Subgroup S–commutativity degrees of finite groups

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Abstract

The so–called subgroup commutativity degree $sd(G)$ of a finite group G is the number of permuting subgroups $(H,K) \in \mathrm{L}(G) \times \mathrm{L}(G)$, where $\mathrm{L}(G)$ is the subgroup lattice of G, divided by $|\mathrm{L}(G)|^2$. It allows to measure how G is far from the celebrated classification of quasihamiltonian groups of K. Iwasawa. Here we generalize $sd(G)$, looking at suitable sublattices of $\mathrm{L}(G)$, and show some new lower bounds. More precisely, we define and study the subgroup S-commutativity degree of a group, which measures the probability that subnormal subgroups commute with maximal subgroups.

1 Introduction and terminology

All groups in the present paper are supposed to be finite. Noting that, given two subgroups H and K of a group G, the product $HK = \{hk \mid h \in H, k \in K\}$ is not always a subgroup of G, one says that the subgroups H and K *permute* if $HK = KH$, or equivalently, if HK is a subgroup of G. The subgroup H is said to be *permutable* (or *quasinormal*) in G if it permutes with every subgroup of G. This notion can be strengthen in various ways, for example one can say that a subgroup H is *S–permutable* (or *S–quasinormal*) in G, if H permutes with all Sylow subgroups of G (for all primes in the set $\pi(G)$ of the prime divisors of $|G|$). Historically, O. Kegel introduced the class of S–permutable subgroups in 1962, to generalize a well–known result of O. Ore of 1939, who proved that permutable

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subgroups are subnormal (see [7, 13] for details). Several authors investigated the topic in the successive years and we mention here only [1, 2, 12, 13] for our aims.

The *subgroup lattice* $\mathrm{L}(G)$ of a group G is the set of all subgroups of G and is a complete bounded lattice with respect to set inclusion, having initial element the trivial subgroup $\{1\}$ and final element G itself (see [6, 13]). Its binary operations $\wedge, \vee$ are defined by $X \wedge Y = X \cap Y$, $X \vee Y = \langle X \cup Y \rangle$, for all $X, Y \in \mathrm{L}(G)$. Furthermore, $\mathrm{L}(G)$ is *modular* if all the subgroups of G satisfy the *modular law*, and the group G is modular whenever $\mathrm{L}(G)$ is modular (see [13, Section 2.1]). This notion is important because of the following concept. A group G is *quasihamiltonian* if all its subgroups are permutable. By a result of K. Iwasawa [13, Theorem 2.4.14], quasihamiltonian groups are classified, but, at the same time, these groups are characterized to be nilpotent and modular (see [13, Exercise 3, p.87]).

Now we recall some terminology from [12], which will be useful in the rest of the paper. Every non–empty subset of subgroups of G generates a sublattice $\mathrm{S}(G)$ of $\mathrm{L}(G)$ in a natural way, closed with respect to $\wedge$ and $\vee$ (see [6] or [12, §1]). The symbol $\mathrm{S}^{\perp}(G)$ denotes the sublattice of $\mathrm{L}(G)$ containing all subgroups H of G which are permutable with all $S \in \mathrm{S}(G)$. It may be helpful to note that $T \subseteq S^{\perp}(G)$ implies $S \subseteq T^{\perp}(G)$.

There is a wide literature when one chooses $\mathrm{S}(G)$ to be equal to the sublattice $\mathrm{M}(G)$ of $\mathrm{L}(G)$ containing all maximal subgroups of G, or to the sublattice $\mathrm{sn}(G)$ containing all subnormal subgroups of G, or also to the sublattice $\mathrm{n}(G)$ containing all normal subgroups of G. Consequently, $\mathrm{L}^{\perp}(G)$ is the sublattice containing all permutable subgroups of G, $\mathrm{M}^{\perp}(G)$ the one containing all subgroups permutable with all maximal subgroups of G and so on for $\mathrm{sn}^{\perp}(G)$ and $\mathrm{n}^{\perp}(G) = \mathrm{L}(G)$. Immediately, the role of the operator $\perp$ appears to be very intriguing for the structure of G and several authors investigated this aspect. For instance, G is quasihamiltonian if and only if $\mathrm{L}(G) = \mathrm{L}^{\perp}(G)$.

Finally let us note that the study of probability aspects of finite groups has enjoyed a rapid evolution in the last years as indicate the recent literature (e. g. [3, 5, 8, 9, 10, 11, 15, 16]). New concepts appear in the form of probability that randomly chosen elements or subgroups satisfy some prescribed condition.

In this setting we will describe in Section 2 a notion of probability on $\mathrm{L}(G)$, starting from groups in which the subgroups in $\mathrm{sn}(G)$ permute with those in $\mathrm{M}(G)$. The generality of the methods (we follow [3, 5, 8, 9, 10, 11, 15]) may be translated in terms of arbitrary sublattices, satisfying a prescribed restriction. Section 3 shows some consequences on the size of $|\mathrm{L}(G)|$.

2 Measure theory on subgroup lattices

The following notion has analogies with [5, Definitions 2.1,3.1,4.1] and [10, Equation 1.1] and will be treated as in [3, 5, 8, 9, 10, 11, 15].

Definition 2.1. *For a group G,*

$$spd(G) = \frac{|\{(X,Y) \in \mathrm{sn}(G) \times \mathrm{M}(G) \mid XY = YX\}|}{|\mathrm{sn}(G)|\ |\mathrm{M}(G)|}, \tag{2.1}$$

is the subgroup S–commutativity degree of G.

It is clear that two isomorphic groups have the same subgroup S–commutativity degree. The value $0 < spd(G) \leq 1$ denotes the probability that a randomly picked pair $(X, Y) \in \mathrm{sn}(G) \times \mathrm{M}(G)$ is permuting, that is, $XY = YX$. The equality (2.1) may be rewritten, introducing the function $\chi : \mathrm{sn}(G) \times \mathrm{M}(G) \rightarrow \{0, 1\}$ defined by

$$\chi(X,Y) = \begin{cases} 1, & \text{if } XY = YX, \\ 0, & \text{if } XY \neq YX, \end{cases} \tag{2.2}$$

in the following form

$$spd(G) = \frac{1}{|\mathrm{sn}(G)|\ |\mathrm{M}(G)|} \sum_{(X,Y)\in \mathrm{sn}(G)\times \mathrm{M}(G)} \chi(X,Y). \tag{2.3}$$

In Definition 2.1 and (2.3), we may replace $\mathrm{sn}(G) \times \mathrm{M}(G)$ with $\mathrm{S}(G) \times \mathrm{T}(G)$, where $\mathrm{S}(G)$ and $\mathrm{T}(G)$ are two arbitrary sublattices of $\mathrm{L}(G)$. For instance, [1, 2] describe the structure of the groups in which the subnormal subgroups permute with all Sylow subgroups (called *PST–groups*). If $\mathrm{Syl}(G)$ is the set of all Sylow subgroups of G, we may consider $\mathrm{S}(G) = \mathrm{sn}(G)$, $\mathrm{T}(G) = \mathrm{Syl}(G)$, and we have already a classification for a group G such that $\mathrm{sn}(G) \subseteq \mathrm{Syl}(G)^{\perp}$.

The formula (2.3) allows us to treat the problem from the point of view of the measure theory on groups. A computational advantage may be found in the calculation of $spd(G_1 \times G_2)$, where G_1 and G_2 are two given groups.

Corollary 2.2. *Let G_i be a family of groups of coprime orders for $i = 1, 2, \ldots, k$. Then $spd(G_1 \times G_2 \times \ldots \times G_k) = spd(G_1)\, spd(G_2)\, \ldots\, spd(G_k)$.*

Proof. The proof is an application of (2.3). We illustrate only the case of two factors. In any lattice, in particular in $\mathrm{L}(G_1 \times G_2)$, we know that $\mathrm{L}(G_1 \times G_2) \neq \mathrm{L}(G_1) \times \mathrm{L}(G_2)$ (see [6] or [13]), but, if $\gcd(|G_1|, |G_2|) = 1$, then $\mathrm{L}(G_1) \cap \mathrm{L}(G_2)$ is trivial, and the above passage is allowed. The same happens for the lattices $\mathrm{M}(G_1 \times G_2)$ and $\mathrm{sn}(G_1 \times G_2)$, whenever $\gcd(|G_1|, |G_2|) = 1$. Therefore

$$\begin{aligned}
&|\mathrm{sn}(G_1 \times G_2)||\mathrm{M}(G_1 \times G_2)|spd(G_1 \times G_2) \\
&\quad = |\mathrm{sn}(G_1)||\mathrm{sn}(G_2)||\mathrm{M}(G_1)||\mathrm{M}(G_2)|spd(G_1 \times G_2) \\
&\qquad = |\mathrm{sn}(G_1)|\ |\mathrm{sn}(G_2)|\ |\mathrm{M}(G_1)|\ |\mathrm{M}(G_2)| \cdot \\
&\qquad\qquad \sum_{((X_1,X_2),(Y_1,Y_2))\in \mathrm{sn}(G_1\times G_2)\times \mathrm{M}(G_1\times G_2)} \chi((X_1,X_2),(Y_1,Y_2)) \\
&\quad = \left(|\mathrm{sn}(G_1)|\ |\mathrm{M}(G_1)| \sum_{(X_1,Y_1)\in \mathrm{sn}(G_1)\times \mathrm{M}(G_1)} \chi(X_1,Y_1) \right) \cdot \\
&\left(|\mathrm{sn}(G_2)|\ |\mathrm{M}(G_2)| \sum_{(X_2,Y_2)\in \mathrm{sn}(G_2)\times \mathrm{M}(G_2)} \chi(X_2,Y_2) \right) = spd(G_1) \cdot spd(G_2).
\end{aligned}$$
■

Corollary 2.2 shows the stability with respect to forming direct products of $spd(G)$; similar results can be found in [3, 5, 8, 10, 11, 15] in different contexts. Another basic property one could investigate is how to relate $spd(G)$ to quotients and subgroups of G.

Let $G = NH$ for a normal subgroup N of G and a subgroup H of G isomorphic to G/N (briefly, $H \simeq G/N$). It is easy to check that $\mathrm{sn}(G/N)$ is lattice isomorphic to $\mathrm{sn}(H)$ (briefly, $\mathrm{sn}(G/N) \sim \mathrm{sn}(H)$) and that $\mathrm{M}(G/N) \sim \mathrm{M}(H)$. Then (2.3) allows us to conclude

$$\sum_{(X,Y)\in \mathrm{sn}(G)\times \mathrm{M}(G)} \chi(X,Y) \geq \sum_{(X/N,Y/N)\in \mathrm{sn}(G/N)\times \mathrm{M}(G/N)} \chi(X/N,Y/N) \tag{2.4}$$
$$= \sum_{(Z,T)\in \mathrm{sn}(H)\times \mathrm{M}(H)} \chi(Z,T).$$

Now, several groups of small order and computational evidences, suggested by [17], show that the following condition may be satisfied:

$$\sum_{(X,Y)\in \mathrm{sn}(G)\times \mathrm{M}(G)} \chi(X,Y) \geq \sum_{(X,Y)\in \mathrm{sn}(N)\times \mathrm{M}(N)} \chi(X,Y). \tag{2.5}$$

We are not saying that the above condition is always true, but that it is satisfied by large classes of groups. Consequently,

$$2\,|\mathrm{sn}(G)|\,|\mathrm{M}(G)|\,spd(G) \geq \sum_{(X,Y)\in \mathrm{sn}(N)\times \mathrm{M}(N)} \chi(X,Y) + \sum_{(Z,T)\in \mathrm{sn}(H)\times \mathrm{M}(H)} \chi(Z,T). \tag{2.6}$$
$$= |\mathrm{sn}(N)|\,|\mathrm{M}(N)|\,spd(N) + |\mathrm{sn}(G/N)|\,|\mathrm{M}(G/N)|\,spd(G/N).$$

Similar techniques have been used by Tărnăuceanu [15] in order to study the *subgroup commutativity degree*

$$sd(G) = \frac{|\{(X,Y) \in \mathrm{L}(G)^2 \mid XY = YX\}|}{|\mathrm{L}(G)|^2} = \frac{1}{|\mathrm{L}(G)|^2} \sum_{(X,Y)\in \mathrm{L}(G)^2} \chi(X,Y). \tag{2.7}$$

Actually, the paper [15] can be seen as a natural extension, in the context of lattice theory, of the concept of *commutativity degree*

$$d(G) = \frac{|\{(x,y) \in G^2 \mid xy = yx\}|}{|G|^2} = \frac{1}{|G|^2} \sum_{x\in G} |C_G(x)|, \tag{2.8}$$

where $C_G(x) = \{g \in G \mid gx = xg\}$. Several contributions on $d(G)$ can be found in [3, 5, 8, 9, 10, 11]. The main strategy of investigation starts with a characterization of the case $d(G) = 1$ (the abelian case), then one notes that a nonabelian group G should have $d(G) \leq \frac{5}{8}$ and successively one studies what happens for the cases which are close to $d(G) = 0$. Upper and lower bounds will then "measure" the distance from known classes of groups. For instance, $d(G) = 1$ if and only if G is abelian, and $sd(G) = 1$ if and only if $\mathrm{L}(G) = \mathrm{L}(G)^{\perp}$. Therefore, the next results are important steps for the rest of the paper.

Corollary 2.3. *In a group G we have $spd(G) = 1$ if and only if $\mathrm{M}(G) \subseteq \mathrm{sn}^{\perp}(G)$.*

Proof. It follows from the above considerations. ∎

Corollary 2.4. *Let G be a group. If G is nilpotent, then $spd(G) = 1$.*

Proof. It follows from Corollary 2.3, noting that $\mathrm{M}(G) \subseteq \mathrm{n}(G) \subseteq \mathrm{sn}^{\perp}(G)$. ■

Corollary 2.5. *In a group G we have $\frac{|\mathrm{sn}(G)|\ |\mathrm{M}(G)|}{|\mathrm{L}(G)|^2}\ spd(G) \leq\ sd(G)$.*

Proof. Since $\mathrm{sn}(G) \times \mathrm{M}(G) \subseteq \mathrm{L}(G)^2$,we have that $\{(X,Y) \in \mathrm{sn}(G) \times \mathrm{M}(G) \mid XY = YX\} \subseteq \{(X,Y) \in \mathrm{L}(G)^2 \mid XY = YX\}$ and then

$$\begin{aligned}|\mathrm{sn}(G)|\ |\mathrm{M}(G)|\ spd(G) &= |\{(X,Y) \in \mathrm{sn}(G) \times \mathrm{M}(G) \mid XY = YX\}| \\ &\leq |\{(X,Y) \in \mathrm{L}(G)^2 \mid XY = YX\}| = |\mathrm{L}(G)|^2\ sd(G)\end{aligned}$$

from which the inequality follows. ■

Corollary 2.4 clarifies the situation for nilpotent groups. Then we proceed to study solvable groups. Unfortunately, these cannot be described as in [15, Proposition 2.4], and different techniques are necessary.

We recall now that an abelian group A of order $n = p_1^{n_1} p_2^{n_2} \dots p_m^{n_m}$, for suitable powers of $p_1, p_2, \dots, p_m \in \pi(A)$, has a canonical decomposition of the form $A \simeq A_1 \times A_2 \times \dots \times A_m$, where $n_1, \dots, n_m$ are positive integers and $A_1, A_2 \dots, A_m$ are the primary factors. It is well–known that, whenever the p_i's are all distinct (as in this case), $|\mathrm{L}(A)| = |\mathrm{L}(A_1)| \cdot |\mathrm{L}(A_2)| \cdot \dots \cdot |\mathrm{L}(A_m)|$. On the other hand, [16, Proposition 3.2] shows that the number of maximal subgroups of the p–group $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \dots \times \mathbb{Z}_{p^{\alpha_k}}$ is equal to $\frac{p^k-1}{p-1}$, for suitable integers $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_k$ and $k \geq 1$.

We will also use the fact that a cyclic group of prime order $\mathbb{Z}_p$ has $\mathrm{sn}(\mathbb{Z}_p) = \mathrm{L}(\mathbb{Z}_p) = \{\{1\}, \mathbb{Z}_p\}$, which is formed by only 2 elements, and $\mathrm{M}(\mathbb{Z}_p) = \{\{1\}\}$, which is formed by the trivial subgroup.

Lemma 2.6. *Let $N = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ be a normal abelian subgroup of a group G with $1 \leq \alpha_1 \leq \alpha_2$. If $G = NH$ with $G/N \simeq H$ of prime order and* (2.5) *is satisfied, then*

$$spd(G) \geq \frac{f(p,\alpha_1,\alpha_2)}{2\ |\mathrm{sn}(G)|\ |\mathrm{M}(G)|},$$

where $f(p,\alpha_1,\alpha_2) = \frac{1}{p^2-2p+1}\Big((\alpha_2-\alpha_1+1)p^{\alpha_1+3} + 2p^{\alpha_1+2} - (\alpha_2-\alpha_1-1)p^{\alpha_1+1} - (\alpha_1+\alpha_2+1)p^2 - 6p + (\alpha_1+\alpha_2+3)\Big)$ *is a function depending on N.*

Proof. Since (2.4) and (2.5) are satisfied, we may apply (2.6) and get

$$spd(G) \geq \frac{|\mathrm{sn}(N)|\ |\mathrm{M}(N)| spd(N) + |\mathrm{sn}(G/N)|\ |\mathrm{M}(G/N)| spd(G/N)}{2\ |\mathrm{sn}(G)|\ |\mathrm{M}(G)|}. \tag{2.9}$$

Since $|\mathrm{sn}(G/N)| = 2$, $|\mathrm{M}(G/N)| = 1$ and $spd(N) = spd(G/N) = 1$, we may apply Corollary 2.4, and we obtain

$$\begin{aligned}&= \frac{|\mathrm{sn}(N)|\ |\mathrm{M}(N)|}{2\ |\mathrm{sn}(G)|\ |\mathrm{M}(G)|} + \frac{2}{2\ |\mathrm{sn}(G)|\ |\mathrm{M}(G)|} \\ &\qquad\qquad = \frac{1}{2\ |\mathrm{sn}(G)|\ |\mathrm{M}(G)|}\left(|\mathrm{sn}(N)|\ |\mathrm{M}(N)| + 2\right).\end{aligned} \tag{2.10}$$

Now [16, Theorem 3.3] implies that $|\mathrm{sn}(N)| = |\mathrm{L}(N)| = \frac{1}{(p-1)^2}[(\alpha_2-\alpha_1+1)p^{\alpha_1+2} - (\alpha_2-\alpha_1-1)p^{\alpha_1+1} - (\alpha_1+\alpha_2+3)p + (\alpha_1+\alpha_2+1)]$, and, as noted above, $|\mathrm{M}(N)| = \frac{p^2-1}{p-1} = p+1$. Hence the right hand side of (2.10) is equal to

$$\frac{1}{2\,|\mathrm{sn}(G)|\,|\mathrm{M}(G)|} \cdot \Big(\frac{p+1}{(p-1)^2}\Big((\alpha_2-\alpha_1+1)p^{\alpha_1+2} - (\alpha_2-\alpha_1-1)p^{\alpha_1+1} - (\alpha_1+\alpha_2+3)p + (\alpha_1+\alpha_2+1)\Big) + 2\Big). \tag{2.11}$$

In order to better write the expression above, let us introduce the coefficients

$$C_1 = \alpha_2 - \alpha_1 + 1;\ C_2 = \alpha_2 - \alpha_1 - 1;\ C_3 = \alpha_1 + \alpha_2 + 3; C_4 = \alpha_1 + \alpha_2 + 1$$

obtaining

$$= \frac{1}{2\,|\mathrm{sn}(G)|\,|\mathrm{M}(G)|}\Big(\frac{p+1}{(p-1)^2}\,(C_1p^{\alpha_1+2} - C_2p^{\alpha_1+1} - C_3p + C_4) + \frac{(p-1)^2}{(p-1)^2}\cdot 2\Big)$$

$$= \frac{1}{2\,|\mathrm{sn}(G)|\,|\mathrm{M}(G)|}\Big(\frac{1}{(p-1)^2}\Big)\Big((p+1)(C_1p^{\alpha_1+2} - C_2p^{\alpha_1+1} - C_3p + C_4) + 2(p-1)^2\Big)$$

$$= \frac{C_1p^{\alpha_1+3} + (C_1-C_2)p^{\alpha_1+2} - C_2p^{\alpha_1+1} + (2-C_3)p^2 + (C_4-C_3-4)p + (C_4+2)}{2\,|\mathrm{sn}(G)|\,|\mathrm{M}(G)|\,(p-1)^2}.$$

Developing the computations in the brackets, we get $f(p,\alpha_1,\alpha_2)$. ■

Lemma 2.6 may be adapted to $sd(G)$ in the following way.

Lemma 2.7. *Let $N = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$ be a normal subgroup of a group G with $1 \le \alpha_1 \le \alpha_2$. If $G = HN$ with $G/N \simeq H$ of prime order and* (2.5) *is satisfied, then*

$$sd(G) \ge \frac{g(p,\alpha_1,\alpha_2)}{2\,|\mathrm{L}(G)|^2},$$

where $g(p,\alpha_1,\alpha_2) = \frac{1}{(p-1)^4}\Big((\alpha_2-\alpha_1+1)p^{\alpha_1+2} - (\alpha_2-\alpha_1-1)p^{\alpha_1+1} - (\alpha_1+\alpha_2+3)p + (\alpha_1+\alpha_2+1)\Big)^2 + 4$ *is a function depending on N.*

Proof. Since (2.4) and (2.5) are satisfied, (2.6) becomes

$$sd(G) \ge \frac{|\mathrm{L}(N)|^2 sd(N) + |\mathrm{L}(G/N)|^2 spd(G/N)}{2\,|\mathrm{L}(G)|^2} \tag{2.12}$$

and, from the assumptions, $|\mathrm{L}(G/N)| = 2$, $sd(G/N) = sd(N) = 1$. But, once again by [16, Theorem 3.3], we have that $|\mathrm{L}(N)|^2 = \frac{1}{(p-1)^4}\Big((\alpha_2-\alpha_1+1)p^{\alpha_1+2} - (\alpha_2-\alpha_1-1)p^{\alpha_1+1} - (\alpha_1+\alpha_2+3)p + (\alpha_1+\alpha_2+1)\Big)^2$. Therefore

$$= \frac{1}{2\,|\mathrm{L}(G)|^2}\Big(\frac{1}{(p-1)^4}\Big((\alpha_2-\alpha_1+1)p^{\alpha_1+2} - (\alpha_2-\alpha_1-1)p^{\alpha_1+1} - (\alpha_1+\alpha_2+3)p + (\alpha_1+\alpha_2+1)\Big)^2 + 4\Big), \tag{2.13}$$

where one sees the function $g(p,\alpha_1,\alpha_2)$, which we were looking for. ■

Let us denote, as usual, by $Fit(G)$ the *Fitting subgroup* of G.

Theorem 2.8. *Let G be a solvable group in which $C = C_G(Fit(G)) = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$, for $1 \leq \alpha_1 \leq \alpha_2$, p a prime and $|G : C|$ a prime.*

(i) If (2.5) *is satisfied, then $spd(G) \geq \frac{f(p,\alpha_1,\alpha_2)}{2\,|\mathrm{sn}(G)|\,|\mathrm{M}(G)|}$, where $f(p, \alpha_1, \alpha_2)$ is a function depending on C.*

(ii) If (2.5) *is satisfied, $sd(G) \geq \frac{g(p,\alpha_1,\alpha_2)}{2\,|\mathrm{L}(G)|^2}$, where $g(p, \alpha_1, \alpha_2)$ is a function depending on C.*

Proof. Since G is solvable, it is well–known that C is an abelian normal subgroup of G. Moreover, by hypothesis, we also have that $C = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}}$, with $1 \leq \alpha_1 \leq \alpha_2$, p prime and G/C is of prime order. Now (i) is an application of Lemma 2.6 and (ii) of Lemma 2.7. ■

The lower bound in Lemma 2.7 for $sd(G)$ is more precise than the following bound, which was the first to be presented in literature.

Corollary 2.9 (See [15], Corollary 2.6)**.** *A group G possessing a normal abelian subgroup of prime index has $|\mathrm{L}(G)|^2\, sd(G) \geq |\mathrm{L}(N)|^2 + 2|\mathrm{L}(N)| + 1$.*

A different restriction is obtained when we multiply up (2.4) and (2.5).

Proposition 2.10. *Let N be a normal subgroup of a group $G = NH$ satisfying* (2.4) *and* (2.5)*. Then*

$$spd(G) \geq \frac{1}{|\mathrm{sn}(G)|\,|\mathrm{M}(G)|}\sqrt{\sum_{\substack{(X,Y)\in \mathrm{sn}(N)\times \mathrm{M}(N)\\ (Z,T)\in \mathrm{sn}(H)\times \mathrm{M}(H)}} \chi(X,Y)\,\chi(Z,T)}.$$

Proof. From (2.4), (2.5) and the Cauchy inequality for numerical series, we have

$$\begin{aligned}|\mathrm{sn}(G)|^2\,|\mathrm{M}(G)|^2\,spd(G)^2 &\geq \sum_{(X,Y)\in \mathrm{sn}(N)\times \mathrm{M}(N)} \chi(X,Y) \cdot \sum_{(Z,T)\in \mathrm{sn}(H)\times \mathrm{M}(H)} \chi(Z,T)\\ &\geq \sum_{\substack{(X,Y)\in \mathrm{sn}(N)\times \mathrm{M}(N)\\ (Z,T)\in \mathrm{sn}(H)\times \mathrm{M}(H)}} \chi(X,Y)\,\chi(Z,T). \qquad (2.14)\end{aligned}$$

Since all the quantities are positive, then, extracting the square root, the result follows. ■

The next result answers in a certain sense to [15, Problem 4.1].

Corollary 2.11. *Let N be a normal subgroup of a group $G = NH$, satisfying* (2.4) *and* (2.5)*. Then*

$$sd(G) \geq \frac{1}{|\mathrm{L}(G)|^2}\sqrt{\sum_{\substack{(X,Y)\in \mathrm{L}(N)^2\\ (Z,T)\in \mathrm{L}(H)^2}} \chi(X,Y)\,\chi(Z,T)}.$$

Proof. Mutatis mutandis, we may argue as in Proposition 2.10. ■

3 Applications and final considerations

The symmetric group on 3 elements $S_3 = \mathbb{Z}_2 \ltimes \mathbb{Z}_3 = \langle a, b \mid a^3 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ has $sd(S_3) = \frac{5}{6}$ (see [15, p.2510]), is metabelian and satisfies the description in Theorem 2.8, since (see below) it is an example of a *primitive group of affine type* [4]. This group was the origin of our investigation. In fact, a primitive group P of affine type is a semidirect product with normal factor $Fit(P)$. Furthermore, $Fit(P)$ turns out to be elementary abelian and $C_P(Fit(P)) = Fit(P)$. This means that Theorem 2.8 gives a good description for the subgroup commutativity degree and for the subgroup S–commutativity degree of such groups. While [5, 9, 10, 11] show that we may classify a group, whenever restrictions on $d(G)$ are given, the problem remains still open for $sd(G)$ and $spd(G)$. We illustrate here just one case, involving $sd(G)$. This is to justify the interest of Section 2 in the new bounds.

Corollary 3.1. *A metabelian group G with $|G'|$ and $|G/G'|$ of prime orders is cyclic, whenever the bound in Corollary 2.9 is achieved with $sd(G) = \frac{5}{6}$.*

Proof. We begin from $|\mathrm{L}(G)|^2 \, \frac{5}{6} = |\mathrm{L}(N)|^2 + 2|\mathrm{L}(N)| + 1$, which becomes $|\mathrm{L}(G)|^2 \, \frac{5}{6} = 4 + 4 + 1 = 9$, then $2 \leq |\mathrm{L}(G)| = \sqrt{\frac{56}{5}} = \sqrt{11.2} < 4$. This implies either $|\mathrm{L}(G)| = 2$ or $|\mathrm{L}(G)| = 3$. In the first case, G is cyclic of prime order. In the second case, G is lattice isomorphic to $\mathbb{Z}_{p^2}$ for a suitable prime p. In both cases G is cyclic. ∎

The control of $|\mathrm{L}(G)|$ was the main ingredient in the previous proof. Unfortunately, formulas for the growth of $\mathrm{L}(G)$ are hard to find out and [14] helps our investigations. The *Möbius number* of $\mathrm{L}(G)$ is a number which allows us to control the size of $|\mathrm{L}(G)|$. In case of a symmetric group S_n, it is denoted by $\mu(1, S_n)$ and was conjectured to be $(-1)^{n-1}$ $(|\mathrm{Aut}(S_n)|/2)$ for all $n > 1$ (see [14, p.1]). For $n \leq 11$, this was proved by H. Pahlings. Recent progresses are summarized below.

Theorem 3.2 (See [14], Theorems 1.6, 1.8, 1.10)**.**

(i) Let p be a prime. Then $\mu(1, S_p) = (-1)^{p-1} \, \frac{p!}{2}$.

(ii) Let $n = 2p$ and p be an odd prime. Then

$$\mu(1, S_n) = \begin{cases} -n!, & \text{if } n-1 \text{ is prime and } p \equiv 3 \mod 4, \\ \frac{n!}{2}, & \text{if } n = 22, \\ -\frac{n!}{2}, & \text{otherwise.} \end{cases}$$

(iii) Let $n = 2^a$ for an integer $a \geq 1$. Then $\mu(1, S_n) = -\frac{n!}{2}$.

Let $\mu(1, G) \in \{\mu(1, S_p), \mu(1, S_n)\}$, being $\mu(1, S_p)$ and $\mu(1, S_n)$ the values in Theorem 3.2 under the given restrictions on n and p. Now we can be more precise.

Corollary 3.3. *Under the assumptions of Theorem 2.8, let G be a solvable group such that $|\mathrm{L}(G)| = \mu(1, G)$. Then $sd(G) \geq \frac{g(p,\alpha_1,\alpha_2)}{2 \, \mu(1,G)^2}$, where $g(p, \alpha_2, \alpha_2)$ is a function depending on $C_G(Fit(G))$.*

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