Global representation of some mixed ultradistributions

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Abstract

Given non empty open subsets Ω of $\mathbb{R}^r$ and Ω' of $\mathbb{R}^s$, and sequences $\mathscr{M}$ and $\mathscr{M}'$, we recall the definition of the space $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$. Given $p \in [1, +\infty[$, we also introduce the space $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(\Omega \times \Omega')$. By use of a basic idea due to Valdivia, we obtain a global representation of the corresponding ultradistributions, i.e. of the elements of the topological duals of these spaces.

1 Introduction

For the notations, we refer to the Paragraphs 2 and 5.

Classically the non quasi-analytic classes of ultradifferentiable functions are defined by use of special sequences of positive numbers or of weights, the basic references are [2] and [1] respectively. These two possibilities lead to similar properties but are not completely equivalent.

In this paper, we adopt the first possibility and continue the study of the locally convex properties of the mixed non quasi-analytic classes of ultradifferentiable functions initiated in [3] to [8]. This time we consider the problem of obtaining a global representation of some mixed ultradistributions, i.e. explicitly of the elements of the topological duals of the spaces $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$ and $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(\Omega \times \Omega')$.

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The paper is based on a method developed by Valdivia in [10], [11] and [12] to get a global representation of the elements of the topological duals of the spaces $\mathscr{D}^{\{\mathscr{M}\}}(\Omega)$ and $\mathscr{D}^{\{\mathscr{M}\}}_{(L_p)}(\Omega)$. In [8] we already used this method to get a global representation of the elements of the dual of the projective limits $\mathscr{D}^{(\mathfrak{M})}(\Omega)$ and $\mathscr{D}^{(\mathfrak{M}')}_{(L_p)}(\Omega)$.

In Paragraph 3, we adapt to our case the basic abstract construction leading to a global representation.

In Paragraph 4, we recall the definition of the space $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$ and obtain a global representation of the elements of its topological dual by means of series using Radon measures on $\Omega \times \Omega'$.

Starting with Paragraph 5, we do the same for the space $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(\Omega \times \Omega')$.

2 Notations

Throughout the paper,
a) r and s belong to $\mathbb{N}$;
b) Ω and Ω' are non empty open subsets of $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively;
c) $\boldsymbol{m} = (m_p)_{p\in\mathbb{N}_0}$ and $\boldsymbol{m}' = (m'_p)_{p\in\mathbb{N}_0}$ are two sequences of real numbers such that $m_0 = m'_0 = 1$, which are increasing and non quasi-analytic, i.e. $\sum_{p=0}^{\infty} 1/m_p < \infty$ and $\sum_{p=0}^{\infty} 1/m'_p < \infty$. The sequences $\mathscr{M} = (M_p)_{p\in\mathbb{N}_0}$ and $\mathscr{M}' = (M'_p)_{p\in\mathbb{N}_0}$ are defined by $M_p = m_0 \dots m_p$ and $M'_p = m'_0 \dots m'_p$ for every $p \in \mathbb{N}_0$.
d) the space $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$ is defined as in [3]. For the sake of clarity, let us briefly recall its definition.

The notation $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),h}(K \times K')$ requires that h is a positive number and that K and K' are non empty compact subsets of $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively. It designates the following Banach space: its elements are the $\mathcal{C}^\infty$-functions φ on $\mathbb{R}^r \times \mathbb{R}^s$ which have their support contained in $K \times K'$ and such that

$$\|\varphi\|_{K\times K',h} := \sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \frac{||\mathrm{D}^{(\alpha,\beta)}\varphi||_{K\times K'}}{h^{|\alpha|+|\beta|}M_{|\alpha|}M'_{|\beta|}} < \infty,$$

endowed with the norm $\|.\|_{K\times K',h}$.

The space $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K')$ is then the inductive limit of the spaces $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),m}(K \times K')$ with $m \in \mathbb{N}$; it is a (DFS)-space.

Finally the space $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$ is the inductive limit of the spaces $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K_n \times K'_n)$ where $(K_n)_{n\in\mathbb{N}}$ and $(K'_n)_{n\in\mathbb{N}}$ are exhaustions of Ω and Ω' respectively by strictly regular compact sets such that $K_n \subset K^\circ_{n+1}$ and $K'_n \subset K'^\circ_{n+1}$ for every $n \in \mathbb{N}$. It is a (DFS)-space.

To be complete, let us recall that a compact subset H of $\mathbb{R}^n$ is *strictly regular* if it has a finite number of connected components and if each of these connected components B verifies the following two properties:
(i) B is *regular*, i.e. $B = B^{\circ -}$;
(ii) there is a positive constant C such that, for every $x, y \in B$, there is a polygonal

path joining x and y in B°, of length $L \leq C\,|x-y|$.
e) the space $\mathscr{D}_{(L^p)}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$ is new; it is defined in Paragraph 5.

Let us also mention that, if E is a locally convex space, E' designates its strong topological dual.

3 Basic construction

Let $X = (X, \|.\|)$ be a Banach space.

Then, for every $n \in \mathbb{N}$, Y_n designates the following Banach space: it is the vector subspace of $X^{\mathbb{N}_0^r \times \mathbb{N}_0^s}$ which elements $\varkappa = (x_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{N}_0^r \times \mathbb{N}_0^s}$ verify

$$\|\varkappa\|_n := \sup_{(\alpha,\beta)\in\mathbb{N}_0^r \times \mathbb{N}_0^s} \frac{\|x_{\alpha,\beta}\|}{n^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|}} < \infty,$$

endowed with the norm $\|.\|_n$.

It is clear that, for every $n \in \mathbb{N}$, Y_n is a vector subspace of Y_{n+1} and that the canonical inclusion map is continuous. Therefore we may consider the inductive limit Y of these spaces; it is clear that Y is a Hausdorff (LB)-space.

Let us improve this knowledge. It is well known that the strong topological dual Y' of Y is a Fréchet space. Moreover, if $B(Y_n)$ denotes the closed unit ball of Y_n, the polar sets U_n of $nB(Y_n)$ in Y' constitute a 0-neighbourhood basis in Y'. So the polar set of any bounded subset B of Y contains some U_n which implies that B is contained in the closure of some $nB(Y_n)$ in Y.

Proposition 3.1. *For every $n \in \mathbb{N}$, $B(Y_n)$ is a closed subset of Y.*
Therefore Y is a Hausdorff regular (LB)*-space.*

Proof. Let $(\varkappa_j)_{j\in J}$ be a generalized sequence of $B(Y_n)$ converging to $\varkappa$ in Y. Then, for every $(\alpha,\beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$ and $v \in X'$, it is clear that

$$u\colon Y \to \mathbb{C}; \quad \varkappa \mapsto \langle x_{\alpha,\beta}, v\rangle$$

is a well defined continuous linear functional. This implies

$$\lim_j \langle x_{\alpha,\beta,j} - x_{\alpha,\beta}, v\rangle = \lim_j \langle \varkappa_j - \varkappa, u\rangle = 0;$$

i.e. the generalized sequence $(x_{\alpha,\beta,j})_{j\in J}$ converges weakly to $x_{\alpha,\beta}$ in X. As the generalized sequence $(x_{\alpha,\beta,j})_{j\in J}$ is made of elements of the closed ball $n^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|} B(X)$, a weakly closed subset of X indeed, we conclude at once. ■

Notation. a) For every $u \in Y'$ and $n \in \mathbb{N}$, we set

$$\|u\|_{(n)} := \sup_{\varkappa\in B(Y_n)} |\langle \varkappa, u\rangle|\,.$$

b) To every $u \in Y'$, we associate for every $(\alpha,\beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$ the following continuous linear functional on X

$$u_{\alpha,\beta}\colon X \to \mathbb{C}; \quad x \mapsto \langle \varkappa, u\rangle$$

where $\varkappa$ is defined by $x_{\alpha,\beta} = x$ and $x_{\gamma,\delta} = 0$ if $(\gamma,\delta) \neq (\alpha,\beta)$.

Proposition 3.2. *For every $u \in Y'$, we have*

$$\sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} n^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|} \left\| u_{\alpha,\beta} \right\| \leq \|u\|_{(n)}, \quad \forall n \in \mathbb{N},$$

as well as

$$\langle \varkappa, u \rangle = \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \langle x_{\alpha,\beta}, u_{\alpha,\beta} \rangle, \quad \forall \varkappa \in Y,$$

the series converging absolutely and uniformly on the bounded subsets of Y.

Proof. As $B(Y_n)$ certainly contains

$$n^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|} \{ \varkappa \colon x_{\alpha,\beta} \in B(X), x_{\gamma,\delta} = 0 \text{ if } (\gamma,\delta) \neq (\alpha,\beta) \}$$

the inequality is clear.

For every $\varkappa \in Y$ and $(\alpha,\beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$, let us denote by $\varkappa^{(\alpha,\beta)}$ the element of Y defined by $x_{\alpha,\beta}^{(\alpha,\beta)} = x_{\alpha,\beta}$ and $x_{\gamma,\delta}^{(\alpha,\beta)} = 0$ if $(\gamma,\delta) \neq (\alpha,\beta)$. Then, for every $\varkappa \in Y_n$, $m \geq 2n$ and $q \in \mathbb{N}$, we certainly have

$$\left\| \varkappa - \sum_{|\alpha|+|\beta|\leq q} \varkappa^{(\alpha,\beta)} \right\|_m = \sup_{|\alpha|+|\beta|>q} \frac{\left\| x_{\alpha,\beta} \right\|}{m^{|\alpha|+|\beta|} M_{|\alpha|} M'_\beta} \leq 2^{-q} \|\varkappa\|_n .$$

This implies that the family $(\varkappa^{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$ is summable to $\varkappa$ in Y_m hence the representation formula as well as the convergence property of the series. ■

Proposition 3.3. *For every family $(z_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$ of elements of X' such that*

$$A_h := \sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} h^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|} \left\| z_{\alpha,\beta} \right\| < \infty, \quad \forall h > 0,$$

there is a unique element u of Y' such that $u_{\alpha,\beta} = z_{\alpha,\beta}$ for every $(\alpha,\beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$.

Proof. For every $\varkappa \in Y_n$ and $(\alpha,\beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$, we certainly have

$$\left| \langle x_{\alpha,\beta}, z_{\alpha,\beta} \rangle \right| \leq \frac{\left\| x_{\alpha,\beta} \right\|}{C} . C \left\| z_{\alpha,\beta} \right\|_{\leq} \frac{A_{2(r+s)n}}{(2(r+s))^{|\alpha|+|\beta|}} \|\varkappa\|_n$$

with $C := (2(r+s)n)^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|}$, hence

$$u \colon Y \to \mathbb{C}; \quad \varkappa \mapsto \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \langle x_{\alpha,\beta}, z_{\alpha,\beta} \rangle$$

is a well defined continuous linear functional on Y.

To conclude we just have then to note that, for every $x \in X$ and $(\alpha,\beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$, if $\varkappa \in Y$ is defined by $x_{\alpha,\beta} = x$ and $x_{\gamma,\delta} = 0$ if $(\gamma,\delta) \neq (\alpha,\beta)$, the representation formula of Proposition 3.2 leads to

$$\langle x, u_{\alpha,\beta} \rangle = \langle \varkappa, u \rangle = \sum_{(\gamma,\delta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \langle x_{\gamma,\delta}, z_{\gamma,\delta} \rangle = \langle x, z_{\alpha,\beta} \rangle . \qquad ■$$

4 Case of the space $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$

From now on we identify the Banach space X of the basic construction with $\mathcal{C}_0(\Omega \times \Omega')$, the space of the continuous functions f on $\Omega \times \Omega'$ "tending to 0 at infinity" (i.e. for every $\varepsilon > 0$, there is a compact subset H of $\Omega \times \Omega'$ such that $\|f\|_{(\Omega\times\Omega')\setminus H} \leq \varepsilon$), endowed with the norm $\|\cdot\|_{\Omega\times\Omega'}$. By the Riesz representation theorem, for every continuous linear functional u on $\mathcal{C}_0(\Omega \times \Omega')$, there is a Borel measure μ on $\Omega \times \Omega'$ such that $\langle u, \cdot\rangle = \int_{\Omega\times\Omega'} \cdot \, d\mu$ on $\mathcal{C}_0(\Omega \times \Omega')$ and $\|u\| = |\mu| (\Omega \times \Omega')$.

As Radon measures will also occur, let us mention the following. Given a non void compact subset H of $\mathbb{R}^k$, $\mathscr{K}(H)$ is the following Banach space: its elements are the continuous functions on $\mathbb{R}^k$ with support contained in H; its norm is $\|.\|_H$; it is topologically isomorphic to $\mathcal{C}_0(H^\circ)$. Then $\mathscr{K}(\Omega)$ is the inductive limit of the spaces $\mathscr{K}(H)$ where H runs through the family of the non void compact subsets of Ω. The Radon measures on Ω are the continuous linear functionals on $\mathscr{K}(\Omega)$. Given a Radon measure u on Ω, $\|u\|\,(H)$ designates the norm of the restriction of u to $\mathscr{K}(H)$.

Notation. For every $n \in \mathbb{N}$, the linear map

$$\xi_n \colon \mathscr{D}^{(\mathscr{M},\mathscr{M}'),n}(K \times K') \to Y_n; \quad \varphi \mapsto (\mathrm{D}^{(\alpha,\beta)}\varphi)_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$$

clearly is an isometry from $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),n}(K \times K')$ onto its image P_n considered as a topological vector subspace of Y_n. We then introduce P as the topological vector subspace $\cup_{n=1}^{\infty} P_n$ of Y and consider the map

$$\xi \colon \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K') \to P; \quad \varphi \mapsto (\mathrm{D}^{(\alpha,\beta)}\varphi)_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}.$$

It is clear that ξ is a continuous linear bijection. In the next result we prove that in fact ξ is a topological isomorphism; its proof makes use of the following result (cf. [11], Proposition 6): *let E be a locally convex space whose dual is a Fréchet space. If every absolutely convex, closed and bounded subset of the vector subspace F of E is locally compact, then F endowed with the $\rho(E, E')$-topology is a* (LB)-*space*, where $\rho(E, E')$ designates the topology on E of the uniform convergence on the absolutely convex compact subsets of E'.

Proposition 4.1. *The spaces $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K')$ and P are topologically isomorphic; more specifically, the map ξ is a topological isomorphism.*

Proof. We know that the strong dual of $Y(= E)$ is a Fréchet space. Let now A be any absolutely convex, closed and bounded subset of $P(= F)$. By Proposition 3.1, there is $n \in \mathbb{N}$ such that A is a bounded subset of P_n. Therefore $\xi_n^{-1}(A) = \xi^{-1}(A)$ is an absolutely convex and bounded subset of $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),n}(K \times K')$ that is closed in $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K')$, hence A is a compact subset of P_{n+1}. Therefore, by use of Proposition 6 of [11], $Q := (P, \rho(Y, Y'))$ is a (LB)-space. If R denotes the inductive limit of the Banach spaces $(P_n)_{n\in\mathbb{N}}$, the identity maps from R into P and from P into Q are of course continuous. Hence the conclusion by use of the closed graph theorem. ■

Proposition 4.2. *Let K and K' be non void compact subsets of Ω and Ω' respectively. For every continuous linear functional S on $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$, there is a family $(\mu_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$ of Borel measures on $\Omega \times \Omega'$ such that*

$$\sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} n^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|} |\mu_{\alpha,\beta}|(\Omega \times \Omega') < \infty, \quad \forall n \in \mathbb{N}, \tag{1}$$

and

$$\langle S, \varphi \rangle = \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \int_{\Omega\times\Omega'} \mathrm{D}^{(\alpha,\beta)} \varphi \, d\mu_{\alpha,\beta}, \quad \forall \varphi \in \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K'), \tag{2}$$

these series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K')$.

Proof. Let us denote by η the map ξ considered as a map from the space $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K')$ into Y. By the previous result we know that its transpose ${}^t\eta$ is surjective.

Now let S^* denote the restriction of S onto $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K')$. As ${}^t\eta$ is surjective, there is $u \in Y'$ such that ${}^t\eta(u) = S^*$ and the conclusion follows at once from Proposition 3.2. ■

In the next result we are going to represent the continuous linear functionals S on $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$, having a compact support contained in $H \times H'$ where H and H' are non void compact subsets of Ω and Ω' respectively.

For this purpose, let us introduce some notations. First of all we chose compact subsets K of Ω and K' of Ω' such that $H \subset K^\circ$ and $H' \subset K'^\circ$. Then we apply Proposition 4.2 and obtain a family $(\mu_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$ of Borel measures on $\Omega \times \Omega'$ verifying the inequalities (1) and giving rise to the representation formula (2), the series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K')$. Next we chose $\psi \in \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$, identically 1 on a neighbourhood of $H \times H'$ and of support contained in $K^\circ \times K'^\circ$. Finally we chose $n_0 \in \mathbb{N}$ such that $\psi \in \mathscr{D}^{(\mathscr{M},\mathscr{M}'),n_0}(K_{n_0} \times K'_{n_0})$.

This leads to the following property. For every $\varphi \in \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K')$, $\psi\varphi$ of course belongs to $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K')$ and verifies $\langle S, \varphi \rangle = \langle S, \psi\varphi \rangle$. Let us analyse $\langle S, \psi\varphi \rangle$: we have

$$\langle S, \psi\varphi \rangle = \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \int_{\Omega\times\Omega'} \sum_{\gamma\leq\alpha,\delta\leq\beta} \binom{\alpha}{\gamma}\binom{\beta}{\delta} \mathrm{D}^{(\gamma,\delta)}\psi \mathrm{D}^{(\alpha-\gamma,\beta-\delta)}\varphi \, d\mu_{\alpha,\beta}$$

and, choosing an integer $p > n_0$ such that $\varphi \in \mathscr{D}^{(\mathscr{M},\mathscr{M}'),p}(K \times K')$, we successively get

$$\sum_{\gamma\leq\alpha,\delta\leq\beta}\binom{\alpha}{\gamma}\binom{\beta}{\delta}\int_{\Omega\times\Omega'}|D^{(\gamma,\delta)}\psi||D^{(\alpha-\gamma,\beta-\delta)}\varphi|\,d|\mu_{\alpha,\beta}|$$
$$\leq \|\psi\|_{n_0}\sum_{\gamma\leq\alpha,\delta\leq\beta}\binom{\alpha}{\gamma}\binom{\beta}{\delta}n_0^{|\gamma|+|\delta|}M_{|\gamma|}M'_{|\delta|}\|D^{(\alpha-\gamma,\beta-\delta)}\varphi\|_{K\times K'}|\mu_{\alpha,\beta}|(\Omega\times\Omega')$$
$$\leq \|\psi\|_{n_0}2^{|\alpha|+|\beta|}\,\|\varphi\|_p\,p^{|\alpha|+|\beta|}M_{|\alpha|}M'_{|\beta|}|\mu_{\alpha,\beta}|(\Omega\times\Omega')$$
$$\leq \frac{\|\psi\|_{n_0}\|\varphi\|_p}{(2(r+s))^{|\alpha|+|\beta|}}\sup_{(\gamma,\delta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}(4p(r+s))^{|\gamma|+|\delta|}M_{|\gamma|}M'_{|\delta|}|\mu_{\gamma,\delta}|(\Omega\times\Omega'),$$

which implies the absolute convergence of the series giving the value of $\langle S,\psi\varphi\rangle$. Setting $\zeta=\alpha-\gamma$ and $\xi=\beta-\delta$, we finally obtain the representation

$$\langle S,\psi\varphi\rangle=\sum_{\substack{(\zeta,\xi)\in\mathbb{N}_0^r\times\mathbb{N}_0^s\\(\gamma,\delta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}}\frac{(\gamma+\zeta)!}{\gamma!\zeta!}\frac{(\delta+\xi)!}{\delta!\xi!}\int_{\Omega\times\Omega'}D^{(\gamma,\delta)}\psi D^{(\zeta,\xi)}\varphi\,d\mu_{\gamma+\zeta,\delta+\xi},$$

this series being absolutely converging.

Theorem 4.3. *Let H and K be compact subsets of Ω such that $H\subset K^\circ$ and let H' and K' be compact subsets of Ω' such that $H'\subset K'^\circ$.*

If $S\in\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega\times\Omega')'$ has its support contained in $H\times H'$, there is a family $(\mu_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$ of Borel measures on $\Omega\times\Omega'$ such that
a) $\sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}n^{|\alpha|+|\beta|}M_{|\alpha|}M'_{|\beta|}|\mu_{\alpha,\beta}|(\Omega\times\Omega')<\infty,\forall n\in\mathbb{N}$;
b) $\mathrm{supp}(\mu_{\alpha,\beta})\subset K\times K',\forall(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s$;
c) $\langle S,\varphi\rangle=\sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}\int_{\Omega\times\Omega'}D^{(\alpha,\beta)}\varphi\,d\mu_{\alpha,\beta},\forall\varphi\in\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega\times\Omega')$,
the series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega\times\Omega'')$.

Proof. The Proposition 4.2 provides a family $(\mu_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$ of Borel measures on $\Omega\times\Omega'$ verifying the inequalities (1) and the representation (2), the series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K\times K')$. So we may use the notations and the representation formula of $\langle S,\psi\varphi\rangle$ we just obtained.

Let us now consider (ζ,ξ) in $\mathbb{N}_0^r\times\mathbb{N}_0^s$. Given $f\in\mathcal{C}_0(\Omega\times\Omega')$, let us consider the expression

$$v_{\zeta,\xi}(f):=\sum_{(\gamma,\delta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}\frac{(\gamma+\zeta)!}{\gamma!\zeta!}\frac{(\delta+\xi)!}{\delta!\xi!}\int_{\Omega\times\Omega'}D^{(\gamma,\delta)}\psi\,f\,d\mu_{\gamma+\zeta,\delta+\xi}.$$

We first observe that this series converges absolutely since

$$\sum_{(\gamma,\delta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}2^{|\gamma+\zeta|}2^{|\delta+\xi|}\,\|\psi\|_{n_0}\,n_0^{|\gamma|+|\delta|}M_{|\gamma|}M'_{|\delta|}\,\|f\|_{\Omega\times\Omega'}\,|\mu_{\gamma+\zeta,\delta+\xi}|(\Omega\times\Omega')$$
$$\leq\|\psi\|_{n_0}\|f\|_{\Omega\times\Omega'}\sum_{(\gamma,\delta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}\frac{1}{(2(r+s))^{|\gamma|+|\delta|}}A$$

with

$$A := \sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} (4n_0(r+s))^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|} |\mu_{\alpha,\beta}|(\Omega\times\Omega') < \infty.$$

This implies that $v_{\zeta,\xi}$ is a continuous linear functional on $\mathcal{C}_0(\Omega\times\Omega')$. Therefore there is a Borel measure $\nu_{\zeta,\xi}$ on $\Omega\times\Omega'$ such that

$$v_{\zeta,\xi}(f) = \int_{\Omega\times\Omega'} f\,d\nu_{\zeta,\xi}, \quad \forall f \in \mathcal{C}_0(\Omega\times\Omega').$$

Summarizing what we obtained so far leads to the following representation: we have

$$\langle S, \psi\varphi\rangle = \sum_{(\zeta,\xi)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \int_{\Omega\times\Omega'} \mathrm{D}^{(\zeta,\xi)}\varphi\,d\nu_{\zeta,\xi}, \quad \forall \varphi \in \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K\times K'),$$

the Borel measures $\nu_{\zeta,\xi}$ on $\Omega\times\Omega'$ having of course their support contained in $\operatorname{supp}(\psi) \subset K\times K'$.

Now for every $(\zeta,\xi) \in \mathbb{N}_0^r\times\mathbb{N}_0^s$, we chose $g \in \mathcal{C}_0(\Omega\times\Omega')$ such that $\|g\|_{\Omega\times\Omega'} \leq 2$ and $\int_{\Omega\times\Omega'} g\,d\nu_{\zeta,\xi} = |\nu_{\zeta,\xi}|(\Omega\times\Omega')$ and observe that, for every integer $n \geq n_0$, we have

$$\begin{aligned} n^{|\zeta|+|\xi|} M_{|\zeta|} M'_{|\xi|} |\nu_{\zeta,\xi}|(\Omega\times\Omega') &= n^{|\zeta|+|\xi|} M_{|\zeta|} M'_{|\xi|} v_{\zeta,\xi}(g) \\ &\leq 2\,\|\psi\|_{n_0} \sum_{(\gamma,\delta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \frac{1}{(2(r+s))^{|\gamma|+|\delta|}} B \end{aligned}$$

with

$$B := \sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} (4n(r+s))^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|} |\mu_{\alpha,\beta}|(\Omega\times\Omega') < \infty.$$

This leads to

$$\sup_{(\zeta,\xi)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} n^{|\zeta|+|\xi|} M_{|\zeta|} M'_{|\xi|} |\nu_{\zeta,\xi}|(\Omega\times\Omega') < \infty, \quad \forall n \in \mathbb{N}.$$

So, for every compact subsets L of Ω and L' of Ω', by the propositions 4.1, 3.2 and 3.3, we know that

$$T_{L,L'}\colon \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(L\times L') \to \mathbb{C}; \quad \varphi \mapsto \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times_0^s} \int_{\Omega\times\Omega'} \mathrm{D}^{(\alpha,\beta)}\varphi\,d\nu_{\alpha,\beta}$$

is a well defined continuous linear functional, the series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(L\times L')$. This implies that

$$T\colon \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega\times\Omega') \to \mathbb{C}; \quad \varphi \mapsto \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times_0^s} \int_{\Omega\times\Omega'} \mathrm{D}^{(\alpha,\beta)}\varphi\,d\nu_{\alpha,\beta}$$

is a well defined continuous linear functional, the series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega\times\Omega')$.

Finally we note that, for every $\varphi \in \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K')$, we have $\langle S, \varphi \rangle = \langle S, \psi\varphi \rangle = \langle T, \varphi \rangle$. So, if $\chi \in \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$ is identically 1 on a neighbourhood of $\operatorname{supp}(\psi)$ and has its support contained in $K^\circ \times K'^\circ$, we successively get

$$\langle T, \varphi \rangle = \langle T, \chi\varphi \rangle = \langle S, \psi\chi\varphi \rangle = \langle S, \psi\varphi \rangle = \langle S, \varphi \rangle$$

for every $\varphi \in \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$. Hence the conclusion. ∎

Theorem 4.4. *Let $(u_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$ be a family of Radon measures on $\Omega \times \Omega'$ such that, for every $n \in \mathbb{N}$ and compact subsets K of Ω and K' of Ω',*

$$\sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} n^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|} ||u_{\alpha,\beta}||_{K\times K'} < \infty.$$

Then

$$S\colon \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega') \to \mathbb{C}; \quad \varphi \mapsto \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \langle u_{\alpha,\beta}, D^{(\alpha,\beta)}\varphi \rangle$$

is a well defined continuous linear functional, the series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$.

Proof. For every $n \in \mathbb{N}$ and $(\alpha, \beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$, let $u^n_{\alpha,\beta}$ be the restriction of $u_{\alpha,\beta}$ to $\mathcal{C}_0(K_n^\circ \times K_n'^\circ)$. The Riesz representation theorem provides then a Borel measure $\mu_{\alpha,\beta,n}$ on $K_n^\circ \times K_n'^\circ$ such that

$$\langle u^n_{\alpha,\beta}, f \rangle = \int_{K_n^\circ\times K_n'^\circ} f\, d\mu_{\alpha,\beta,n}, \quad \forall f \in \mathcal{C}_0(K_n^\circ \times K_n'^\circ),$$

and

$$||u^n_{\alpha,\beta}||_{K_n\times K_n'} = |\mu_{\alpha,\beta,n}|(K_n^\circ \times K_n'^\circ)|.$$

Therefore, by the propositions 3.3, 3.2 and 4.1,

$$S^n\colon \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K_n \times K_n') \to \mathbb{C}; \quad \varphi \mapsto \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \int_{K_n^\circ\times K_n'^\circ} \mathrm{D}^{(\alpha,\beta)}\varphi\, d\mu_{\alpha,\beta,n}$$

is a well defined continuous linear functional, the series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K_n \times K_n')$.

For every $\varphi \in \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$, there is $n \in \mathbb{N}$ such that $\operatorname{supp}(\varphi) \subset K_n^\circ \times K_n'^\circ$. As we then have $\langle S^n, \varphi \rangle = \langle S^{n+1}, \varphi \rangle$, it is easy to conclude. ∎

Theorem 4.5. *For every continuous linear functional S on the space $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$, there is a family $(u_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$ of Radon measures on $\Omega \times \Omega'$ such that*

$$\sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} n^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|} ||u_{\alpha,\beta}||_{K_n\times K_n'} < \infty$$

for every $n \in \mathbb{N}$ and

$$\langle S, \varphi \rangle = \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \langle u_{\alpha,\beta}, D^{(\alpha,\beta)}\varphi \rangle, \quad \forall \varphi \in \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega'),$$

the series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$.

Proof. Let $\{\mathcal{O}_m : m \in \mathbb{N}\}$ and $\{\mathcal{O}'_m : m \in \mathbb{N}\}$ be locally finite open covers of Ω and Ω' respectively, every $\mathcal{O}_m$ being relatively compact in Ω and every $\mathcal{O}'_m$ being relatively compact in Ω'. Let moreover $\{\psi_m : m \in \mathbb{N}\}$ be a $\mathscr{D}^{(\mathscr{M})}(\Omega)$-partition of unity on Ω subordinate to $\{\mathcal{O}_m : m \in \mathbb{N}\}$ and $\{\psi'_m : m \in \mathbb{N}\}$ be a $\mathscr{D}^{(\mathscr{M}')}(\Omega')$-partition of unity on Ω' subordinate to $\{\mathcal{O}'_m : m \in \mathbb{N}\}$.

For every m, $m' \in \mathbb{N}$, $\psi_m(x)\psi'_{m'}(y)S$ belongs then to $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')'$ and has its support contained in the product of a compact subset of $\mathcal{O}_m$ by a compact subset of $\mathcal{O}'_{m'}$. By Proposition 4.3, there is then a family $(\mu^{m,m'}_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$ of Borel measures on $\Omega \times \Omega'$ such that

$$\begin{aligned}
&\sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} n^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|} |\mu^{m,m'}_{\alpha,\beta}|(\Omega \times \Omega') < \infty, \quad \forall n \in \mathbb{N};\\
&\operatorname{supp}(\mu^{m,m'}_{\alpha,\beta}) \subset \mathcal{O}_m \times \mathcal{O}'_{m'}, \quad \forall (\alpha,\beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s;\\
&\langle \psi_m(x)\psi'_{m'}(y)S, \varphi\rangle = \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \int_{\Omega\times\Omega'} \mathrm{D}^{(\alpha,\beta)}\varphi \, d\mu^{m,m'}_{\alpha,\beta},
\end{aligned}$$

for every $\varphi \in \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$, the series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$.

Now, for every $(\alpha,\beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$, $u_{\alpha,\beta} := \sum_{m,m'\in\mathbb{N}} \mu^{m,m'}_{\alpha,\beta}$ clearly defines a Radon measure on $\Omega \times \Omega'$. Moreover, for every $n \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that $\operatorname{supp}(\mu^{m,m'}_{\alpha,\beta}) \cap (K_n \times K'_n) = \emptyset$ if m, $m' \geq N$; this leads to

$$||u_{\alpha,\beta}||_{K_n\times K'_n} \leq \sum_{m,m'\leq N} |\mu^{m,m'}_{\alpha,\beta}|(\Omega \times \Omega')$$

hence

$$\sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} n^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|} ||u_{\alpha,\beta}||_{K_n\times K'_n} < \infty.$$

So we may apply Theorem 4.4 and obtain that

$$T : \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega') \to \mathbb{C}; \quad \varphi \mapsto \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \langle u_{\alpha,\beta}, \mathrm{D}^{(\alpha,\beta)}\varphi\rangle$$

is a continuous linear functional, the series converging absolutely and uniformly on the bonded subsets of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$.

As it is a direct matter to prove that $\langle T, .\rangle = \langle S, .\rangle$ on $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$, we conclude at once. ∎

5 Case of the space $\mathscr{D}_{(L^p)}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$

Given $p \in [1, \infty]$, $L^p(\Omega \times \Omega')$ and $\mathscr{L}^p(\Omega \times \Omega')$ designate the classical Lebesgue spaces and for $f \in \widetilde{f} \in L^p(\Omega \times \Omega')$, we set

$$\|f\|_p = ||\widetilde{f}||_p = (\int_{\Omega \times \Omega'} |f(x,y)|^p \, dx\, dy)^{1/p} \quad \text{if } 1 \leq p < \infty$$

and

$$\|f\|_\infty = ||\widetilde{f}||_\infty = \sup \text{ess}\{|f(x,y)| : (x,y) \in \Omega \times \Omega'\}.$$

From now on, in this paragraph, for $p \in [1, \infty[$, we identify the Banach space X of the basic construction with $L^p(\Omega \times \Omega')$.

Given $p \in [1, \infty[$, we now adapt the introduction by Schwartz of the space $\mathscr{D}_{L^p}(\mathbb{R}^k)$ (cf. [9], p. 199) to our setting.

The notation $\mathscr{D}_{(L^p)}^{(\mathscr{M},\mathscr{M}'),h}(K \times K')$ requires that K and K' are non empty compact subsets of $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively and that h is a positive number. It designates the following Banach space: its elements are the C^∞-functions φ on $\mathbb{R}^r \times \mathbb{R}^s$ having their support contained in $K \times K'$ and such that

$$|\varphi|_{p,h} := \sup_{(\alpha,\beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s} \frac{||\mathrm{D}^{(\alpha,\beta)}\varphi||_p}{h^{|\alpha|+|\beta|} M_{|\alpha|} M'_{|\beta|}} < \infty;$$

its norm is $|\cdot|_{p,h}$.

The space $\mathscr{D}_{(L^p)}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K')$ is then the inductive limit of the spaces $\mathscr{D}_{(L^p)}^{(\mathscr{M},\mathscr{M}'),n}(K \times K')$ for $n \in \mathbb{N}$ and the space $\mathscr{D}_{(L^p)}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$ is the inductive limit of the spaces $\mathscr{D}_{(L^p)}^{\{\mathscr{M},\mathscr{M}'\}}(K_n \times K'_n)$. Clearly these spaces are Hausdorff (LB)-spaces. In fact we have a lot more.

Proposition 5.1. *For every $n \in \mathbb{N}$, the closed unit ball B_n of the space $\mathscr{D}_{(L^p)}^{(\mathscr{M},\mathscr{M}'),n}(K \times K')$ is a compact subset of $\mathscr{D}_{(L^p)}^{(\mathscr{M},\mathscr{M}'),n+1}(K \times K')$.*

So $\mathscr{D}_{(L^p)}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K')$ and $\mathscr{D}_{(L^p)}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$ are regular (LB)*-spaces.*

Proof. We first establish that, for every $(\alpha, \beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$, the set $\{\mathrm{D}^{(\alpha,\beta)}\varphi \colon \varphi \in B_n\}$ is uniformly bounded; as a consequence, these sets will also be equicontinuous. Indeed, setting $\mathbf{1}_k = (1, \ldots, 1) \in \mathbb{N}_0^k$, we know that, for every $\varphi \in B_n$, we have

$$\mathrm{D}^{(\alpha,\beta)}\varphi(x,y) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_r} \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_s} \mathrm{D}^{(\alpha+\mathbf{1}_r, \beta+\mathbf{1}_s)}\varphi(t)\, dt$$

hence, with q defined by $1/p + 1/q = 1$,

$$|\mathrm{D}^{(\alpha,\beta)}\varphi(x,y)| \leq \mu(K \times K')^{1/q} n^{|\alpha|+|\beta|+r+s} M_{|\alpha|+r} M'_{|\beta|+s} \, |\varphi|_{p,n}$$

where μ is the Lebesgue measure.

So, given a sequence $(\varphi_m)_{m\in\mathbb{N}}$ of B_n, the Arzela-Ascoli theorem provides a subsequence $(\varphi_{m(k)})_{k\in\mathbb{N}}$ as well as a function $\varphi \in C^\infty(\mathbb{R}^r \times \mathbb{R}^s)$ such that, for every $(\alpha,\beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$, the sequence $(\mathrm{D}^{(\alpha,\beta)}\varphi_{m(k)})_{k\in\mathbb{N}}$ converges uniformly on $\mathbb{R}^r \times \mathbb{R}^s$ to $\mathrm{D}^{(\alpha,\beta)}\varphi$. Therefore it is clear that φ belongs to B_n.

To conclude, it suffices to prove that the sequence $(\varphi_{m(k)})_{k\in\mathbb{N}}$ converges to φ in $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),n+1}_{(L^p)}(K \times K')$. Given $\varepsilon > 0$, we first fix $m_0 \in \mathbb{N}$ such that $(n/(n+1))^{m_0} \leq \varepsilon/4$ and next chose $k_0 \in \mathbb{N}$ such that

$$\mu(K \times K')^{1/p}||\mathrm{D}^{(\alpha,\beta)}\varphi_{m(k)} - \mathrm{D}^{(\alpha,\beta)}\varphi||_{\mathbb{R}^r\times\mathbb{R}^s} \leq \frac{\varepsilon}{2}$$

for every $(\alpha,\beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$ such that $|\alpha| \leq m_0$, $|\beta| \leq m_0$ and $k \geq k_0$. Then, for $k \geq k_0$, we successively obtain

$$\begin{aligned}
&||\varphi_{m(k)} - \varphi||_{p,n+1} \\
&\leq \sup_{|\alpha|,|\beta|\leq m_0} ||\mathrm{D}^{(\alpha,\beta)}\varphi_{m(k)} - \mathrm{D}^{(\alpha,\beta)}\varphi||_p + \sup_{|\alpha|+|\beta|\geq m_0} \frac{||\mathrm{D}^{(\alpha,\beta)}\varphi_{m(k)} - \mathrm{D}^{(\alpha,\beta)}\varphi||_p}{(n+1)^{|\alpha|+|\beta|}M_{|\alpha|}M'_{|\beta|}} \\
&\leq \mu(K \times K')^{1/p} \sup_{|\alpha|,|\beta|\leq m_0} ||\mathrm{D}^{(\alpha,\beta)}\varphi_{m(k)} - \mathrm{D}^{(\alpha,\beta)}\varphi||_{\mathbb{R}^r\times\mathbb{R}^s} \\
&\quad + \frac{\varepsilon}{4}(|\varphi_{m(k)}|_{p,n} + |\varphi|_{p,n}) \leq \varepsilon.
\end{aligned}$$

Hence the conclusion. ■

Notation. For every $n \in \mathbb{N}$, the linear map

$$\xi_n \colon \mathscr{D}^{(\mathscr{M},\mathscr{M}'),n}_{(L^p)}(K \times K') \to Y_n; \quad \varphi \mapsto (\widetilde{D}^{(\alpha,\beta)}\varphi)_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$$

clearly is an isometry from $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),n}_{(L^p)}(K \times K')$ onto its image P_n considered as a topological vector subspace of Y_n. We then introduce P as the topological vector subspace $\cup_{n=1}^\infty P_n$ of Y and consider the map

$$\xi \colon \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(K \times K') \to P; : \varphi \mapsto (\widetilde{D}^{(\alpha,\beta)}\varphi)_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}.$$

It is clear that ξ is a continuous linear bijection. Acting as in the proof of Proposition 4.1 leads to the following property.

Proposition 5.2. *The spaces $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(K \times K')$ and P are topologically isomorphic; more specifically ξ is a topological isomorphism.* ■

The following property justifies the fact that the elements of the topological dual of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(\Omega \times \Omega')$ may be considered as ultradistributions.

Proposition 5.3. *For every $p \in [1,\infty[$ and $n \in \mathbb{N}$, the inclusion maps*

$$\begin{aligned}
I_n \colon \mathscr{D}^{(\mathscr{M},\mathscr{M}'),n}(K \times K') &\to \mathscr{D}^{(\mathscr{M},\mathscr{M}'),n}_{(L^p)}(K \times K'), \\
I \colon \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K') &\to \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(K \times K')
\end{aligned}$$

and

$$J\colon \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega\times\Omega') \to \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(\Omega\times\Omega')$$

are well defined continuous linear maps. Moreover J has a dense image.

Proof. It is a direct matter to establish that these maps are well defined, continuous and linear.

To conclude, let us prove the following deeper property: every element φ of $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),n}_{(L^p)}(K_n\times K'_n)$ is the limit in $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),n+1}_{(L^p)}(K_{n+1}\times K'_{n+1})$ of a sequence $(\varphi_m)_{m\in\mathbb{N}}$ of $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),n+1}(K_{n+1}\times K'_{n+1})$.

Setting $B_k(d) := \{x\in\mathbb{R}^k\colon\ |x|\le d\}$ for $d>0$, we first chose a strictly decreasing sequence of positive numbers converging to 0 and such that $(K_n\times K'_n)+(B_r(d_1)\times B_s(d_1))\subset K^{\circ}_{n+1}\times K'^{\circ}_{n+1}$. We next chose a sequence $(\psi_m)_{m\in\mathbb{N}}$ such that, for every $m\in\mathbb{N}$, ψ_m belongs to $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),n}(B_r(d_m)\times B_s(d_m)))$, $\int_{\mathbb{R}^r\times\mathbb{R}^s}\psi_m(x,y)\,dx\,dy=1$ and $\psi_m(x,y)\ge 0$ for every $(x,y)\in\mathbb{R}^r\times\mathbb{R}^s$. We then set $\varphi_m=\varphi\star\psi_m$ for every $m\in\mathbb{N}$. It is a direct matter to check that φ_m belongs to $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),n}(K_{n+1}\times K'_{n+1})$.

Let us prove that this sequence $(\varphi_m)_{m\in\mathbb{N}}$ converges to φ in the space $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),n+1}_{(L^p)}(K_{n+1}\times K'_{n+1})$. Given $\varepsilon>0$, we first fix $n_0\in\mathbb{N}_0$ such that $(n/(n+1))^{n_0}\,\|\varphi\|_{p,n}\le\varepsilon/4$. As we know that, for every $(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s$, the sequence $(\mathrm{D}^{(\alpha,\beta)}\varphi_m)_{m\in\mathbb{N}}$ converges uniformly on $\mathbb{R}^r\times\mathbb{R}^s$ to $\mathrm{D}^{(\alpha,\beta)}\varphi$, we next fix $m_0\in\mathbb{N}$ such that

$$||\mathrm{D}^{(\alpha,\beta)}\varphi_m-\mathrm{D}^{(\alpha,\beta)}\varphi||_{\mathbb{R}^r\times\mathbb{R}^s}\le\frac{\varepsilon}{2\mu(K_{n+1}\times K'_{n+1})^{1/p}}$$

hence $||\mathrm{D}^{(\alpha,\beta)}\varphi_m-\mathrm{D}^{(\alpha,\beta)}\varphi||_p\le\varepsilon/2$ for every $m\ge m_0$ and $(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s$ such that $|\alpha|,|\beta|\le n_0$, μ designating the Lebesgue measure. Let us remark now that, given $m\in\mathbb{N}$ and $(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s$, we have $\mathrm{D}^{(\alpha,\beta)}\varphi\in L^p(\mathbb{R}^r\times\mathbb{R}^s)$ and $\psi_m\in L^1(\mathbb{R}^s\times\mathbb{R}^s)$ hence $\mathrm{D}^{(\alpha,\beta)}\varphi_m=(\mathrm{D}^{(\alpha,\beta)}\varphi)\star\psi_m\in L^p(\mathbb{R}^r\times\mathbb{R}^s)$ with

$$||\mathrm{D}^{(\alpha,\beta)}\varphi_m||_p\le||\mathrm{D}^{(\alpha,\beta)}\varphi||_p||\psi_m||_1=||\mathrm{D}^{(\alpha,\beta)}\varphi||_p.$$

This leads directly to $||\varphi_m-\varphi||_{p,n+1}\le\varepsilon$ for every $m\ge m_0$. Hence the conclusion. ∎

Proposition 5.4. *If $\mathscr{M}$ and $\mathscr{M}'$ are stable under differential operators, then, for every $p\in[1,\infty[$, the canonical injection from $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(K\times K')$* [resp. $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega\times\Omega')$] *into $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(K\times K')$* [resp. $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(\Omega\times\Omega')$] *is a topological isomorphism.*

Proof. As $\mathscr{M}$ and $\mathscr{M}'$ are stable under differential operators, there are constants A, $H>0$ such that $M_{p+r}\le AH^{rp}M_p$ and $M'_{p+r}\le AH^{sp}M'_p$ for every $p\in\mathbb{N}_0$.

To conclude, we just need to prove that, for $t=\sup\{r,s\}$ and every $n\in\mathbb{N}$, the inclusion map from $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),n}_{(L^p)}(K\times K')$ into $\mathscr{D}^{(\mathscr{M},\mathscr{M}'),H^t n}(K\times K')$ is well defined, continuous and linear.

Let us set $\mathbf{1}_k = (1, \ldots, 1) \in \mathbb{N}_0^k$ and chose $C > 0$ such that $K \times K' \subset [-C, C]^{r+s}$. For every $\varphi \in \mathscr{D}_{(L^p)}^{(\mathscr{M},\mathscr{M}'),n}(K \times K')$, $(\alpha, \beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s$ and $(x, y) \in \mathbb{R}^r \times \mathbb{R}^s$, we then have

$$\mathrm{D}^{(\alpha,\beta)}\varphi(x, y) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_r} \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_s} \mathrm{D}^{(\alpha+\mathbf{1}_r,\beta+\mathbf{1}_s)}\varphi(t)\, dt$$

hence

$$||\mathrm{D}^{(\alpha,\beta)}\varphi||_{K\times K'} \leq (2C)^{(r+s)/q}||\mathrm{D}^{(\alpha+\mathbf{1}_r,\beta+\mathbf{1}_s)}\varphi||_p$$

with q defined by $1/p + 1/q = 1$. This leads easily to

$$\|\varphi\|_{K\times K',H^t n} \leq A^2(2C)^{(r+s)/q}n^{r+s}|\varphi|_{p,n}.$$

Hence the conclusion. ■

Proceeding as in the proofs of the Proposition 4.2 and the Theorems 4.3, 4.4 and 4.5 leads directly to the following results.

Proposition 5.5. *Let K and K' be non void compact subsets of Ω and Ω' respectively.*

For every continuous linear functional S on $\mathscr{D}_{(L^p)}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$, there is a family $(g_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$ of elements of $\mathscr{L}^q(\Omega \times \Omega')$ such that

$$\sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} n^{|\alpha|+|\beta|}M_{|\alpha|}M'_{|\beta|}||g_{\alpha,\beta}||_q < \infty, \quad \forall n \in \mathbb{N},$$

and

$$\langle S, \varphi\rangle = \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \int_{\Omega\times\Omega'} D^{(\alpha,\beta)}\varphi.g_{\alpha,\beta}\, dx\, dy, \quad \forall\varphi \in \mathscr{D}_{(L^p)}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K'),$$

these series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}_{(L^p)}^{\{\mathscr{M},\mathscr{M}'\}}(K \times K')$.

Theorem 5.6. *Let H and K be compact subsets of Ω such that $H \subset K^\circ$ and let H' and K' be compact subsets of Ω' such that $H' \subset K'^\circ$.*

If $S \in \mathscr{D}_{(L^p)}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')'$ has its support contained in $H \times H'$, there is a family $(g_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$ of elements of $\mathscr{L}^q(\Omega \times \Omega')$ such that
a) $\sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} n^{|\alpha|+|\beta|}M_{|\alpha|}M'_{|\beta|}||g_{\alpha,\beta}||_q < \infty, \forall n \in \mathbb{N};$
b) $\mathrm{supp}(g_{\alpha,\beta}) \subset K \times K', \forall(\alpha, \beta) \in \mathbb{N}_0^r \times \mathbb{N}_0^s;$
c) $\langle S, \varphi\rangle = \sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \int_{\Omega\times\Omega'} D^{(\alpha,\beta)}\varphi.g_{\alpha,\beta}\, dx\, dy, \forall\varphi \in \mathscr{D}_{(L^p)}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega'),$
the series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}_{(L^p)}^{\{\mathscr{M},\mathscr{M}'\}}(\Omega \times \Omega')$.

Theorem 5.7. *Let $(g_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$ be a family of elements of $\mathscr{L}^q_{\mathrm{loc}}(\Omega \times \Omega')$ such that, for every $n \in \mathbb{N}$ and compact subsets K of Ω and K' of Ω',*

$$\sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} n^{|\alpha|+|\beta|}M_{|\alpha|}M'_{|\beta|}||g_{\alpha,\beta}|_{K\times K'}||_q < \infty.$$

Then

$$S\colon \mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(\Omega\times\Omega')\to\mathbb{C};\quad \varphi\mapsto\sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}\int_{\Omega\times\Omega'}D^{(\alpha,\beta)}\varphi.g_{\alpha,\beta}\,dx\,dy$$

is a well defined continuous linear functional, the series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(\Omega\times\Omega')$.

Theorem 5.8. *For every continuous linear functional* S *on the space* $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(\Omega\times\Omega')$, *there is a family* $(g_{\alpha,\beta})_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}$ *of elements of* $\mathscr{L}^q_{\mathrm{loc}}(\Omega\times\Omega')$ *such that*

$$\sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} n^{|\alpha|+|\beta|}M_{|\alpha|}M'_{|\beta|}||g_{\alpha,\beta}|_{K\times K'}||_q<\infty$$

for every $n\in\mathbb{N}$ *and compact subsets* K *and* K' *of* Ω *and* Ω' *respectively, as well as*

$$\langle S,\varphi\rangle=\sum_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}\int_{\Omega\times\Omega'}D^{(\alpha,\beta)}\varphi.g_{\alpha,\beta}\,dx\,dy,\quad\forall\varphi\in\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(\Omega\times\Omega'),$$

the series converging absolutely and uniformly on the bounded subsets of $\mathscr{D}^{\{\mathscr{M},\mathscr{M}'\}}_{(L^p)}(\Omega\times\Omega')$.

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