"Easy" Representations and the QSF property for groups

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Abstract

We define the class of *easily-representable groups* as the class of those finitely presented groups Γ admitting an *inverse representation* (which, roughly, is a map from some 2-complex to a certain singular 3-manifold $M^3(\Gamma)$ associated to Γ, satisfying several topological properties) for which the set of double points is closed. Our main result is that easily-representable groups are QSF (i.e. quasi-simply filtered).

1   Introduction and definitions

In this paper we will deal only with finitely presented groups Γ. For our purpose a very special kind of presentations will be used. To each finitely presented group $\Gamma = \langle S \mid \mathcal{R} \rangle$ with a finite set of generators S and a finite set of relators $\mathcal{R}$, one can associate a compact SINGULAR 3-manifold $M^3(\Gamma)$ by the following kind of procedure.

Start with a smooth 3-dimensional handlebody H of genus g corresponding to the generators $s_1, \ldots, s_g \in S$. Then to each relator $R_i \in \mathcal{R}, i = 1, \ldots r$, we can associate a curve on the boundary of H, $S_i^1 \subset \partial H$. More explicitly, we consider a smooth generic immersion α

$$(1) \qquad \qquad \bigcup_{i=1}^{r} S_i^1 \xrightarrow{\ \alpha\ } \partial H$$

for which we will take the immersed regular neighborhood

$$(2) \qquad\qquad \bigcup_{i=1}^{r} S_i^1 \times [-\epsilon, \epsilon] \xrightarrow{\alpha} \partial H.$$

Then, the $M^3(\Gamma)$ is gotten by adding r handles of index two to H, along (2). Without any loss of generality, after possibly enlarging g and r, one may assume here, if one wishes to do so, that each individual S_i^1 is embedded. Anyway $M^3(\Gamma)$ has singular points where it fails to be a manifold. The singular set Sing $M^3(\Gamma) \subset M^3(\Gamma)$ is a disjoint union of little squares $S \subset \partial H$, the connected components of the image of the set of double points of (2) in ∂H, and we will call these kind of singularities *immortal* so as to distinguish them from the mortal singularities to appear later. In the neighborhood of each singularity $x \in$ int S, our $M^3(\Gamma)$ can be described as $Y \times \mathbb{R}^2$, where Y is the wedge of three half-lines, or alternatively as

(3)   the union of three copies of the upper half-space $\mathbb{R}_+^3$

                                        along their common boundary.

The central notion of this paper will be the *(inverse) representation of* Γ. Contrary to the standard terminology, whereby "representations" mean morphisms $\Gamma \to \cdots$, our representations will be arrows of the form $\cdots \to \widetilde{M}^3(\Gamma) = \{$the universal covering space of $M^3(\Gamma)\}$, an object which, up to quasi-isometry, is the same thing as Γ itself. Such representations were already defined in [6, 13], but since we want this paper to be, as much as it is possible, independently readable, we will review here the basics necessary.

Consider first some (non necessarily locally finite) simplicial complex of dimension 2 or 3 and a non-degenerate simplicial map

$$(4) \qquad\qquad X \xrightarrow{F} Y$$

where Y may be $M^3(\Gamma)$ or $\widetilde{M}^3(\Gamma)$ or even a smooth manifold M^3.

We will call *(mortal) singularities* of F, the points $x \in X$ in the neighborhood of which F fails to be immersive. Their set is denoted Sing$(F) \subset X$.

On the set X, two equivalence relations $\Psi(F) \subset \Phi(F) \subset X \times X$ will be considered, namely

- $\Phi(F)$ which is the set of pairs $(x_1, x_2) \in X \times X$ with $F(x_1) = F(x_2)$;

- $\Psi(F)$ which is the "smallest" equivalence relation compatible with F, killing all the mortal singularities, so that the induced map $X/\Psi(F) \longrightarrow Y$ is an immersion.

It is shown in [7] that this definition of $\Psi(f)$ makes sense and we will not pursue this issue longer here. In the specific situation considered later in this paper, things will anyway be concrete.

Definition 1.1. *A* GSC *(resp.* QSF*)-**representation for** Γ is a non-degenerate simplicial map*

$$(5) \qquad\qquad X^2 \xrightarrow{f} \widetilde{M}^3(\Gamma)$$

with the following features:

(5-1) X^2 *is* GSC, *i.e.* **geometrically simply connected** *(resp.* QSF, *i.e.* **quasi-simply filtered***), and what this means will be soon recalled.*

(5-2) $\Psi(f) = \Phi(f)$; *and in such a case one says that the representation* (5) *is **zippable**.*

(5-3) f *is "essentially" surjective, in the sense that one can get $\widetilde{M}^3(\Gamma)$ from $\overline{fX^2} \subset \widetilde{M}^3(\Gamma)$ by adding cells (possibly infinitely many) of dimension 2 and 3.*

Now the notion GSC is well-known in differential topology, where it means the existence of a handlebody decomposition with handles of index 1 and 2 in cancelling position; but it also makes sense in the context of cell-complexes (see e.g. [13, 14]).

We remind now the reader the following notion, due to S. Brick and M. Mihalik (see [1, 15]). We are now in the simplicial category, and a locally compact space X is called QSF (i.e. **quasi-simply filtered**) if for every compact $k \subset X$ there is another (abstract) simply-connected compact K, endowed with an inclusion i from k, coming with a (continuous) simplicial map f

$$(6)$$

exhibiting the *Dehn-type property*: $M_2(f) \cap i(k) = \varnothing$.

NOTATIONS. For any map $X \xrightarrow{h} Y$ we denote $M_2(h) = \{x \in X$ such that $\#\{h^{-1}h(x)\} > 1\}$ while $M^2(h) \subset X \times X$ is the set of pairs (x_1, x_2), with $x_1 \neq x_2$, such that $h(x_1) = h(x_2)$.

We will not discuss here more on this notion for which there is a vast literature [1, 3, 15]. It suffices to stress the following facts:

- The notion of QSF is well-defined for (finitely presented) discrete groups Γ, in the sense that if $\Gamma = \pi_1 M$ for some finite complex M such that $\widetilde{M}$ is QSF, then any other finite complex N with $\pi_1 N = \Gamma$ has the property that $\widetilde{N}$ is also QSF.

- There is a trivial implication X is GSC $\Rightarrow$ X is QSF.

- The first author (D.O.) in collaboration with Louis Funar have proved the following in [3]: A group Γ is QSF IF AND ONLY IF there is a smooth compact manifold M such that $\pi_1 M = \Gamma$ and $\widetilde{M}$ is GSC.

Definition 1.2. *A 2-dimensional representation* (5) *is called **easy** if* $Im(f)$ *(i.e.* fX^2*)* $\subset$ $\widetilde{M}^3(\Gamma)$ *and* $M_2(f) \subset X^2$ *are closed subsets. In such a case we call the group* Γ ***easily-representable***.

Before discussing this notion a bit further, we will state the main result of this paper.

Theorem 1.3. *If* Γ *admits an easy* GSC *(or* QSF*)-representation, then* Γ *is* QSF.

Remarks:

- It suffices to prove the Theorem only for a QSF-representation.

- The technology we will use for proving our Theorem could actually be also used for a stronger result, of which we will give here the general flavor. Start with a simply connected, locally finite 3-complex Y^3 with singularities (i.e. non-manifold points) which are sufficiently "gentle"; without trying to be more specific, let us say this means singularities like those which $\widetilde{M}^3(\Gamma)$ has (or only slightly more general). Then, what our technology can prove is that if such an Y^3 admits an easy GSC (QSF)-representation, then it is QSF.

  The point is that, in the present paper, the fact that $\widetilde{M}^3(\Gamma)$ admits a free Γ-action plays no role. But, one of the features of this notion of representation is that it lends itself to X admitting such an action and f being equivariant. This plays an essential role in [12, 13].

Concerning easy-representations, here are some comments:

- Gromov-hyperbolic groups (as well as some other geometric classes of groups) are easily-representable. This can be read between the lines in papers like [8, 9].

- Using the full Thurston geometrization, proved by G.Perelman, one should certainly be able to show that all $\pi_1 M^3$'s admit easy representations.

- For a finitely presented group Γ, it should be (almost)-equivalent (in the sense of [3]), to be QSF or to be easily-representable. But this issue requires a lengthier discussion which we will not undertake here.

- Nobody has actually ever seen a group which does not admit one. The second author (V.P.) conjectures that: *All finitely presented groups* Γ *admit easy representations*.

2 Proof of the *"Easy"* Theorem

We start with an easy QSF-representation for our group Γ and notice that, since $M_2(f)$ and fX^2 are, by hypothesis, closed subsets, the fX^2 is a locally finite simplicial complex such that

$$(7) \qquad \widetilde{M}^3(\Gamma) = fX^2 \cup \{\text{cells of dimension 2 and 3}\}.$$

Claim A: We can factor the map $f : X^2 \longrightarrow fX^2$ as an infinite sequence of *elementary zipping moves*

$$(8) \qquad X^2 = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots fX^2 \subset \widetilde{M}^3(\Gamma)$$

each of which is either

(8-1) to begin with, an elementary move $O(0), O(1), O(2)$ like in the Figures 5, 2 and 4 of [10]. These moves $O(i \leq 2)$ are "acyclic", in the sense that they are homotopy equivalences. Because $\widetilde{M}^3(\Gamma)$ is singular, we have to add to the $O(i \leq 2)$ another acyclic move occurring when the zipping hits S and has to stop, like in (9-2) below. This movement "$O(S)$" changes a mortal singularity into an immortal one. The $X_{i \geq 1}$'s can have such.

or (8-2) movements $O(3)$ like in the Figure 1 below (with less details this figure occurs also as Figures 6 and 7 in [10]). These movements are, homotopically, additions of 2-cells.

Something very close to this CLAIM A is done in the paper [10] (and also in [11]) by the second author (V.P.) and in the review by D. Gabai [4].

But something similar can be done in the present context, with two caveats. Firstly, $\widetilde{M}^3(\Gamma)$ has now singularities and that issue will be discussed below. Secondly, we are now in an infinite, non-compact context. But that should not make any problem.
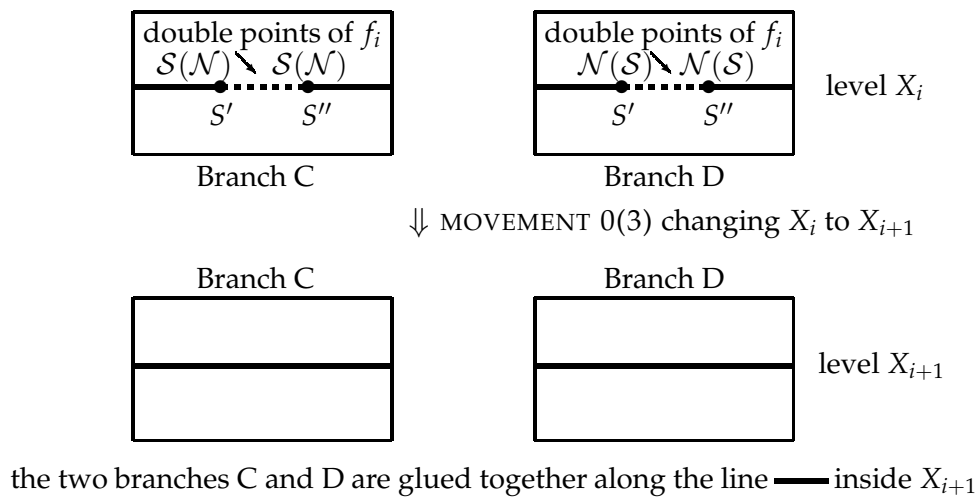

FIGURE 1. Movement $O(3)$ where the mortal singularities $S', S'' \in \mathrm{Sing}\,(f_i)$ enter in a frontal collision and kill each other. The letters "$\mathcal{S}$", and "$\mathcal{N}$" smeared in their neighborhoods on the respective branches C, D, correspond to some abstract desingularization at level X_i. As written here, the $O(3)$ movement is, by definition, COHERENT. If, let us say at S'', we change $\mathcal{S}(\mathcal{N})$ into $\mathcal{N}(\mathcal{S})$ on C and hence $\mathcal{N}(\mathcal{S})$ into $\mathcal{S}(\mathcal{N})$ on D, then we get the NON-COHERENT movement $O(3)$.

A more pedantic way of describing (8) is the following. The f defines the equivalence relation $\Phi(f) \subset X^2 \times X^2$ and so does every finite piece $X \to X_i$ define a similar $\Phi_i \subset X^2 \times X^2$, with $\Phi_i \subset \Phi_{i+1}$. The factorization in (8) simply means that $\Phi(f) = \cup_i \Phi_i$.

It should be noted that the various X_i, $i \geq 1$, are not necessarily QSF (certainly not GSC in the case of a GSC-representation), and so the $X_i \to \widetilde{M}^3(\Gamma)$, $i \geq 1$, are not representations, in the sense defined before.

So let us consider now three half spaces U_1, U_2, U_3 of $\widetilde{M}^3(\Gamma)$ having in common exactly an immortal singularity $S = \partial U_1 \cap \partial U_2 \cap \partial U_3$. Any double line in $fM_2(f)$, starting let us say in U_1 and reaching at S, either continue transversally through S or stops short. This leads to the following two possible local models.

(9-1) There are two smooth sheets $A, B \subset X^2$ homeomorphic to $\mathbb{R}^2$ such that f injects them inside $U_1 \cup U_2$ where fA, fB, S are three planes in general position (coordinate planes). Here the zipping proceeds through S, without paying any attention to U_3.

(9-2) The map f injects A and B into $U_1 \cup U_2$, $U_1 \cup U_3$ respectively, and the zipping stops at S. Here S generates an immortal singularity for fX^2. There are no mortal singularities for fX^2.

We will state now our second claim, the proof of which will be given later on.

Claim B: For each $X_{i \geq 0}$ in (8), and for each dimension $n \geq 5$, we can chose a smooth n-dimensional regular neighborhood,

$$(10) \qquad\qquad \Theta^n(X_i)$$

satisfying the QSF property, and such that the following things happen:

(11) the sequence (8) of quotient space projections can be changed into a sequence of smooth embeddings

$$(S_\infty) \qquad \Theta^n(X^2) = \Theta^n(X_0) \underset{j_o}{\subset} \Theta^n(X_1) \underset{j_1}{\subset} \Theta^n(X_2) \underset{j_2}{\subset} \cdots, \text{ where}$$

(11-1) if $X_l \to X_{l+1}$ is an acyclic $O(i)$ move, then $\Theta^n(X_l) \underset{j_l}{\hookrightarrow} \Theta^n(X_{l+1})$ is a compact, smooth Whitehead dilatation;

(11-2) if $X_l \to X_{l+1}$ is an $O(3)$ move, then $\Theta^n(X_l) \underset{j_l}{\hookrightarrow} \Theta^n(X_{l+1})$ is the addition of a handle of index 2;

(12) the $\Theta^n\left(M^3(\Gamma)\right)$ and $\Theta^n\left(\widetilde{M}^3(\Gamma)\right)$ will also make sense, and $\Theta^n\left(M^3(\Gamma)\right)^{\sim} = \Theta^n\left(\widetilde{M}^3(\Gamma)\right)$.

Remark:

- It is a feature of our choice of (10) that we have (11) and (12). We will sum this up by saying that (10) is *canonical*.

2.1 The final argument

Since $M_2(f) \subset X^2$ is a closed subset, there is a locally finite covering of X^2 by open subsets U such that each $U \subset \widetilde{M}^3(\Gamma)$ can meet only finitely many elementary zipping moves from (8). It follows that we can put together the infinite sequence of embeddings $((S_\infty))$ into a smooth non-compact n-manifold with non-empty boundary $\cup_{i=0}^{\infty}\Theta^n(X_i)$. Here, for any local piece of $\Theta^n(X_i)$ and any $i < j$, only finitely many of the intersections $\Theta^n(X_i) \cap \left(\Theta^n(X_{j+1}) - \Theta^n(X_j)\right)$ are non-empty.

Since for every compact subset $k \subset \cup_{i=0}^{\infty}\Theta^n(X_i)$ there is an i such that $k \subset \Theta^n(X_i)$ that is QSF, it follows that $\cup_{i=0}^{\infty}\Theta^n(X_i)$ is also QSF. Furthermore, because of $M_2(f) \subset X^2$ being closed, we have, inside $\Theta^n\left(\widetilde{M}^3(\Gamma)\right)$, the equality of sets

$$(13) \qquad \bigcup_{i=0}^{\infty} \Theta^n(X_i) = \Theta^n(fX^2).$$

Again, since $fX^2 \subset \widetilde{M}^3(\Gamma)$ is closed, so is $\Theta^n(fX^2) \subset \Theta^n\left(\widetilde{M}^3(\Gamma)\right)$. All this, together with (5-3), implies that

$$(13\text{-}1) \qquad \Theta^n\left(\widetilde{M}^3(\Gamma)\right) = \Theta^n(fX^2) \cup \{\text{handles of index 2 and 3}\}.$$

Hence $\Theta^n\left(\widetilde{M}^3(\Gamma)\right)$ is QSF, which, together with (12), implies that Γ is QSF.

Lemma 2.1. *Let V^n be a smooth non-compact manifold with boundary $\partial V \neq \emptyset$ and let also $N^n = V^n \cup \{a\ \lambda\text{-handle } H^\lambda, \text{ with } \lambda > 1\}$. If V^n is QSF then N^n is also QSF.*

Proof. Start with a compact $k \subset N^n$ and let $k_1 = \overline{k - H^\lambda} \subset V^n$. Then go to $\partial H^\lambda = S^{\lambda-1} \times B^n \subset \partial V^n$, the attaching zone, and apply the QSF of V^n for the compact $k_2 = k_1 \cup \partial H^\lambda \subset V^n$.

We have that $\partial H^\lambda \subset K_2 - M_2(j)$, and the compact $K = K_2 \cup H^\lambda$ is simply connected, contains k and comes with a map into N having the Dehn-property. This ends the proof. ∎

To complete the proof of our Theorem, it remains to show how one constructs the $\Theta^n(\ldots)$, with all the features above. Here we will borrow very heavily on the very initial part of the technology from [4, 10]. But everything necessary for our present aim is explained here, and this paper is, at this point, essentially self-contained.

We start by considering any map

$$(14) \qquad Y^2 \xrightarrow{\ f\ } \widetilde{M}^3(\Gamma), \text{ where}$$

(14-1) Y^2 is a locally finite simplicial complex;

(14-2) the non-immersive points, i.e. the $\sigma \in \text{Sing}\,(f) \subset Y^2$, are like in the Figure 1 of [10]; they are called *undrawable type singularities*;

(14-3) we have to assume now that Y^2 has both mortal singularities $\text{Sing}\,(f)$ and immortal singularities $\text{Sing}\,(Y^2)$, disjoined from each other. The $\text{Sing}\,(Y^2)$ are like the ones from (3), and such that $f(\text{Sing}\,(Y^2)) \subset \text{Sing}\,\widetilde{M}^3(\Gamma)$; locally f is here injective. So, for any $\sigma \in \text{Sing}\,(f) + \text{Sing}\,(Y^2)$, the germ of (Y^2, σ) consists of two branches, say P_1 and P_2, partially glued together at the source, like in the Figure 1 of [10].

We will define now (abstract) desingularizations for our map (14). By definition, such an **abstract desingularization** is a map ϕ as follows

(15) $\{\text{the set of branches } P_1, P_2 \text{ for each } \sigma \in \text{Sing}\,(f) + \text{Sing}\,(Y^2)\} \overset{\phi}{\longrightarrow} \{\mathcal{S}, \mathcal{N}\}$

(where $\{\mathcal{S}, \mathcal{N}\}$ is the alphabet with two letters $\mathcal{S}, \mathcal{N}$), and such that for each individual singularity σ and its $P_1 = P_1(\sigma), P_2 = P_2(\sigma)$, we have

$$\phi\big(P_1(\sigma)\big) \neq \phi\big(P_2(\sigma)\big).$$

The branch P coming with $\phi = \mathcal{S}$ will be called *specified*, the other one *non-specified*.
Consider now an elementary zipping move $O(i \leq 3)$ or $O(S)$, like the ones occurring in (S_∞)

(16)

$$
\begin{array}{ccc}
Y^2 & \hookrightarrow & Y_1^2 \\
 & \searrow^{f} \quad \swarrow^{f_1} & \\
 & \widetilde{M}^3(\Gamma). &
\end{array}
$$

Lemma 2.2. *Any given abstract desingularization ϕ for (Y^2, f) **propagates canonically** into an abstract desingularization ϕ_1 for (Y_1^2, f_1).*

Proof. The cases $O(0)$ and $O(1)$ can be simply treated just looking at the Figures 5 and 2 of [10]. In the cases $O(2), O(3)$ one simply keeps by decree $\phi_1 = \phi$ for the singularities not killed by the local move, and by decree too, one ignores the killed ones. The case $O(S)$ is obvious. ∎

Remarks:

- In the context of (8), consider some arbitrarily given desingularization ϕ at level $X^2 = X_0$, where $\text{Sing}\,(X_0) = \varnothing$. According to Lemma 2.2, this propagates canonically through the whole of (8) inducing at each stage a desingularization which we continue to denote ϕ, for all X_i's including $X_\omega \overset{\text{def}}{=} fX^2$. At the final level $\text{Sing}\,\big(X_\omega \overset{f_\omega}{\to} \widetilde{M}^3(\Gamma)\big) = \varnothing$, while $\text{Sing}\,X_\omega \subsetneqq X_\omega \cap \text{Sing}\,\widetilde{M}^3(\Gamma) \neq \varnothing$, generically speaking. At intermediary levels we find both $\text{Sing}\,\big(X_i \to \widetilde{M}^3(\Gamma)\big) \neq \varnothing$ (mortal case) and $\text{Sing}\,X_i \neq \varnothing$ (immortal case).

- When the propagation of ϕ from X_o to X_ω is considered, the $O(3)$'s (and $O(2)$'s) are not in the way, we can always redirect the zipping flow so that it reaches to Sing X_ω before performing $O(3)$ (and $O(2)$). This is suggested in the next drawing.
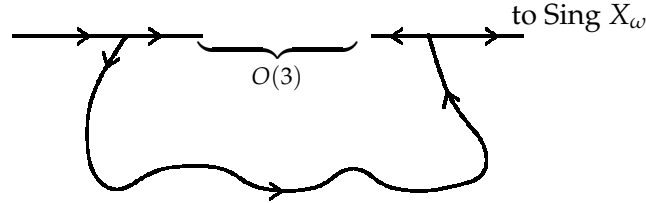

FIGURE 2: Redirected zipping flow.

So, without any loss of generality, the zipping strategy (8) is such that the $O(3)$'s are corralled at the very end.

To any abstract desingularization ϕ for (Y^2, f_1) there is a canonically attached *geometric desingularization*

$$\check{Y}^2 = \check{Y}^2(\phi)$$

(17)
$$\downarrow \pi(\phi)$$

$$Y^2$$

with the following feature:

(17-1) any singularity $\sigma \in$ Sing $(f) +$ Sing Y^2 is blown up into a circle inside that local branch P_1 or P_2 (let us suppose it is in P_1), which is coming with $\phi = \mathcal{S}$. This is suggested in the figure below.
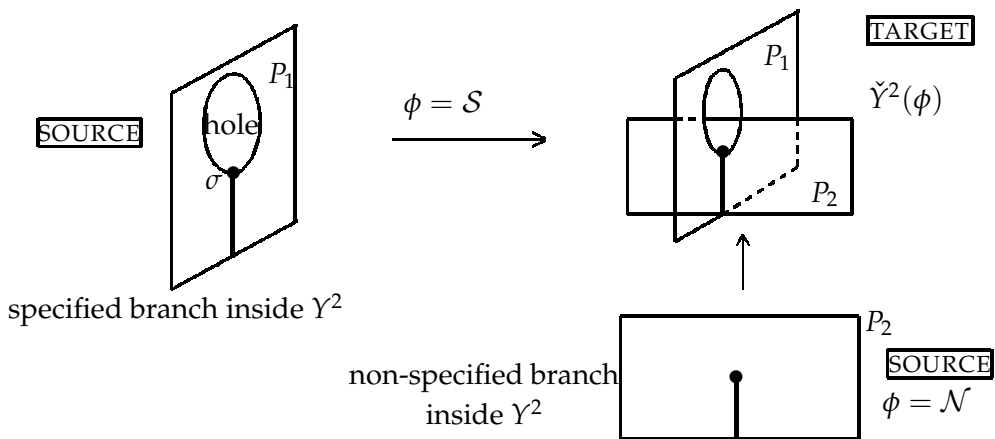

FIGURE 3: Geometric desingularization $\check{Y}^2(\phi) \xrightarrow{\pi} Y^2$. Actually, with the notations of the next Section 2.2, we see here the local situation $\check{K}^2(\sigma) \xrightarrow{\pi} K^2(\sigma)$.

Notice that $\check{Y}^2(\phi)$ in (17) has a canonical smooth 3-dimensional regular neighborhood which comes with an immersion into $\widetilde{M}^3(\Gamma)$, guided by f in (14); we will denote it by $\Theta^3\big(\check{Y}^2(\phi)\big)$.

Together with the abstract desingularization (15) comes not only the geometric desingularization (17) but also a ϕ-dependent 4-dimensional smooth regular neighborhood $\Theta^4(Y^2,\phi)$, together with a smooth embedding

$$(18) \qquad\qquad \Theta^3\big(\check{Y}^2(\phi)\big) \subset \partial\Theta^4(Y^2,\phi).$$

The pairs of type $\big(\Theta^4(Y^2,\phi),\Theta^3\big(\check{Y}^2(\phi)\big)\big)$ are well defined for local pieces and one can glue them in a natural way so as to generate the global objects. What we will do with a bit more details below is to implement this little program of going from local to global.

2.2 Construction of $\Theta^4(Y^2,\phi)$

For each $\sigma \in \mathrm{Sing}\,(f) + \mathrm{Sing}\,(Y^2) \subset Y^2$ we consider (like in the Figure 1 of [10]) the *undrawable model*

$$(19) \qquad\qquad K^2(\sigma) \overset{\text{def}}{=} P_1(\sigma) \underset{\frac{1}{2}L}{\cup} P_2(\sigma) \subset Y^2.$$

This determines, with a $\check{K}^2(\sigma)$ to be defined below, related decompositions

$$(20) \quad Y^2 = Y^2 \text{ (non-singular)} \cup \sum_\sigma K^2(\sigma) \quad \text{and}$$

$$\check{Y}^2(\phi) = Y^2 \text{ (non-singular)} \cup \sum_\sigma \check{K}^2(\sigma).$$

To Y^2 (non-singular), which is DEFINED by the formula (20), (and that we will denote by Y^2_{NS}), corresponds a smooth 3-manifold

$$(21) \qquad\qquad \Theta^3\big(Y^2_{NS}\big) \subset \Theta^3\big(\check{Y}^2(\phi)\big),$$

while to $K^2(\sigma)$ corresponds, to begin with, a $\check{K}^2(\sigma) \subset \check{Y}^2(\phi)$, like in the upper right corner of Figure 3, and a $\Theta^3\big(\check{K}^2(\sigma)\big) \subset \Theta^3\big(\check{Y}^2(\phi)\big)$ which is a copy of $S^1 \times D^2$. [For typographic simplicity's sake we have omitted to add a "ϕ" to $\check{K}^2(\sigma)$ too.]

There are really two cases for $\check{K}^2(\sigma)$, only one of which (the I below) is displayed in Figure 3, namely

(22)   CASE I: $\phi(P_1) = \mathcal{S}, \quad \phi(P_2) = \mathcal{N}$ and

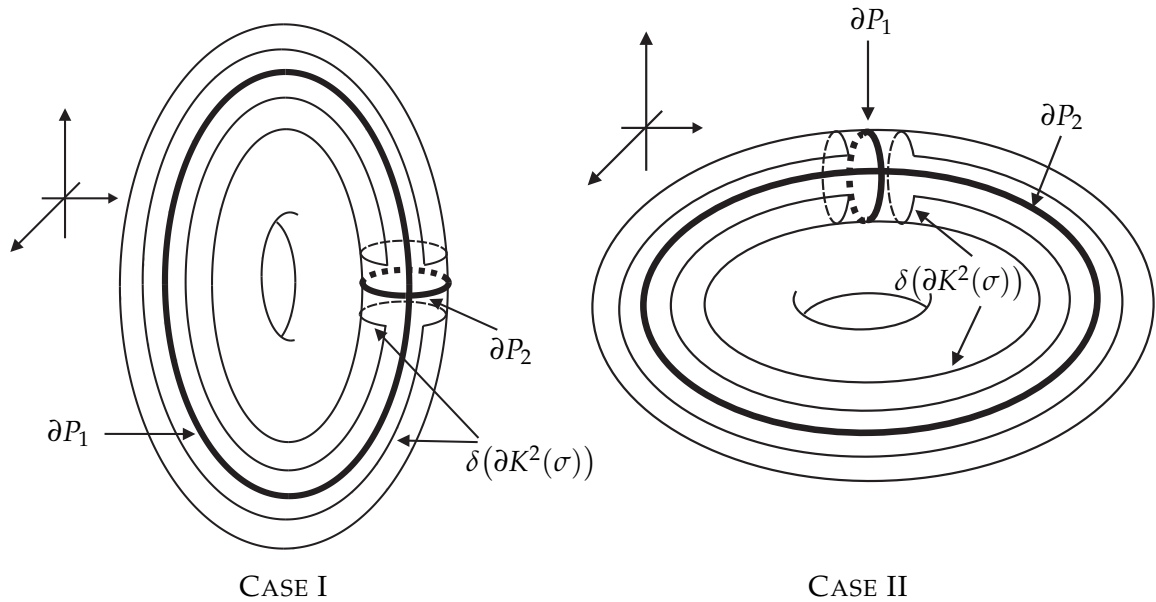   CASE II: $\phi(P_1) = \mathcal{N}, \quad \phi(P_2) = \mathcal{S}$

CASE I   CASE II

FIGURE 4. We see here the $\Theta^3\big(\check{K}^2(\sigma)\big)$ in the two cases from (22). Inside $\partial\Theta^3\big(\check{K}^2(\sigma)\big)$ lives $\partial K^2(\sigma) = \partial P_1 \vee \partial P_2 \subset \delta\big(\partial K^2(\sigma)\big) \subset \partial\Theta^3\big(\check{K}^2(\sigma)\big)$, where $\delta\big(\partial K^2(\sigma)\big)$ is the 2-dimensional regular neighborhood of $\partial K^2(\sigma) \subset \partial\Theta^3\big(\check{K}^2(\sigma)\big)$.

To the second formula in (20) corresponds also a reconstruction formula for $\Theta^3\big(\check{Y}^2(\phi)\big)$, namely

(23) $\Theta^3\big(\check{Y}^2(\phi)\big) \;=\; \Theta^3\big(Y^2_{NS}\big) \cup \sum_{\sigma} \Theta^3\big(\check{K}^2(\sigma)\big).$

Here, with a $\delta\big(\partial K^2(\sigma)\big) \supset \partial P_1 \vee \partial P_2 = \partial K^2(\sigma)$ like in Figure 4, we have embeddings

(23-1) $\partial\Theta^3\big(Y^2_{NS}\big) \supset \delta\big(\partial K^2(\sigma)\big) \subset \partial\Theta^3\big(\check{K}^2(\sigma)\big)$

along which the pieces in (23) are to be glued.
For each σ we consider now a copy of B^4

(24) $\Theta^4\big(K^2(\sigma),\phi\big) \supset \partial\Theta^4 \underset{i}{\supset} \Theta^3\big(\check{K}^2(\sigma)\big),$

with the embedding i like in the Figure 3 (think of it as representing $\Theta^3 \subset \mathbb{R}^3 \cup \{\infty\} = \partial\Theta^4$).

     We finally define

(25) $\Theta^4(Y^2,\phi) \;=\; \Theta^3\big(Y^2_{NS}\big) \times [0,1] \;\cup\; \sum_{\sigma} \Theta^4\big(\check{K}^2(\sigma),\phi\big),$

where, for each σ, one should glue

$$\delta\big(\partial K^2(\sigma)\big) \times [0,1] \subset \partial\Big(\Theta^3\big(Y_{NS}^2\big) \times [0,1]\Big)$$

coming from the left-hand side of (23-1) with the

$$\delta\big(\partial K^2(\sigma)\big) \times [0,1] \subset \partial\Theta^4\big(K^2(\sigma), \phi\big)$$

defined by the right-hand side of (23-1), where $\delta(\partial K^2) = \delta(\partial K^2) \times \{0\}$, and which is outgoing with respect to $\Theta^3\big(\check{K}^2(\sigma)\big) \subset \partial\Theta^4$. ∎

In the context of (16) and of Lemma 2.2, we consider now $\Theta^4(Y^2, \phi)$ and $\Theta^4\big(Y_1^2, \phi \text{ (induced)}\big)$.

Lemma 2.3.

1. *In the cases $O(0), O(1), O(2), O(S)$ we have a canonical embedding*

   (26) $$\Theta^4(Y^2, \phi) \subset \Theta^4(Y_1^2, \phi)$$

   which is just a compact smooth Whitehead dilatation.

2. *In the* COHERENT $O(3)$ *case we have again an embedding (26), this time it is the addition of a handle on index 2.*

3. *There is NO embedding in the* NON-COHERENT $O(3)$ *case (see the schematic Figure 5).*

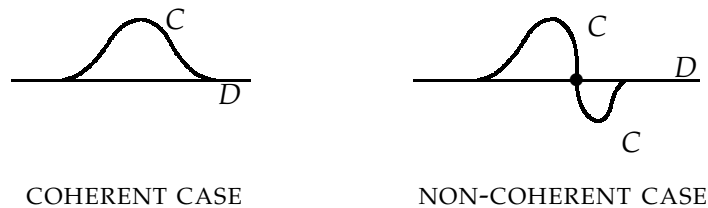COHERENT CASE   NON-COHERENT CASE

FIGURE 5. In the non-coherent case we see the well known intersection $\mathbb{R}^2 \pitchfork \mathbb{R}^2 \subset \mathbb{R}^4$, standard obstruction in 4-dimensional topology.

Lemma 2.4. *Let $p \geq 1$ and consider*

(27) $$\Theta^{n=p+4}(Y^2, \phi) = \Theta^4(Y^2, \phi) \times B^p.$$

1. *Up to diffeomorphism, (27) is no longer ϕ-dependent, and we will just denote it by $\Theta^n(Y^2)$.*

2. *With this Θ^n all the features in the* CLAIM B *are satisfied.*

Proof. In the context of the Figure 4, we have two *distinct* embeddings

$$(28) \qquad \delta\big(\partial K^2(\sigma)\big) \times [0,1] \underset{i_{II}}{\overset{i_I}{\rightrightarrows}} \partial \Theta^4\big(K^2(\sigma), \phi\big).$$

Here the source is $\big(S^1 \times S^1 - \text{int} D^2\big) \times [0,1]$ and the target is S^3. When we move to $\Theta^{n \geq 5}$, then the corresponding embeddings

$$(29) \qquad \delta\big(\partial K^2(\sigma)\big) \times [0,1] \times B^p \subset \partial \Theta^n\big(K^2(\sigma)\big)$$

are now smoothly isotopic. ∎

We consider now singular 3-manifolds V^3, of which $M^3(\Gamma)$ and $\widetilde{M}^3(\Gamma)$ are examples, with immortal singularities S. The singular local model is gotten from the Figure 1 of [10] by changing each P_1, P_2 into a thin $P_i \times [-\epsilon, \epsilon]$. These two flat rectangular boxes are then appropriately glued together, so as to change the σ (again in the Figure 1 of [10]) into a small square S.

Claim C: Our claim is that all the little theory above, which has started at (14), extends to V^3's (no f are needed now). This is quite well explained in [4].

With arbitrary abstract desingularizations ϕ for $M^3(\Gamma)$ and Φ for $\widetilde{M}^3(\Gamma)$ one can define now $\Theta^n\big(M^3(\Gamma), \phi\big), \Theta^n\big(\widetilde{M}^3(\Gamma), \Phi\big)$ and then the canonical, desingularization-independent ($p \geq 1$)

$$(30) \qquad \Theta^{n=p+4}\big(M^3(\Gamma)\big) = \Theta^4\big(M^3(\Gamma), \phi\big) \times B^p$$

$$\Theta^{n=p+4}\big(\widetilde{M}^3(\Gamma)\big) = \Theta^4\big(\widetilde{M}^3(\Gamma), \Phi\big) \times B^p.$$

These verify the functorial property (12). ∎

Final comment. So, as explained in [4], there are really two theories, one for 2-dimensional objects like $Y^2 = X^2 \xrightarrow{f} \widetilde{M}^3(\Gamma)$ and another one of 3-dimensional objects like $V^3 = \widetilde{M}^3(\Gamma)$.

In the present paper the only necessary bridge between the two is (13-1). In real life one has to go much deeper into the connection between $\Theta^n(Y^2)$ and $\Theta^n(V^3)$.

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