{D}$.

On the other hand, it follows easily that

$$\frac{zF_\alpha'(z)}{F_\alpha(z)}=\frac{1+(1-3\alpha^2)z^2+\alpha^2z^4}{(1-\alpha^2z^2)(1-z^2)}.$$

A straightforward computation shows that for $0 < \theta < \pi$,

$$\mathrm{Re}\left(\frac{e^{i\theta}F'_\alpha(e^{i\theta})}{F_\alpha(e^{i\theta})}\right) = \frac{A(\theta)}{|1-\alpha^2e^{2i\theta}|^2\,|1-e^{2i\theta}|^2},$$

where

$$A(\theta) = 4\alpha^2(\alpha^2 - \cos 2\theta)(1-\cos 2\theta).$$

Therefore, $A(\theta) < 0$ if $0 < \alpha^2 < \cos 2\theta < 1$, i.e. $|\theta| < (1/2)\arccos(\alpha^2) < \pi/4$. This observation shows that for each $\alpha \in (-1,1)\backslash\{0\}$, the function F_α is not starlike in $\mathbb{D}$ although F_α is locally univalent in $\mathbb{D}$.

The above example motivates the following conjectures

Conjecture 1. **(a)** *The function F defined by* (3) *is not necessarily univalent in $\mathbb{D}$ whenever $f, g \in \mathcal{S}$ such that $((f(z)+g(z))/z) \neq 0$ in $\mathbb{D}$.*

(b) *The function F defined by* (3) *is univalent in $\mathbb{D}$ whenever $f, g \in \mathcal{C}$ such that $((f(z)+g(z))/z) \neq 0$ in $\mathbb{D}$.*

Theorem 1 may be generalized in the following form.

Theorem 3. *Let $f_k \in \mathcal{S}$ for $k = 1, \dots, m$ and $\sum_{k=1}^m \frac{z}{f_k(z)} \neq 0$ for $z \in \mathbb{D}$. Define F by*

$$\frac{z}{F(z)} = \frac{1}{m}\sum_{k=1}^m \frac{z}{f_k(z)}. \tag{5}$$

Then we have

(a) *G defined by $G(z) = r^{-1}F(rz)$ belongs to $\mathcal{U}(\lambda)$ for $0 < r \leq \sqrt{\lambda/(1+\lambda)}$. In particular, F is univalent in the disk $|z| < 1/\sqrt{2}$.*

(b) *G belongs to $\mathcal{S}^*$ for $0 < r \leq \sqrt{\lambda/(1+\lambda)}$, with*

$$\lambda = 1 - \frac{1}{m}\left|\sum_{k=1}^m \frac{f_k''(0)}{2}\right|.$$

In particular, F is starlike (univalent) in the disk $|z| < 1/\sqrt{2}$ whenever $f_k''(0) = 0$ for each $k = 1, \dots, m$.

The idea of the proof of Theorem 2 can be used to prove the following general result.

Theorem 4. *Let $f_k \in \mathcal{U}(\lambda_k)$ $(0 < \lambda_k \leq 1)$ for $k = 1, \dots, m$, $\sum_{k=1}^m \frac{z}{f_k(z)} \neq 0$ for $z \in \mathbb{D}$ and F be defined by* (5)*. Then G defined by $G(z) = r^{-1}F(rz)$ belongs to $\mathcal{U}(\lambda)$ whenever*

$$0 < r \leq \sqrt{\frac{-K^2 + K\sqrt{K^2+4}}{2}} \quad \text{with } K = \sqrt{\frac{m\lambda^2}{\sum_{k=1}^m \lambda_k}}. \tag{6}$$

In particular, if $f_k \in \mathcal{U}$ for $k = 1, \dots, m$, then $G \in \mathcal{U}$ for $0 < r \leq \sqrt{\frac{\sqrt{5}-1}{2}}$; that is F is univalent in the disk $|z| < \sqrt{\frac{\sqrt{5}-1}{2}}$.

The proof of Theorems 1, 2, 3, and 4 are presented in Section 3.

2 Preliminary Lemmas

For the proofs of our results, we need the following lemmas.

Lemma 1. *Let $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a non-vanishing analytic function on $\mathbb{D}$ and let f be of the form* (1)*. Then, we have the following:*

(a) *If $\sum_{n=2}^{\infty}(n-1)|b_n| \leq \lambda$, then $f \in \mathcal{U}(\lambda)$.*

(b) *If $\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1 - |b_1|$, then $f \in \mathcal{S}^*$.*

(c) *If $f \in \mathcal{U}(\lambda)$, then $\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \leq \lambda^2$.*

The conclusion **(a)** in Lemma 1 is from [6, 7] whereas the **(b)** is due to Reade et al. [11, Theorem 1]. Finally, as $f \in \mathcal{U}(\lambda)$, we have

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 \right| = \left| -z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1 \right| = \left| \sum_{n=2}^{\infty}(n-1)b_n z^n \right| \leq \lambda$$

and so **(c)** follows from Prawitz' theorem which is an immediate consequence of Gronwall's area theorem. This may be also obtained as a consequence of Parseval's relation.

Next we recall the following result due to Obradović and Ponnusamy [9].

Lemma 2. *Let $f \in \mathcal{A}$ have the form*

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots \quad \text{with } b_n \geq 0 \text{ for all } n \geq 2 \tag{7}$$

and for all z in a neighborhood of $z = 0$. Then the following conditions are equivalent.

(a) *$f \in \mathcal{S}$,*

(b) *$\dfrac{f(z)f'(z)}{z} \neq 0$ for $z \in \mathbb{D}$,*

(c) *$\displaystyle\sum_{n=2}^{\infty}(n-1)b_n \leq 1$,*

(d) *$f \in \mathcal{U}$.*

This lemma helps to compare the relation between results here and the earlier work of the authors in [9], in particular.

3 Proofs

Proof of Theorem 1. Let $f, g \in \mathcal{S}$. Then f and g can be written in the form

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots \quad \text{and} \quad \frac{z}{g(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{8}$$

Further, as $f, g \in \mathcal{S}$, the well-known Gronwall's Area Theorem [3, Theorem 11 on p.193 of Vol. 2] gives

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2 \leq 1 \text{ and } \sum_{n=2}^{\infty}(n-1)|c_n|^2 \leq 1. \tag{9}$$

From (3), we may rewrite F in the form

$$\frac{z}{F(z)} = \frac{1}{2}\left(\frac{z}{f(z)} + \frac{z}{g(z)}\right) = 1 + \sum_{n=1}^{\infty}\frac{b_n + c_n}{2}z^n.$$

For $0 < r \leq 1$, we define G by $G(z) = r^{-1}F(rz)$ so that

$$\frac{z}{G(z)} = \frac{z}{r^{-1}F(rz)} = 1 + \sum_{n=1}^{\infty}\frac{b_n + c_n}{2}r^n z^n.$$

In order to prove that F is univalent in $|z| < 1/\sqrt{2}$, it suffices to show that $G \in \mathcal{U}$ for $0 < r \leq 1/\sqrt{2}$. According to Lemma 1**(a)** (compare with Lemma 2**(c)**), it suffices to show that

$$S := \sum_{n=2}^{\infty}(n-1)\left|\frac{b_n + c_n}{2}\right| r^n \leq 1 \tag{10}$$

for $0 < r \leq 1/\sqrt{2}$. By (9) and the Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty}(n-1)|b_n|r^n \leq \left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}(n-1)r^{2n}\right)^{\frac{1}{2}} \leq \frac{r^2}{1-r^2}.$$

and similarly, we obtain that

$$\sum_{n=2}^{\infty}(n-1)|c_n|r^n \leq \frac{r^2}{1-r^2}.$$

As $|b_n + c_n| \leq |b_n| + |c_n|$, the last two inequalities gives that

$$S \leq \frac{r^2}{1-r^2}.$$

Thus, $S \leq 1$ whenever $r^2 \leq 1 - r^2$, i.e. if $r \leq 1/\sqrt{2}$. Thus, $G \in \mathcal{U}$ and we complete the proof of the first part. The proof of the second part is a consequence of Lemma 1**(a)** and solving the inequality $r^2 \leq \lambda(1 - r^2)$. The final part, namely, $G \in \mathcal{S}^*$, follows by setting $\lambda = 1 - |b_1 + c_1|/2$ and applying Lemma 1**(b)**. In other words, $F(|z| < r_0)$ is a starlike domain, where $r_0 = \sqrt{\lambda/(1+\lambda)}$ with $\lambda = 1 - |b_1 + c_1|/2$. A computation gives

$$r_0 = \sqrt{1 - (2/(4 - |b_1 + c_1|))}. \qquad \blacksquare$$

Proof of Theorem 2. Let $f \in \mathcal{U}(\lambda_1)$ and $g \in \mathcal{U}(\lambda_2)$, and have the form (8). By Lemma 1**(c)**, we have

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \leq \lambda_1^2 \text{ and } \sum_{n=2}^{\infty}(n-1)^2|c_n|^2 \leq \lambda_2^2. \tag{11}$$

As in the proof of Theorem 1, for G belonging to $\mathcal{U}(\lambda)$, it suffices to show by Lemma 1**(a)** that

$$T = \sum_{n=2}^{\infty}(n-1)\left|\frac{b_n + c_n}{2}\right| r^n \leq \lambda$$

under the condition (4). Now, by (11) and the Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty}(n-1)|b_n|r^n \leq \left(\sum_{n=2}^{\infty}(n-1)^2|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}r^{2n}\right)^{\frac{1}{2}} \leq \frac{\lambda_1 r^2}{\sqrt{1-r^2}},$$

and similarly, we obtain that

$$\sum_{n=2}^{\infty}(n-1)|c_n|r^n \leq \frac{\lambda_2 r^2}{\sqrt{1-r^2}}.$$

In view of the last two inequalities, it follows that

$$T \leq \left(\frac{\lambda_1 + \lambda_2}{2}\right)\frac{r^2}{\sqrt{1-r^2}}.$$

It can be easily seen that the last expression is less than or equal to λ if and only if r satisfies the inequality (4). This means that the function $G \in \mathcal{U}(\lambda)$ under the condition (4), which is equivalent to saying that $F \in \mathcal{U}$ (and hence univalent) in the disk

$$|z| < \sqrt{\frac{-K^2 + K\sqrt{K^2+4}}{2}}, \quad K = \sqrt{\frac{2\lambda^2}{\lambda_1 + \lambda_2}}.$$

The proof of the main part is complete. Setting $\lambda_1 = \lambda_2 = \lambda = 1$, it follows that if $f, g \in \mathcal{U}$, then $G \in \mathcal{U}$ for $0 < r \leq \sqrt{\frac{\sqrt{5}-1}{2}}$. In particular, we obtain that F is univalent in the disk $|z| < \sqrt{\frac{\sqrt{5}-1}{2}} \approx 0.78615$. ∎

Proof of Theorem 3. Let $f_k \in \mathcal{S}$ for $k = 1, \ldots, m$. Then we may represent $z/f_k(z)$ in power series form

$$\frac{z}{f_k(z)} = 1 + \sum_{n=1}^{\infty} b_n^{(k)} z^n \tag{12}$$

so that

$$\frac{z}{F(z)} = 1 + \sum_{n=1}^{\infty} B_n^{(m)} z^n, \quad B_n^{(m)} = \frac{1}{m}\sum_{k=1}^{m} b_n^{(k)}. \tag{13}$$

Now, we consider the function G defined by

$$\frac{z}{G(z)} = \frac{z}{r^{-1}F(rz)} = 1 + \sum_{n=1}^{\infty} B_n^{(m)} r^n z^n.$$

Moreover, as in the proof of Theorem 1, it follows that

$$\sum_{n=2}^{\infty}(n-1)\left|b_n^{(k)}\right|^2 \leq 1$$

and

$$\sum_{n=2}^{\infty}(n-1)\left|b_n^{(k)}\right|r^n \leq \frac{r^2}{1-r^2} \tag{14}$$

for $k=1,\ldots,m$. According to Lemma 1**(a)**, G belongs to $\mathcal{U}(\lambda)$ if

$$S_m := \sum_{n=2}^{\infty}(n-1)\left|B_n^{(m)}\right|r^n \leq \lambda$$

for $0<r\leq\sqrt{\lambda/(1+\lambda)}$. By the definition of $B_n^{(m)}$, (14) and the triangle inequality, we find that

$$S_m \leq \frac{1}{m}\sum_{k=1}^{m}\left(\sum_{n=2}^{\infty}(n-1)\left|b_n^{(k)}\right|r^n\right) \leq \frac{r^2}{1-r^2}.$$

Thus, $S_m\leq\lambda$ whenever $r^2\leq\lambda(1-r^2)$. This gives the condition $r\leq\sqrt{\lambda/(1+\lambda)}$ and we complete the proof of the first part.

For the proof of part **(b)**, the role of $\lambda=1-|b_1+c_1|/2$ in the proof of Theorem 1 will be replaced by

$$\lambda = 1-\frac{1}{m}\left|\sum_{k=1}^{m}\frac{f_k''(0)}{2}\right|.$$

■

Proof of Theorem 4. We simply follow the method of proof of Theorem 2 with required modifications. Assume the hypotheses that $f_k\in\mathcal{U}(\lambda_k)$ $(k=1,\ldots,m)$ and F is defined by (5), where f_k and F have the power series form given by (12) and (13), respectively.

By Lemma 1**(c)**, we have

$$\sum_{n=2}^{\infty}(n-1)^2\left|b_n^{(k)}\right|^2 \leq \lambda_k^2, \quad k=1,\ldots,m. \tag{15}$$

As in the proof of Theorem 3, for $G(z)=r^{-1}F(rz)$ belonging to $\mathcal{U}(\lambda)$, it suffices to show by Lemma 1**(a)** that

$$T_m = \sum_{n=2}^{\infty}(n-1)\left|B_n^{(m)}\right|r^n \leq \lambda$$

under the condition (6). Now, by (15) and the arguments used in the proof of Theorem 2, we have

$$\sum_{n=2}^{\infty}(n-1)\left|b_n^{(k)}\right|r^n \leq \frac{\lambda_k r^2}{\sqrt{1-r^2}}$$

for each $k=1,\ldots,m$. Using this and the triangle inequality, it follows that

$$T_m \leq \frac{1}{m}\sum_{k=1}^{m}\left(\sum_{n=2}^{\infty}(n-1)\left|b_n^{(k)}\right|r^n\right) \leq \frac{r^2}{\sqrt{1-r^2}}\left(\frac{1}{m}\sum_{k=1}^{m}\lambda_k\right).$$

It can be easily seen that the last expression is less than or equal to λ if and only if r satisfies the inequality (6). This means that the function $G\in\mathcal{U}(\lambda)$ under the condition (6) and the rest of the conclusions follow easily from this observation.

■

4 Discussion

In Theorem 4, it is possible to remove the hypothesis that $\sum_{k=1}^{m} \frac{z}{f_k(z)} \neq 0$ for $z \in \mathbb{D}$. For example, if $f_k \in \mathcal{U}(\lambda_k)$ with $f_k''(0) = 0$ for $k = 1, \ldots, m$, then it is known that [7, 8]

$$\mathrm{Re}\left(\frac{f_k(z)}{z}\right) > \frac{1}{1+\lambda_k} \geq \frac{1}{2} \quad \text{for } z \in \mathbb{D}.$$

In particular, we obtain that $\mathrm{Re}\,(f_k(z)/z) > 0$ in $\mathbb{D}$ and for each $k = 1, \ldots, m$. Thus, $\mathrm{Re}\,(z/f_k(z)) > 0$ in $\mathbb{D}$ and for each $k = 1, \ldots, m$ so that the assumption that $\sum_{k=1}^{m} \frac{z}{f_k(z)} \neq 0$ obviously holds for $z \in \mathbb{D}$. In this case, this observation gives that F defined by (5) is univalent and starlike in the disk $|z| < \sqrt{\frac{\sqrt{5}-1}{2}}$ whenever $f_k \in \mathcal{U}$ with $f_k''(0) = 0$ for $k = 1, \ldots, m$. Moreover, in some special situations, one can improve Theorem 1. For instance, we have

Theorem 5. *Let $f, g \in \mathcal{S}$ have the form* (7) *with $b_n \geq 0$ and $c_n \geq 0$ for all $n \geq 2$. Then the function F defined by* (3) *belongs to $\mathcal{S}$. In particular, if $b_1 + c_1 = 0$, then F is also starlike in $\mathbb{D}$.*

Proof. By Lemma 2, we have $f, g \in \mathcal{U}$ and

$$\sum_{n=2}^{\infty} (n-1) b_n \leq 1 \ \text{ and } \ \sum_{n=2}^{\infty} (n-1) c_n \leq 1.$$

The last two coefficient conditions imply that

$$\sum_{n=2}^{\infty} (n-1) \frac{b_n + c_n}{2} \leq 1$$

and therefore, F defined by (3) is univalent in $\mathbb{D}$.

If $b_1 + c_1 = 0$, then according to Lemma 1**(b)** the function F defined by (3) is starlike in $\mathbb{D}$. ∎

Using Theorem 5, one may state a general result as in Theorems 3 and 4. Also, it is an open problem to determine the exact radii in Theorems 1 and 2.

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