Weak orthogonal sequences in L^2 of a vector measure and the Menchoff-Rademacher Theorem

E. Jiménez Fernández E. A. Sánchez Pérez*

Abstract

Consider a positive Banach lattice valued vector measure $\mathbf{m} : \Sigma \to X$, its space of 2-integrable functions $L^2(\mathbf{m})$ and a sequence S in it. We analyze the notion of weak $\mathbf{m}$-orthogonality for such an S in these spaces and we prove a Menchoff-Rademacher Theorem on the almost everywhere convergence of series in them. In order to do this, we provide a criterion for determining when there is a functional $0 \leq x' \in X'$ such that S is orthogonal with respect to the scalar positive measure $\langle \mathbf{m}, x' \rangle$. As an application, we use the representation of $\ell-$sums of L^2-spaces as spaces $L^2(\mathbf{m})$ for a suitable vector measure $\mathbf{m}$ centering our attention in the case of c_0-sums.

1 Introduction

The representation theorem for 2-convex order continuous Banach lattices with a weak unit establishes that such an space can be always identified (isomorphically and in order) with a space $L^2(\mathbf{m})$ of 2-integrable functions with respect to a positive Banach lattice valued vector measure $\mathbf{m}$ on a σ-algebra (see [11, Proposition 2.4] or [20, Proposition 3.3]). Although these spaces are not in general Hilbert spaces, the integration structure of the spaces $L^2(\mathbf{m})$ provides

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several extensions of the notion of orthogonality. Some theoretical results and applications have been already obtained in this setting. The notion of (strong) orthogonality with respect to a vector measure $\mathbf{m}$ in spaces $L^2(\mathbf{m})$ of 2-integrable functions has been defined an studied in a series of papers in the last ten years ([12, 13, 14, 21, 25]). Roughly speaking, it is defined by imposing *simultaneously* orthogonality with respect to *all* the elements of the family of scalar measures defined by the vector measure. In this paper we consider the weaker version given by the definition of weak $\mathbf{m}$-orthogonality: a sequence $(f_i)_i$ is weak $\mathbf{m}$-orthogonal if there is an element $x' \in X'$ such that S is orthogonal with respect to $\langle \mathbf{m}, x' \rangle$. In this paper we prove a criterion for characterizing such sequences and we give some relevant examples. As an application, we prove a Menchoff-Rademacher type theorem on almost everywhere convergence of series. We finish the paper analyzing some concrete cases regarding c_0-sums of Lebesgue spaces.

The paper is organized in five sections. After some preliminaries (Section 2), in the third section we prove our main result on weak orthogonality, which characterize weak $\mathbf{m}$-orthogonal sequences in $L^2(\mathbf{m})$. Some effort has been made for providing examples and applications of the criterion in several settings, in order to show that this kind of orthogonality of sequences of $L^2(\mathbf{m})$ extends in a relevant way some geometric aspects of the Hilbert space theory.

Although several properties and applications of orthogonal series with respect to a vector measure are known ([13, 14, 21]), the question of the almost everywhere convergence of series defined by such functions has not been studied yet. Following this research and as an application of the criterion for weak orthogonality, the results of this paper provide also reasonable answers to the problems concerning almost everywhere convergence of (strongly orthogonal) series that appear in [21, 25]. From the methodological point of view, we follow the technique that is used in [26] to study the almost everywhere convergence of series. In Section III.H of this book it is shown that there exists a deep link between the evaluation of 2-summing norms for a special class of operators between sequence spaces and the problems concerning almost everywhere convergence of series; the origin of this idea can be already found in [2, Section 4] (see also [22]). Recently, related techniques have been used in [4, 8, 9], for instance for proving generalizations of the Menchoff-Rademacher Theorem for vector valued Banach function spaces ([9]). We explain the required version of this argument for our work in the proof of Theorem 4.2. In Section 4 we prove this theorem, that establishes the requirements for obtaining the almost everywhere convergence of series defined by weak orthogonal sequences. Finally, in Section 5 we provide a technique to construct non trivial examples of weak $\mathbf{m}$-orthogonal sequences $(f_i)_i$ in concrete Banach lattices such that the requirement $\sum_{i=1}^{\infty} a_i^2 < \infty$ on the sequence of scalar coefficients implies the a.e. convergence of the series $\sum_{i=1}^{\infty} a_i f_i$.

2 Preliminaries

In what follows we introduce several concepts and results that are needed to define weak $\mathbf{m}$-orthogonal sequences. Let (Ω, Σ) be a measurable space and X a Banach lattice. Throughout the paper $\mathbf{m} : \Sigma \to X$ will be a positive countably additive vector measure, i.e $m(A) \geq 0$ for all $A \in \Sigma$. For each element $x' \in X'$ the formula $\langle \mathbf{m}, x' \rangle(A) := \langle \mathbf{m}(A), x' \rangle$, $A \in \Sigma$, defines a (countably additive) scalar measure. We write $|\langle \mathbf{m}, x' \rangle|$ for its variation, i.e. $|\langle \mathbf{m}, x' \rangle|(A) := \sup \sum_{B \in \Pi} |\langle \mathbf{m}(B), x' \rangle|$ for $A \in \Sigma$, where the supremum is computed over all finite measurable partitions Π of A. The semivariation of $\mathbf{m}$ is the extended nonnegative function $\|\mathbf{m}\|$ whose value on a set $A \in \Sigma$ is given by:

$$\|\mathbf{m}\|(A) = \sup\{|\langle \mathbf{m}, x' \rangle|(A) : x' \in X', \|x'\| \leq 1\}.$$

As usual, we say that a sequence of functions converges $|\langle \mathbf{m}, x' \rangle|$-almost everywhere if it converges pointwise in a set $A \in \Sigma$ such that $|\langle \mathbf{m}, x' \rangle|(\Omega \setminus A) = 0$. A sequence converges $\mathbf{m}$-almost everywhere if it converges in a set A that satisfies that the semivariation of $\mathbf{m}$ in $\Omega \setminus A$ is 0. The Bartle, Dunford and Schwartz theorem (see [10, Ch.I,2, Corollary 6]) produces a finite nonnegative real-valued measure μ on Σ such that $\mathbf{m} \ll \mu$ (i.e. $\mathbf{m}$ is μ-continuous). Furthermore, it is known that there exists always an element $x' \in X'$ such that $\mathbf{m} \ll |\langle \mathbf{m}, x' \rangle|$ satisfies this property. We call such a scalar measure a Rybakov measure for $\mathbf{m}$ (see [10, Ch.IX,2]). If $|\langle \mathbf{m}, x' \rangle|$ is a Rybakov measure for $\mathbf{m}$, then it is known that a sequence of functions converges $\mathbf{m}$-almost everywhere if and only if it converges $|\langle \mathbf{m}, x' \rangle|$-almost everywhere. Notice that if $\mathbf{m}$ is positive and x' is a positive element of the Banach lattice X', then $|\langle \mathbf{m}, x' \rangle| = \langle \mathbf{m}, x' \rangle$. We write $(X')^+_{\mathbf{m}}$ for the set of these elements.

Let (Ω, Σ, μ) be a σ-finite measure space. Following the definition in [16, p.28], a Banach space $X(\mu)$ of (classes of) locally μ-integrable real functions is said to be a Banach function space over μ (Köthe function space) if it satisfies the next two properties.

- If f is measurable and $g \in X(\mu)$ such that $|f(w)| \leq |g(w)|$ $\mu-$a.e. on Ω, then $f \in X(\mu)$ and $\|f\| \leq \|g\|$.
- If $A \in \Sigma$, and $\mu(A) < \infty$, then the characteristic function χ_A belongs to $X(\mu)$.

The space $L^1(\mathbf{m})$ of integrable functions with respect to $\mathbf{m}$ is a Banach function space over any Rybakov measure μ for $\mathbf{m}$ (see [5, 16]). The elements of this space are (classes of μ-a.e. measurable) functions f that are integrable with respect to each scalar measure $\langle \mathbf{m}, x' \rangle$, and for every $A \in \Sigma$ there is an element $\int_A f d\mathbf{m} \in X$ such that $\langle \int_A f d\mathbf{m}, x' \rangle = \int_A f d\langle \mathbf{m}, x' \rangle$ for every $x' \in X'$. When no explicit reference is needed, we write $\int f d\mathbf{m}$ instead of $\int_\Omega f d\mathbf{m}$. The reader can find the definitions and fundamental results concerning the space $L^1(\mathbf{m})$ in [5, 15]. In this paper we deal with sequences of functions in $L^2(\mathbf{m})$. This space is defined as the set of (classes of) real functions that satisfy that $|f|^2 \in L^1(\mathbf{m})$ with the norm

$$\|f\|_{L^2(\mathbf{m})} := \sup_{x' \in B_{X'}} \Big(\int |f|^2 d|\langle \mathbf{m}, x' \rangle| \Big)^{\frac{1}{2}}, \qquad f \in L^2(\mathbf{m})$$

(see [11, 21, 24]). If $\langle \mathbf{m}, x' \rangle$ is a Rybakov measure for $\mathbf{m}$, then the inclusion map $i_{x'} : L^2(\mathbf{m}) \to L^2(|\langle \mathbf{m}, x' \rangle|)$ is well-defined and continuous; in fact, even if x' do not define a Rybakov measure this identification map is well-defined and continuous, although it is not injective. In this work we only need some particular properties of the functions in $L^2(\mathbf{m})$. For instance, if f, g are functions in $L^2(\mathbf{m})$, we use the fact that the product fg is $\mathbf{m}$-integrable (see [11, 20] or [24]). Thus, the following definitions make sense. We write as usual δ_{ij} to be 0 if $i \neq j$ and 1 if $i = j$ for every couple of natural numbers i, j. We say that a sequence $(f_i)_i$ of functions of $L^2(\mathbf{m})$ is (strongly) $\mathbf{m}$-orthonormal if $\| \int f_i f_j \, d\mathbf{m} \| = \delta_{ij}$, $i, j \in \mathbb{N}$. This kind of sequences has been studied in [13, 14, 21, 25]. In this paper we deal with a weaker version of orthogonality. Recall that we always assume that the vector measure is positive.

Definition 2.1. *A sequence of functions $S = (f_i)_i$ in $L^2(\mathbf{m})$ is weak $\mathbf{m}$-orthogonal if there is an element $0 \leq x' \in (X')^+_{\mathbf{m}}$ such that $\int f_i^2 d\langle \mathbf{m}, x' \rangle > 0$ for all $i \in \mathbb{N}$, and S is orthogonal in $L^2(\langle \mathbf{m}, x' \rangle)$, i.e. for all $i \neq j$*

$$\langle \int f_i f_j d\mathbf{m}, x' \rangle = \int f_i f_j d\langle \mathbf{m}, x' \rangle = 0.$$

For such a sequence we also say that is orthogonal with respect to $\langle \mathbf{m}, x' \rangle$ when an explicit reference to the scalar measure $\langle \mathbf{m}, x' \rangle$ is convenient.

We will use standard Banach and function space notation; our main references are [10, 16, 26]. If $1 \leq p \leq \infty$, we write p' for the (extended) real number satisfying $1/p + 1/p' = 1$. Let E and F be Banach lattices (see [16, 1.a.1] for the definition) and $1 \leq p < \infty$. An operator $T : E \to F$ is p-concave if for every finite set $x_1, x_2, ..., x_n \in E$ there is a constant $K > 0$ such that

$$(\sum_{i=1}^n \| T(x_i) \|^p)^{\frac{1}{p}} \leq K \| (\sum_{i=1}^n | x_i |^p)^{\frac{1}{p}} \| . \tag{2.1}$$

The infimum of such constants K is the p-concavity constant of the operator. An operator $T : E \to F$ is p-convex if for each finite set $x_1, x_2, ..., x_n \in E$ there exists a constant $K > 0$ such that

$$\| (\sum_{i=1}^n | T(x_i) |^p)^{\frac{1}{p}} \| \leq K (\sum_{i=1}^n \| x_i \|^p)^{\frac{1}{p}}. \tag{2.2}$$

As in the case of p-concavity, the infimum of such constants K is the p-convexity constant of T. A Banach lattice E is p-concave (p-convex) if the identity map $Id : E \to E$ is p-concave (p-convex). Throughout the paper we will consider Banach function spaces as Banach lattices with the usual μ-a.e. order. For the aim of simplicity, we will assume that the corresponding p-concavity/p-convexity constants of the spaces are 1; it is known that each r-convex and s-concave Banach lattice, $1 \leq r \leq s \leq \infty$, can be renormed equivalently so that with the new norm, the r-convexity and s-concavity constants are both equal to 1 (see [16, 1.d.8]).

Let X and Y be Banach spaces and let X' be the dual space of X. An operator $T : X \to Y$ is 2-absolutely summing if there exists a constant $C > 0$ such that for every finite sequence $x_1, ..., x_n \in X$,

$$(\sum_{i=1}^n \|T(x_i)\|^2)^{\frac{1}{2}} \leq C \sup\{(\sum_{i=1}^n |\langle x_i, x' \rangle|^2)^{\frac{1}{2}} : x' \in X', \|x'\| \leq 1\}. \tag{2.3}$$

We define the 2-summing norm of T as

$$\pi_2(T)=\inf\{C\text{: (2.3) holds for all } (x_i)_{i=1}^n \subset X,\ \ n \in \mathbb{N}\}.$$

We write as usual ℓ^p, $1 \leq p < \infty$, and c_0 for the classical sequence spaces, and $\|.\|_p$, $\|.\|_0$ for the corresponding norms. The sequence spaces that we deal with (L, ℓ ...) are assumed to be such kind of spaces. Thus, we will consider spaces of real functions on the standard measure space on the set of natural numbers $\mathbb{N}$ with an unconditional normalized basis with unconditional constant 1. We will write e_i, $i \in \mathbb{N}$, for the elements of the canonical basis of the space. Moreover, we also assume that its dual space can be represented as a sequence space, i.e. its elements can be written as sequences $(\tau_i)_i$, and the duality is given by $\langle (\tau_i), (\lambda_i) \rangle = \sum_{i=1}^\infty \tau_i \lambda_i$, $(\lambda_i)_i \in L$. For instance, this always happens when the space is σ-order continuous (see the comments that follow Definition 1.b.17 in [16]). We will use the following construction, for the particular case of sequence spaces (i.e. the measure is the discrete one on the set of the natural numbers). If $L^0(\mu)$ is the space of (classes of μ-a.e. equal) real measurable functions, $0 < r < \infty$ and $E(\mu)$ is a Banach function space, we define the r-power of $E(\mu)$ as the space

$$E(\mu)_{[r]} := \{x \in L^0(\mu) : |x|^{1/r} \in E(\mu)\}$$

endowed with the (quasi-)norm $\|x\|_{E_{[r]}} := \||x|^{1/r}\|_E^r$. The space $E(\mu)_{[r]}$ is always a Köthe function space when $0 < r \leq 1$ and for $r > 1$ it is so whenever $E(\mu)$ is r-convex; in this case the expression above gives a norm if the r-convexity constant is 1 (see [7, 20] for the basic properties of r-powers of Köthe function spaces). For instance, the space $L^2(\mathbf{m})$ above can be written as the $1/2$-power of $L^1(\mathbf{m})$, i.e. $L^2(\mathbf{m}) = (L^1(\mathbf{m}))_{[1/2]}$.

3 Weak m-orthogonal sequences

In this section we give a characterization of when, given a sequence S in $L^2(\mathbf{m})$ there is an element $x' \in B_{X'}$ such that S is orthogonal with respect to $\langle \mathbf{m}, x' \rangle$. It is easy to find examples of sequences that satisfy this property.

Example 3.1. • *Consider the Lebesgue measure space $([0,1], \Sigma, dx)$ and the power series of the exponential function $e^x = \sum_{n=0}^\infty \frac{1}{n!} x^n$. It converges (uniformly) on the interval $[0,1]$ so we can define the positive vector measure $\nu : \Sigma \to c_0$ as $\nu(A) = (\int_A x^n\, dx)_{n=0}^\infty$. It is clearly countably additive and then the corresponding space $L^2(\nu)$ is well-defined.*

Consider now the sequence of functions $f_i = \sqrt{2} e^{-x/2} \sin(2\pi i x)$, $i \in \mathbb{N}$, and note that for each i,

$$\lim_n \langle \int f_i^2 d\nu, e_n \rangle = \lim_n \int x^n f_i^2\, dx = \lim_n \int 2x^n e^{-x} \sin^2(2\pi i x)\, dx = 0,$$

and

$$\|f_i\|_{L^2(\nu)} = \|\big(\int x^n f_i^2\, dx\big)_{n=0}^\infty\|_{c_0}^{1/2} = \sup_n \big(\int x^n f_i^2\, dx\big)^{1/2} = \big(\int f_i^2\, dx\big)^{1/2} \leq \sqrt{2}.$$

This means in particular that $(f_i) \subseteq L^2(\nu)$. *Take the norm one sequence* $x_0' := (\frac{1}{n!e})_{n=0}^{\infty} \in \ell^1 = (c_0)'$. *A direct calculation shows that* $\int f_i f_j d\langle \nu, x_0' \rangle = \delta_{i,j}\frac{1}{e}$.

- *Let* $\gamma := arg\sinh(\frac{1}{2})$. *Take the Lebesgue measure space* $([-\gamma,\gamma],\Sigma,dx)$. *The power series of the function* $\cosh x$ *is equal to* $\sum_{k\geq 0}\frac{x^{2k}}{(2k)!}$ *that converges uniformly on* $[-\gamma,\gamma]$. *The positive countably additive vector measure* $\nu(A) = (\int_A x^{2n}dx)_{n=0}^{\infty} \in c_0$ *is then well defined. Consider the sequence* $(f_m)_m := (\cos(2m\pi\sinh x))_m$. *Let* $(e_n)_n$ *be the canonical basis of* $(c_0)' = \ell^1$, *and note that for each m the integrals in the interval* $[-\gamma,\gamma]$ *are*

$$\lim_n \langle \int f_m^2 d\nu, e_n \rangle = \lim_n \int x^{2n} f_m^2 dx = \lim_n \int x^{2n}\cos^2(m\sinh x)dx = 0,$$

and for all m,

$$\|f_m\|_{L^2(\nu)} = \|(\int x^{2n} f_m^2 dx)_{n=0}^{\infty}\|_{c_0}^{1/2}$$
$$= \sup_n (\int x^{2n} f_m^2 dx)^{1/2} \leq (\int f_m^2 dx)^{1/2} \leq \sqrt{2\gamma}.$$

This means in particular that $(f_m)_m \subset L^2(\nu)$. *Take the norm one sequence* $x_0' := (\frac{1}{\cosh(1)(2n)!})_{n=0}^{\infty} \in \ell^1 = (c_0')$. *A direct calculation shows that*

$$\int f_n f_m d\langle \nu, x_0' \rangle = \frac{1}{2\cosh(1)}\delta_{m,n}.$$

In what follows we provide a characterization of the situation given in the example above, i.e. when we can find an element x' such that the sequence S is orthogonal in the space $L^2(\langle \mathbf{m}, x' \rangle)$. Let us introduce first some notation. Let $\mathbf{m} : \Sigma \to X$ be a positive vector measure and take a sequence $S = (f_i)_i \subseteq L^2(\mathbf{m})$ and a sequence of positive real numbers $\Delta = (\varepsilon_i)_i$. Then we write B_S for the convex weak* compact subset

$$B_{S,\Delta} := B_{X'} \cap (X')^+_{\mathbf{m}} \cap \{x' : \langle \int f_i^2 d\mathbf{m}, x' \rangle \leq \varepsilon_i\}.$$

Let us define the following continuous seminorm on $L^1(\mathbf{m})$.

$$\|f\|_{B_{S,\Delta}} := \sup_{x' \in B_{S,\Delta}} \left(\int f\, d\langle \mathbf{m}, x' \rangle\right).$$

For every $i, j \in \mathbb{N}$, $i \neq j$, let us write

$$\varphi_{i,j,\theta}(w) := \big(f_i(w) + \theta f_j(w)\big)^2, \quad w \in \Omega,$$

where $\theta \in \{-1, 1\}$. Notice that $0 \leq \varphi_{i,j,\theta} \in L^1(\mathbf{m})$.

For instance, in Example 3.1 the sequence Δ is $\Delta = (\varepsilon_i)_i = (\frac{1}{e})_i$, and so $\frac{1}{e}B_{\ell^1} \cap (\ell^1)^+_{\mathbf{m}} \subset B_{S,\Delta}$. Thus, $\|\cdot\|_{B_{S,\Delta}}$ is equivalent to the norm of $L^2(\nu)$ and it can be easily checked that $\langle (\frac{1}{n!e})_{n=0}^{\infty}, \int f_i^2 d\nu \rangle \leq \frac{1}{e}$ for all i.

In the following result the scalar product notation $\langle (\gamma_k), (\delta_k) \rangle := \sum_{k=1}^{n} \gamma_k \delta_k$ for finite sequences $(\gamma_k)_{k=1}^n$ and $(\delta_k)_{k=1}^n$ is used.

Theorem 3.2. *Let $\mathbf{m} : \Sigma \to X$ be a positive vector measure. Consider a sequence $(f_i)_i \subseteq L^2(\mathbf{m})$ and a sequence of positive real numbers $\Delta = (\varepsilon_i)_i$. The following statements are equivalent.*

(1) For every finite sequence of non negative real numbers $(\gamma_k)_k$ such that $\sum_k \gamma_k = 1$, indexes $i_k, j_k \in \mathbb{N}$, $i_k \neq j_k$, and $\theta_k \in \{-1, 1\}$,

$$\langle (\gamma_k), (\varepsilon_{i_k} + \varepsilon_{j_k}) \rangle \leq \| \langle (\gamma_k), (\varphi_{i_k, j_k, \theta_k}) \rangle \|_{B_{S,\Delta}}.$$

(2) There is an element $0 \leq x_0' \in B_{X'}$ such that S is orthogonal with respect to $\langle \mathbf{m}, x_0' \rangle$ and $\int f_i^2 d\langle \mathbf{m}, x_0' \rangle = \varepsilon_i$ for every $i \in \mathbb{N}$.

Proof. Let us prove first that (1) implies (2). Consider the family of functions $\phi : B_{S,\Delta} \to \mathbb{R}$ given by

$$\phi(x') = \sum_{k=1}^{n} \gamma_k (\varepsilon_{i_k} + \varepsilon_{j_k}) - \sum_{k=1}^{n} \gamma_k \langle \int \varphi_{i_k, j_k, \theta_k} d\mathbf{m}, x' \rangle,$$

where $\gamma_1, ..., \gamma_n$ is a family of non negative real numbers such that $\sum_k \gamma_k = 1$. Each such a function is convex, weak* continuous and the set of all these functions is concave. Moreover, taking into account that the functions are weak* continuous and that $B_{S,\Delta}$ is weak* compact, the inequality in (1) gives an element $x_\phi' \in B_{S,\Delta}$ such that $\phi(x_\phi') \leq 0$. Ky Fan Lemma gives an element x_0' such that $\phi(x_0') \leq 0$ for all such functions. Consequently for every $(\gamma_k)_{k=1}^n$ and $(\varphi_{i_k, j_k, \theta_k})_{k=1}^n$ we have that

$$\langle (\gamma_k), (\varepsilon_{i_k} + \varepsilon_{j_k}) \rangle \leq \langle \, (\gamma_k), (\int \varphi_{i_k, j_k, \theta_k} d\langle \mathbf{m}, x_0' \rangle) \, \rangle.$$

In particular, for each couple $i, j \in \mathbb{N}$, $i \neq j$, taking $\gamma_1 = 1$, $\varphi_{i,j,1}$ and $\varphi_{i,j,-1}$ we obtain

$$\varepsilon_i + \varepsilon_j \leq \int (f_i^2 + f_j^2 + 2\theta f_i f_j) d\langle \mathbf{m}, x_0' \rangle \leq \varepsilon_i + \varepsilon_j + 2\theta \int f_i f_j d\langle \mathbf{m}, x_0' \rangle, \quad \theta \in \{-1, 1\}.$$

Therefore, $\int f_i f_j d\langle \mathbf{m}, x_0' \rangle = 0$ for each pair $i \neq j$. Fix now three different indexes $i, j, k \in \mathbb{N}$. We have the inequalities

$$\varepsilon_r + \varepsilon_s \leq \int f_r^2 d\langle \mathbf{m}, x_0' \rangle + f_s^2 d\langle \mathbf{m}, x_0' \rangle \leq \varepsilon_r + \varepsilon_s,$$

and so the equalities

$$\varepsilon_r + \varepsilon_s = \int f_r^2 d\langle \mathbf{m}, x_0' \rangle + f_s^2 d\langle \mathbf{m}, x_0' \rangle$$

for different r and s; $r, s \in \{i, j, k\}$. This implies that $\int f_r^2 d\langle \mathbf{m}, x_0' \rangle = \varepsilon_r$ for each $r \in \{i, j, k\}$ and finishes the proof.

The proof of (2) $\to$ (1) is a straightforward calculation; suppose that there is an element x_0' as in (2) and take non negative real numbers $\varepsilon_1, ..., \varepsilon_n$ such that

$\sum_k \varepsilon_k = 1$. Consider a sequence of functions $(\varphi_{i_k,j_k,\theta_k})_{k=1}^n$. Then

$$\langle(\gamma_k),(\varepsilon_{i_k}+\varepsilon_{j_k})\rangle = \sum_{k=1}^n \gamma_k(\varepsilon_{i_k}+\varepsilon_{j_k}) = \sum_{k=1}^n \gamma_k\Big(\int f_{i_k}^2 d\langle\mathbf{m},x_0'\rangle + \int f_{j_k}^2 d\langle\mathbf{m},x_0'\rangle\Big)$$
$$= \sum_{k=1}^n \gamma_k\Big(\int f_{i_k}^2 d\langle\mathbf{m},x_0'\rangle + \int f_{j_k}^2 d\langle\mathbf{m},x_0'\rangle + 2\theta_k\int f_{i_k}f_{j_k} d\langle\mathbf{m},x_0'\rangle\Big)$$
$$\leq \sup_{x'\in B_{S,\Delta}} |\langle\sum_{k=1}^n \gamma_k \int (f_{i_k}^2+f_{j_k}^2+2\theta_k f_{i_k}f_{j_k})\, d\mathbf{m}\ ,\ x'\rangle)| = \|\langle(\gamma_k)_k,(\varphi_{i_k,j_k,\theta_k})_k\rangle\|_{B_{S,\Delta}}. \quad \blacksquare$$

Remark 3.3. *For particular cases, the condition given in part (1) of Theorem 3.2 can be written in a simpler way. Consider a positive vector measure $\nu:\Sigma\to\ell^1$ and take the sequence Δ given by $(\|\int f_i^2 d\nu\|)_i$, i.e. $\varepsilon_i = \|\int f_i^2 d\nu\|$ for all i. The positivity of ν and the 1-concavity of ℓ^1 implies that the condition (1) in Theorem 3.2 is equivalent to the inequality*

$$\|\int f_i^2 d\nu\|_{\ell^1} + \|\int f_j^2 d\nu\|_{\ell^1} \leq \|\int (f_i+\theta f_j)^2 d\nu\|_{\ell^1}$$

for all $i,j\in\mathbb{N}$, $i\neq j$, and $\theta\in\{-1,1\}$.

In Example 3.1, the element of the dual space that defines the measure was explicitly computed. However, sometimes this is not possible and then the characterization theorem given above becomes useful. This is the situation that is shown in the following example.

Example 3.4.

- *Let (Ω,Σ,μ) be a probability space. Consider a relatively weakly compact sequence $(g_k)_k\subset L^1(\mu)$ defined by norm one positive sequences. Let us define the vector measure $\nu:\Sigma\to\ell^\infty$ given by the expression $\nu(A):=(\int_A g_k d\mu)_k$. It is well defined, and since the set is uniformly integrable, it is countably additive. Take a sequence $S:=(f_i)_i\in L^2(\nu)$ that satisfies the following properties. For every finite subset $I_0\subseteq\mathbb{N}$ there is an index $n\in\mathbb{N}$ such that $\{f_i : i\in I_0\}$ is orthonormal in $L^1(g_n d\mu)$. Let us show that this is enough to prove that condition (1) in Theorem 3.2 is satisfied.*

  Take the sequence $\Delta:=(\varepsilon_i)_i$, where $\varepsilon_i = 1$ for every i. Then the set $B_{S,\Delta}$ is just $B_{(\ell^\infty)'}\cap(\ell^\infty)^+$. For every $i,j\in\mathbb{N}$, $i\neq j$ and $\theta\in\{-1,1\}$, recall that

  $$\varphi_{i,j,\theta} := \big(f_i+\theta f_j\big)^2.$$

  So we have to prove the inequality

  $$2\leq \|\langle(\gamma_k),(\varphi_{i_k,j_k,\theta_k})\rangle\|_{B_{S,\Delta}}$$

  for $\gamma_k>0$ such that $\sum_{k=1}^m \gamma_k = 1$ and $\{f_{i_k}, f_{j_k}\in S : i_k, j_k\in I_0\}$ for a finite set I_0. But this is a direct consequence of the requirements of $(f_i)_i$ and the definition of the ℓ^∞ norm; we find an index n such that

  $$\|\sum_{k=1}^n \gamma_k \int \varphi_{i_k,j_k,\theta_k}\|_{\ell^\infty} \geq \sum_{k=1}^n \gamma_k \int (f_{i_k}+\theta_k f_{j_k})^2 g_n d\mu$$

$$= \sum_{k=1}^{n} \gamma_k \int (f_{i_k}^2 + f_{j_k}^2) g_n d\mu = 2.$$

Therefore, by Theorem 3.2 there is an element $x' \in B_{(\ell^\infty)'}$ such that S is orthonormal when considered as a sequence in $L^2(|\langle \nu, x' \rangle|)$. Notice also that, although the element x' do not belong in general to ℓ^1 and cannot be identified with a sequence, the measure $|\langle \nu, x' \rangle|$ is absolutely continuous with respect to μ, so there is an integrable function such that $|\langle \nu, x' \rangle|(A) = \int_A g_0 d\mu$ for every $A \in \Sigma$.

- *A concrete example of the situation above is given by the following elements. Take the Lebesgue space $([0,1], \Sigma, dx)$ and the functions $g_n := 2\sin^2(2^{n-1}\pi x)$, $n \in \mathbb{N}$. Consider the Rademacher functions $f_i := sgn(\sin(2^i \pi x)$, $i \in \mathbb{N}$. A direct computation shows that $\int f_i^2 g_n dx = 1$ for all $i, n \in \mathbb{N}$, and that for all $i, j \leq n$, $\int f_i f_j g_n dx = 0$ if $i \neq j$. Consequently, the inequality in Theorem 3.2 is satisfied, and there is a measure $|\langle \nu, x' \rangle|$ such that $(f_i)_i$ is an orthonormal sequence in $L^2(|\langle \nu, x' \rangle|)$.*

4 A Menchoff-Rademacher Theorem for weak $\mathbf{m}$-orthogonal sequences

Let (Ω, Σ, μ) be a finite measure space, and consider an orthonormal sequence $(f_i)_i$ of real functions in $L^2(\mu)$ and a sequence or real numbers $(a_i)_i$. The Menchoff-Rademacher Theorem is the main result concerning μ-almost everywhere convergence of the series $\sum_{i=1}^\infty a_i f_i$, and establishes that it converges μ-a.e. if

$$\sum_{i=1}^{\infty} |a_i|^2 log^2(i+1) < \infty,$$

see [18, 19, 23]. Although for particular (even complete) orthonormal sequences this result can be improved (for instance, it is enough that $\sum_{i=1}^\infty |a_i|^2 < \infty$ for the Haar and the trigonometric sequences, see 1.6.1 in [1] and [3]), it is optimal if we consider *any* orthonormal sequence.

In this section we study the almost everywhere convergence of functional series defined by (real valued) functions that are weak **m**-orthogonal for a vector measure **m**. Recall that the vector measure is supposed to be positive through all the paper. We develop a technique for generalizing the arguments that proves the Menchoff-Rademacher Theorem in our setting (see [26, III.H.22] for the scalar measure case). It provides the adequate elements for proving more specialized versions of this theorem depending on the properties of the space where the vector measure is defined. Let $\mathbf{m} : \Sigma \to X$ be a positive vector measure over (Ω, Σ). Our aim is to obtain conditions on a weak **m**-orthogonal sequence $(f_i)_i$ and a sequence of real numbers $(a_i)_i$ to assure $\langle \mathbf{m}, x' \rangle$-almost everywhere convergence of the series $\sum_{i=1}^\infty a_i f_i$ for a certain $x' \in (X')^+_{\mathbf{m}}$.

Let $x' \in (X')^+_{\mathbf{m}}$. Let $(f_i)_i$ be a sequence that is orthogonal with respect to $\langle \mathbf{m}, x' \rangle$. Consider Banach sequence spaces L and M over the standard measure space on $\mathbb{N}$ with canonical (normalized) basis $(e_i)_i$. If s is a natural number, we write Φ_s for the function $\Phi_s : \Omega \to L$ given by the formula

$$\Phi_s(\omega) := \sum_{i=1}^{s} f_i(\omega) e_i, \qquad \omega \in \Omega.$$

Note that this function belongs to the Bochner space $L^2(\langle \mathbf{m}, x' \rangle, L)$ for every $x' \in (X')^+_{\mathbf{m}}$, since $f_i \in L^2(\mathbf{m})$ for every $i \in \mathbb{N}$, and then each such a function can be considered as a (class of) function(s) of $L^2(\langle \mathbf{m}, x' \rangle)$ (see the comments after the definition of $L^2(\mathbf{m})$ in Section 2).

Definition 4.1. *Consider a Banach sequence space and a sequence of real numbers $a := (a_i)_i$. We denote by $\sigma_{a,L}$ the operator $\sigma_{a,L} : L \to \ell^\infty$ given by*

$$\sigma_{a,L}((\lambda_i)_i) := ((\sum_{i=1}^{n} a_i \lambda_i))_{n=1}^{\infty}, \qquad (\lambda_i)_i \in L,$$

if it is well defined and continuous. We also write $\sigma^N_{a,L}$ for the operator $\sigma^N_{a,L} : L \to \ell^\infty$ defined as $\sigma_{a(N),L}$, where $a(N)_i = a_i$ for every $i \geq N$, and 0 otherwise.

The sequence spaces occurring in the following theorem are supposed to satisfy the requirements that has been explained in Section 2.

Theorem 4.2. *Let $x' \in B_{X'} \cap (X')^+_{\mathbf{m}}$. Consider a sequence of real numbers $a = (a_i)_i$ and a sequence $(f_i)_i$ that is orthogonal with respect to $\langle \mathbf{m}, x' \rangle$. Let L be a 2-concave sequence space and let M be a sequence space such that $(L')_{[2]} = M'$. Suppose that*

(1) *there is a constant K such that $\|(\langle \int f_i^2 d\mathbf{m}, x' \rangle)_{i=1}^{s}\|_M < K$ for every $s \in \mathbb{N}$ and*

(2) *the operators $\sigma^N_{a,L} : L \to \ell^\infty$ are 2-summing and $\lim_{N\to\infty} \pi_2(\sigma^N_{a,L}) = 0$.*

Then the series $\sum_{i=1}^{\infty} a_i f_i$ converges $\langle \mathbf{m}, x' \rangle$-a.e.

Proof. First we prove the following *claim*: suppose that the sequence space L is 2-concave, and M satisfies $(L')_{[2]} = M'$. Let Y be a Banach space and let $T : L \to Y$ be a 2-summing operator. If $x' \in B_{X'} \cap (X')^+_{\mathbf{m}}$, then for every natural number s,

$$\|T\Phi_s\|_{L^2(\langle \mathbf{m}, x' \rangle, Y)} \leq \pi_2(T) \|(\langle \int f_i^2 d\mathbf{m}, x' \rangle)_{i=1}^{s}\|_M^{\frac{1}{2}}.$$

To prove this, first note that the elements of the space $(L')_{[2]}$ are sequences $\tau = (\tau_i)_i$ that satisfy that there is a sequence $z' = (z'_i)_i \in L'$ such that for every $i \in \mathbb{N}$, $|z'_i|^2 = |\tau_i|$. Since L' is 2-convex (see [16, Proposition 1.d.4(iii)]), $(L')_{[2]}$ is a Banach space with norm $\|\tau\|_{(L')_{[2]}} := \|(\tau_i)_i\|_{(L')_{[2]}} = \|(|\tau_i|^{1/2})_i\|^2_{L'}$ (recall that we assume for simplicity that the 2-concavity constant of L is 1 and then the 2-convexity constant of L also equals 1; see [6] and [16, Proposition 1.d.4(iii)]). Since T is 2-summing, a direct calculation (see [26, Proposition III.F.33,b)]) gives

$$\|T\Phi_s\|^2_{L^2(\langle \mathbf{m}, x' \rangle, Y)} \leq \pi_2^2(T) \sup_{z' \in B_{L'}} \int |\langle \Phi_s(\omega), z' \rangle|^2 d\langle \mathbf{m}, x' \rangle.$$

Since $(f_i)_i$ is orthogonal with respect to $\langle \mathbf{m}, x' \rangle$, the inequality above can be written as

$$\|T\Phi_s\|^2_{L^2(\langle \mathbf{m},x'\rangle,Y)} \leq \pi_2^2(T) \sup_{z' \in B_{L'}} \sum_{i=1}^{s} |z'_i|^2 \int f_i^2 d\langle \mathbf{m}, x' \rangle;$$

all the integrals in this expression are positive, so we also obtain

$$\sup_{z' \in B_{L'}} \sum_{i=1}^{s} |z'_i|^2 \int f_i^2 d\langle \mathbf{m}, x' \rangle = \sup_{\tau \in B_{(L')^2}} \sum_{i=1}^{s} |\tau_i| \int f_i^2 d\langle \mathbf{m}, x' \rangle$$

$$= \|(\langle \int f_i^2 d\mathbf{m}, x' \rangle)_{i=1}^s\|_M < K.$$

This gives the desired inequality and proves the claim.

Now we just need to show that the requirements for the operators $\sigma_{a,L}^N$ are enough to apply an standard almost everywhere convergence criterion (see for instance [26, III.H.22]). A direct calculation shows that for every natural number s, the pointwise evaluation

$$\omega \mapsto \|\sigma_{a,L}(\Phi_s(\omega))\|_\infty \tag{4.1}$$

gives the sequence of functions

$$g_s(\omega) := \|\sigma_{a,L}(\Phi_s(\omega))\|_\infty = \max_{n=1,\dots,s} |\sum_{i=1}^{n} a_i f_i(\omega)|.$$

Now we apply the claim for $Y = \ell^\infty$ and $T = \sigma_{a,L}$. Then the sequence of norms

$$(\|g_s\|_{L^2(\langle \mathbf{m},x'\rangle)})_{s=1}^\infty = (\| \, \|\sigma_{a,L}(\Phi_s(\omega))\|_\infty \, \|_{L^2(\langle \mathbf{m},x'\rangle)})_{s=1}^\infty$$

$$= \Big(\Big(\int \max_{1\leq n\leq s} |\sum_{i=1}^{n} a_i f_i(\omega)|^2 d\langle \mathbf{m}, x' \rangle\Big)^{\frac{1}{2}}\Big)_{s=1}^\infty$$

is uniformly bounded, since for every s,

$$\|g_s\|_{L^2(\langle \mathbf{m},x'\rangle)} \leq \pi_2(\sigma_{a,L})K.$$

Thus, the Monotone Convergence Theorem gives that the function

$$h_1(\omega) := \sup_{n\geq 1} |\sum_{i=1}^{n} a_i f_i(\omega)|$$

is also in $L^2(\langle \mathbf{m}, x' \rangle)$ and $\|h_1\|_{L^2(\langle \mathbf{m},x'\rangle)} \leq \pi_2(\sigma_{a,L})K$. If we consider the operators $\sigma_{a,L}^N$ instead of $\sigma_{a,L}$ in (4.1), we obtain the same result for each $N \in \mathbb{N}$, i.e.

$$h_N(\omega) := \sup_{n\geq N} |\sum_{i=N}^{n} a_i f_i(\omega)|$$

also belongs to $L^2(\langle \mathbf{m}, x' \rangle)$ for every $N \in \mathbb{N}$ and $\|h_N\|_{L^2(\langle \mathbf{m},x'\rangle)} \leq \pi_2(\sigma_{a,L}^N)K$.

Thus, condition (2) in the statement of the theorem implies that the sequence $(h_N)_{N=1}^{\infty}$ converges to 0 in $L^2(\langle \mathbf{m}, x' \rangle)$, and then there is a subsequence that converges $\langle \mathbf{m}, x' \rangle$-a.e. to 0. This clearly implies that the sequence itself converges $\langle \mathbf{m}, x' \rangle$-a.e. to 0 since it is decreasing. Hence, for $\langle \mathbf{m}, x' \rangle$-a.e. every $\omega \in \Omega$ and $\varepsilon > 0$ there is a natural number R such that for every $N \geq R$,

$$|\sum_{i=1}^{N} a_i f_i(\omega) - \sum_{i=1}^{R-1} a_i f_i(\omega)| \leq \sup_{n \geq R} |\sum_{i=R}^{n} a_i f_i(\omega)| = h_R(\omega) < \varepsilon$$

and then the series $\sum_{i=1}^{\infty} a_i f_i(\omega)$ converges $\langle \mathbf{m}, x' \rangle$-a.e. ■

Note that the almost everywhere convergence with respect to a measure defined by a positive element x' do not provide $\mathbf{m}$-almost everywhere convergence, since such measures *are not* in general Rybakov measures. This means that such a measure can have more null sets in the σ-algebra Σ.

Remark 4.3. *The requirements on L in Theorem 4.2 show that the problem of the almost everywhere convergence of weak* $\mathbf{m}$*-orthogonal series is closely related to the calculus of estimates of* 2*-summing norms for the operators* $\sigma_{a,L}^N : L \to \ell^\infty$ *for suitable sequence spaces L. The canonical examples of such spaces are sequence spaces that satisfy that the inclusions* $\ell^1 \subseteq L \subseteq \ell^2$ *are well defined and continuous; take* $L = \ell^p$ *for* $1 \leq p \leq 2$*. Then* $(L')_{[2]} = (\ell^{p'})_{[2]} = \ell^{p'/2}$*. If we consider* $1 \leq q \leq \infty$ *such that* $1/q = 1/p - 1/p'$*, then the space M satisfying* $M' = ((\ell^p)')_{[2]}$ *is* ℓ^q *(*c_0 *if* $p = 2$*). In the following section we develop the case* $p = 1$*, for which* $(L')_{[2]} = (\ell^\infty)_{[2]} = \ell^\infty$*, and then* $M = \ell^1$*.*

Let us finish this section by giving two estimates for these norms (we give the estimates for $\sigma_{a,L}$*, the ones for* $\sigma_{a,L}^N$ *are obtained with the same arguments). Note that a direct application of the following inequalities to Theorem 4.2 provides formulas involving the sequence "a" that can be directly computed.*

(1) The first one comes from an application of Grothendieck´s Theorem and can be used for the case of operators $\sigma_{a,\ell^1} : \ell^1 \to \ell^\infty$ that are still continuous when defined as $\sigma_{a,\ell^2} : \ell^2 \to \ell^\infty$. Consider a sequence $a \in \ell^2$. In this case we can write a factorization of σ_{a,ℓ^1} as $\sigma_{a,\ell^2} \circ id$, where $id : \ell^1 \to \ell^2$ is the inclusion map. This map is 1-summing (see for instance 17.14 in [7]), which implies that it is also 2-summing, and thus σ_{a,ℓ^1} is so. Moreover,

$$\pi_2(\sigma_{a,\ell^1}) \leq \pi_2(id) \|\sigma_{a,\ell^2}\| \leq (\sum_{i=1}^{\infty} a_i^2)^{\frac{1}{2}},$$

(see e.g Exercise 11.5 in [7] for the estimate of $\pi_2(id)$). Of course, the same argument can be used for general $\sigma_{a,L} : L \to \ell^\infty$ whenever it can be factored through $id : \ell^1 \to \ell^2$.

(2) For the second one the argument is similar, but using the fact that the operator σ_{b,ℓ^1} is integral —we write $\iota(T)$ for the integral norm of an operator T—, where $b = (1/log(i+1))_i$ (see the reference to the Bennet-Maurey-Nahoum Theorem in [8, Section 4], and [7] for the definition and properties of integral operators). Suppose that the sequence "a" satisfies that

$$\|(a_i log(i+1))_i\|_{L'} < \infty.$$

This requirement is the natural generalization of the Menchoff-Rademacher condition for a.e. convergence. We can obtain the factorization through ℓ^1 given by $\sigma_{a,L} = \sigma_{b,\ell^1} \circ D_c$, where D_c is the diagonal operator defined by the sequence $c_i = a_i log(i+1)$, since

$$\pi_2(\sigma_{a,L}) \leq \iota(\sigma_{a,L}) \leq \iota(\sigma_{b,\ell^1})\|D_c\| \leq \iota(\sigma_{b,\ell^1})\|(a_i log(i+1))_i\|_{L'}.$$

The same factorization can be used for every sequence b such that σ_{b,ℓ^1} is 2-summing.

The following results combine Theorem 3.2, Theorem 4.2 and the remark above to give several criteria for $\langle \mathbf{m}, x' \rangle$-a.e. convergence of sequences in $L^2(\mathbf{m})$.

Corollary 4.4. *Let $m : \Sigma \to X$ be a positive vector measure and consider a sequence $(f_i)_i \subseteq L^2(m)$. Let L be a 2-concave sequence space and let M be a sequence space such that $(L')_{[2]} = M'$. Let $\Delta = (\varepsilon_i)_i$ such that $(\varepsilon_i)_{i=1}^s \in M$ for all $s \in \mathbb{N}$, $a = (a_i)_i$ a sequence of real numbers and suppose that*

(1) *the operators $\sigma_{a,L}^N : L \to \ell^\infty$ are 2-summing and $\lim_{N\to\infty} \pi_2(\sigma_{a,L}^N) = 0$ and*

(2) *for every finite sequence of non negative real numbers $(\gamma_k)_k$ such that $\sum_k \gamma_k = 1$, indexes $i_k, j_k \in \mathbb{N}$, $i_k \neq j_k$, and $\theta_k \in \{-1, 1\}$,*

$$\langle (\gamma_k), (\varepsilon_{i_k} + \varepsilon_{j_k}) \rangle \leq \| \langle (\gamma_k), (\varphi_{i_k,j_k,\theta_k}) \rangle \|_{B_{S,\Delta}}.$$

Then there is an element $x' \in X'$ such that
1. the sequence $(\frac{f_i}{\sqrt{\varepsilon_i}})_i$ is orthonormal in $L^2(\langle \mathbf{m}, x' \rangle)$ and
2. the series $\sum_{i=1}^\infty a_i f_i$ converges $\langle \mathbf{m}, x' \rangle$-a.e.

Corollary 4.5. *Let $\mathbf{m} : \Sigma \to X$ be a positive vector measure and consider a sequence $(f_i)_i \subseteq L^2(m)$ such that $(\| \int f_i^2 d\mathbf{m} \|)_{i=1}^s \in M$ for all $s \in \mathbb{N}$. Let L be a 2-concave sequence space and let M be a sequence space such that $(L')_{[2]} = M'$. Let $a = (a_i)_i$ be a sequence of real numbers and suppose that*

(1) *the operators $\sigma_{a,L}^N : L \to \ell^\infty$ are 2-summing and $\lim_{N\to\infty} \pi_2(\sigma_{a,L}^N) = 0$ and*

(2) *for every finite sequence of non negative real numbers $(\gamma_k)_k$ such that $\sum_k \gamma_k = 1$, indexes $i_k, j_k \in \mathbb{N}$, $i_k \neq j_k$, and $\theta_k \in \{-1, 1\}$,*

$$\langle (\gamma_k), (\varepsilon_{i_k} + \varepsilon_{j_k}) \rangle \leq \| \langle (\gamma_k), (\varphi_{i_k,j_k,\theta_k}) \rangle \|.$$

Then there is an element $x' \in X'$ such that
1. the sequence $(\frac{f_i}{\sqrt{\varepsilon_i}})_i$ is orthonormal in $L^2(\langle \mathbf{m}, x' \rangle)$ and
2. the series $\sum_{i=1}^\infty a_i f_i$ converges $\langle \mathbf{m}, x' \rangle$-a.e.

Example 4.6. • *For the case of ℓ^1-valued measures and $L = \ell^2$, we obtain using Remark 3.3 that the result is similar to the one that holds for scalar measures. Let $\mathbf{m} : \Sigma \to \ell^1$ be a positive vector measure and consider a sequence $(f_i)_i \subseteq L^2(\mathbf{m})$ of norm one functions. Let $a = (a_i)_i$ be a sequence of real numbers and suppose that*

(1) *the operators* $\sigma^N_{a,\ell^2} : \ell^2 \to \ell^\infty$ *are* 2*-summing and* $\lim_{N\to\infty} \pi_2(\sigma^N_{a,\ell^2}) = 0$, *and*

(2) $\sqrt{2} \leq \|f_i + \theta f_j\|_{L^2(\mathbf{m})}$ *for all* $i, j \in \mathbb{N}$, $i \neq j$, *and* $\theta \in \{-1, 1\}$.

Then Corollary 4.5 gives an element $x' \in \ell^\infty$ *such that the sequence* $(f_i)_i$ *is orthonormal in* $L^2(\langle \mathbf{m}, x' \rangle)$ *and the series* $\sum_{i=1}^\infty a_i f_i$ *converges* $\langle \mathbf{m}, x' \rangle$*-a.e. For example, a direct calculation shows that for* $x' = (1, 1, 1, ...)$, *the result holds.*

- *Let us show an application regarding Example 3.4 also for* $L = \ell^2$. *Let* $(e_i)_i$ *be the canonical basic sequence in* ℓ^∞ *and take an* ℓ^∞ *valued (countably additive) vector measure* ν, *a sequence of functions* $(f_i)_i \in L^2(\mathbf{m})$ *such that* $\int f_i^2 d\langle \nu, e_j \rangle = 1$ *for every* $i, j \in \mathbb{N}$ *and a sequence* $a = (a_i)_i$ *such that the operators* $\sigma^N_{a,\ell^2} : \ell^2 \to \ell^\infty$ *are* 2*-summing with* $\lim_{N\to\infty} \pi_2(\sigma^N_{a,\ell^2}) = 0$. *Assume also that for every finite sequence of non negative real numbers* $(\gamma_k)_k$ *such that* $\sum_k \gamma_k = 1$, *indexes* $i_k, j_k \in \mathbb{N}$, $i_k \neq j_k$, *and* $\theta_k \in \{-1, 1\}$,

$$2 \leq \sup_i \Big| \int \sum_k \gamma_k (f_{i_k} + \theta_k f_{j_k})^2 d\langle \nu, e_i \rangle \Big|.$$

Then by Corollary 4.5 there is an element $x' \in (\ell^\infty)'$ *such that the sequence* $(f_i)_i$ *is orthonormal in* $L^2(\langle \mathbf{m}, x' \rangle)$ *and the series* $\sum_{i=1}^\infty a_i f_i$ *converges* $\langle \mathbf{m}, x' \rangle$*-a.e.*

5 Almost everywhere convergence in c_0-sums of $L^2(\mu)$ spaces.

In this section we use the representation of $\ell-$sums of L^2-spaces as spaces $L^2(\mathbf{m})$ for a suitable $\mathbf{m}$ to apply our results. In particular, we develop the case of c_0-sums of L^2-spaces.

Let be (Ω, Σ, μ) a finite measure space and consider a disjoint partition $(E_i)_i \subset \Sigma$ of Ω. Consider the sequence space c_0. We define a countably additive vector measure $\mathbf{n} : \Sigma \to c_0$ by

$$\mathbf{n}(E) := \sum_{i=1}^{\infty} \mu(E \cap E_i) e_i,$$

where e_i is the canonical basis of c_0.

Let $\mu_i = \mu \mid_{E_i}$ be the restriction of μ to the subset E_i. We will denote by $\oplus_{c_0} L^2(\mu_i)$ the space of (classes of μ-a.e. equal) measurable functions f such that

(1) $f\chi_{E_i} \in L^2(\mu_i)$, and

(2) $(\| f\chi_{E_i} \|_{L^2(\mu_i)})_{i=1}^\infty \in c_0$.

The (lattice) norm for this space is given by

$$\|f\|_{\oplus_{c_0} L^2(\mu_i)} := \sup_i \|f\|_{L^2(\mu_i)}, \qquad f \in \oplus_{c_0} L^2(\mu_i).$$

The following result show that we can identify the spaces $L^2(\mathbf{n})$ and $\oplus_{c_0} L^2(\mu_i)$; for related examples, see [21, Example 8], [25, Example 10], [13, Example 4] and [20, Example 6.47].

Proposition 5.1. *The natural identification map between $L^2(\mathbf{n})$ and $\oplus_{c_0} L^2(\mu_i)$ is an order isometry.*

Proof. We begin by proving that the identification $f \mapsto (f\chi_{E_i})_i \in \oplus_{c_0} L^2(\mu_i)$, $f \in L^2(\mathbf{n})$ defines an isometry. Assume that $f \in L^2(\mathbf{n})$. Then for every $i_0 \in \mathbb{N}$, $e_{i_0} \in \ell^1 = (c_0)'$ and $\langle \mathbf{n}, e_{i_0} \rangle = \mu_{i_0}$. Therefore $f\chi_{E_{i_0}} \in L^2(\langle \mathbf{n}, e_{i_0} \rangle) = L^2(\mu_{i_0})$. Furthermore, note that for every $f \in L^2(\mathbf{n})$ and $\varepsilon > 0$ there is a natural number $n \in \mathbb{N}$ such that $\| \int_{\cup_{i=n}^{\infty} E_i} |f|^2 d\mathbf{n} \| < \varepsilon$, since $\langle \int_{\cup_{i=n}^{\infty} E_i} |f|^2 d\mathbf{n}, e_j \rangle = 0$ for any $1 \leq j < n$ and $\int |f|^2 d\mathbf{n} \in c_0$. Then

$$\| f \|_{L^2(\mathbf{n})} = \| \int | f |^2 \, d\mathbf{n} \|_{c_0}^{\frac{1}{2}} = \lim_{n\to\infty} \| \int \sum_{i=1}^{n} |f|^2 \chi_{E_i} d\mathbf{n} \|_{c_0}^{\frac{1}{2}} =$$

$$= \lim_{n\to\infty} \| (\int_{E_i} | f |^2 \, d\mu_i)_{i=1}^{n} \|_{c_0}^{\frac{1}{2}} = \| ((\int_{E_i} | f |^2 \, d\mu_i)^{\frac{1}{2}})_{i=1}^{\infty} \|_{c_0} = \|f\|_{\oplus_{c_0} L^2(\mu_i)}.$$

Let us show now that if $(f_i)_{i=1}^{\infty} \in \oplus_{c_0} L^2(\mu_i)$ then $f := \sum_{i=1}^{\infty} f_i \chi_{E_i} \in L^2(\mathbf{n})$. If $x' = (\lambda_i)_{i=1}^{\infty} \in \ell^1$, the scalar measure $\langle \mathbf{n}, x' \rangle$ is given by $\langle \mathbf{n}, x' \rangle(A) := \sum_{i=1}^{\infty} \lambda_i \mu(E_i \cap A)$, $A \in \Sigma$. Clearly the functions $g_n := \sum_{i=1}^{n} |f_n|^2 \chi_{E_i}$, $n \in \mathbb{N}$, converge pointwise to $|f|^2$ and

$$\lim_{n\to\infty} | \int g_n \, d|\langle \mathbf{n}, x' \rangle| \, | = \lim_{n\to\infty} | \sum_{i=1}^{n} |\lambda_i| \int_{E_i} |f_i|^2 d\mu_i |$$

$$\leq \| (\lambda_i)_i \|_{\ell^1} \cdot \| (\int_{E_i} |f|^2 d\mu_i)_i \|_{c_0} < \infty.$$

Then the Monotone Convergence Theorem gives that f is scalarly integrable, and

$$\sup_{x' \in B_{\ell^1}} \int |f|^2 d|\langle \mathbf{n}, x' \rangle| < \infty.$$

A direct calculation shows that the formula $\int |f|^2 d\mathbf{n} = (\int |f_i|^2 d\mu_i)_i \in c_0$ provides the integral of the function $|f|^2$ and $\|f\|_{L^2(\mathbf{n})} = \|(\int |f_i|^2 d\mu_i)_i\|_{c_0}^{\frac{1}{2}}$. ∎

In this context we can apply all the results of Section 3. The proofs of the following corollaries are straightforward applications of Theorem 3.2, Theorem 4.2 with $M = L = \ell^1$ and Remark 4.3 (1).

Corollary 5.2. *Let $X(\mu) = \oplus_{c_0} L^2(\mu_i)$ be the c_0-sum of the spaces $L^2(\mu_i)$, $i \in \mathbb{N}$. Let $(a_i)_i \in \ell^2$, and assume that there exists an element $x' \in (\ell^1)^+_{\mathbf{m}}$ such that $(f_i)_{i=1}^{\infty}$ is orthogonal with respect to $\langle \mathbf{n}, x' \rangle$. If $\|((\langle \int f_i^2 d\mathbf{n}, x' \rangle))_{i=1}^{\infty} \|_{\ell^1} < \infty$, then $\sum_{i=1}^{\infty} a_i f_i$ converges $\langle \mathbf{n}, x' \rangle$-a.e.*

Corollary 5.3. *Let $X(\mu) = \oplus_{c_0} L^2(\mu_i)$ be the c_0-sum of the spaces $L^2(\mu_i)$, $i \in \mathbb{N}$. Let $(a_i)_i \in \ell^2$, and assume that there is a sequence of positive real numbers $\Delta = (\varepsilon_i) \in (\ell^1)^+_{\mathbf{m}}$ satisfying the inequalities in (1) of Theorem 3.2 for the vector measure $\mathbf{n}$. Then there is a sequence $0 \leq x' \in B_{\ell^1}$ such that $(f_i)_i$ orthogonal with respect to $\langle \mathbf{n}, x' \rangle$ and $\sum_{i=1}^{\infty} a_i f_i$ converges $\langle \mathbf{n}, x' \rangle$-a.e.*

Let us finish the paper with a particular example of a sequence that satisfies these corollaries.

Example 5.4. *Let $([0,1],\Sigma,\mu)$ be the Lebesgue measure space. We consider the following partition of the interval $[0,1]$.*

$$E_1=[0,\frac{1}{2}],E_2=[\frac{1}{2},\frac{3}{4}],E_3=[\frac{3}{4},\frac{7}{8}],...,E_n=[\frac{2^{n-1}-1}{2^{n-1}},\frac{2^n-1}{2^n}],\dots\ .$$

For each $i=1,2,...$, take a $\mu|_{E_i}$-orthogonal sequence $(g_i^j)_{j=1}^\infty$ satisfying

$$\int g_i^j g_i^k d\mu_i \quad = \quad \begin{cases} 0 \ \textit{if} \ \ k\neq j \\ 2 \ \textit{if} \ \ k=j \ . \end{cases}$$

Now we define for each $n\in\mathbb{N}$ the function f_n by $f_n:=\sum_{k=1}^\infty \lambda_k^n g_k^n \chi_{E_k}$, where the scalar numbers λ_k^n are given by

$$\lambda_k^n \quad = \quad \begin{cases} \frac{1}{2^{\frac{n-k+1}{2}}} \ \textit{if} \ \ k<n \\ \frac{1}{2^{\frac{k-n+1}{2}}} \ \textit{if} \ \ k\geq n. \end{cases}$$

Let $\alpha := (\alpha_k)_{k=1}^\infty = (\frac{1}{2^k})_{k=1}^\infty \in B_{\ell^1}$. It is easy to see that the sequence $(f_n)_{n=1}^\infty$ is orthogonal with respect to any such measure $\langle \mathbf{n},\alpha\rangle$. Thus,

$$\|(\langle\int f_n^2 d\mathbf{n},\alpha\rangle)\|_{\ell^1}=\sum_{n=1}^\infty|\langle\int f_n^2 d\mathbf{n},\alpha\rangle|=\sum_{n=1}^\infty(\sum_{k=1}^\infty\langle\int f_n^2 d\mathbf{n},e_k'\rangle\alpha_k)=\sum_{n=1}^\infty(\sum_{k=1}^\infty(\lambda_k^n)^2\alpha_k),$$

and therefore,

$$\|(\langle\int f_n^2 d\mathbf{n},\alpha\rangle)\|_{\ell^1}=\sum_{n=1}^\infty(\sum_{k=1}^{n-1}(\lambda_k^n)^2\alpha_k+\sum_{k=n}^\infty(\lambda_k^n)^2\alpha_k)$$

$$=\sum_{n=1}^\infty(\sum_{k=1}^{n-1}\frac{1}{2^{n-k+1}}\alpha_k+\sum_{k=n}^\infty\frac{1}{2^{k-n+1}}\alpha_k)$$

$$=\frac{1}{2}\sum_{n=1}^\infty(\sum_{k=1}^{n-1}\frac{2^k}{2^n}\alpha_k+\sum_{k=n}^\infty\frac{2^n}{2^k}\alpha_k)=\frac{1}{2}\sum_{n=1}^\infty(\sum_{k=1}^{n-1}\frac{1}{2^n}+\sum_{k=n}^\infty\frac{2^n}{2^{2k}})$$

$$=\frac{1}{2}\sum_{n=1}^\infty\sum_{k=1}^{n-1}\frac{1}{2^n}+\frac{1}{2}\sum_{n=1}^\infty\sum_{k=n}^\infty\frac{2^n}{2^{2k}}<\infty.$$

Thus, $\|(\langle\int f_n^2 d\mathbf{n},\alpha\rangle)\|_{\ell^1}$ is bounded. Remark 4.3 (1) provides the required condition on $(a_i)_i$ and so $\sum_{i=1}^\infty a_i f_i$ converges $\langle\mathbf{n},\alpha\rangle$-a.e.

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Instituto Universitario de Matemática Pura y Aplicada.
Universidad Politécnica de Valencia.
Camino de Vera s/n. 46022 Valencia. Spain.
e-mail: edjimfer@mat.upv.es ,easancpe@mat.upv.es