Entire functions that share fixed points with finite weights

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Abstract

With the aid of weighted sharing method we study the uniqueness of entire functions concerning general nonlinear differential polynomials sharing fixed points. The results of the paper improve and generalize some results due to Zhang [18] and Qi-Dou [12].

1 Introduction, Definitions and Results

Let f be a nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [7], [16] and [17]. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. A meromorphic function $a(z)(\not\equiv \infty)$ is called a small function with respect to f, provided that $T(r,a) = S(r,f)$.

We say that two meromorphic function f and g share a small function $a(z)$ CM (counting multiplicities) if $f - a$ and $g - a$ have the same set of zeros with the same multiplicities and we say that f and g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities. A finite value z_0 is a fixed point of $f(z)$ if $f(z_0) = z_0$ and we define

$$E_f = \{z \in \mathbb{C} : f(z) = z, counting\ multiplicities\}.$$

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Let $a \in C \cup \{\infty\}$. For a positive integer p we denote by $N(r,a;f \mid\geq p)$ the counting function of those a-points of f whose multiplicities are not less than p, where each a-point is counted according to its multiplicity. $\overline{N}(r,a;f \mid\geq p)$ is defined similarly, where in counting the a-points of f we ignore the multiplicities. We denote by $N_p(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. That is

$$N_p(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + ... + \overline{N}(r,a;f \mid\geq p).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

In 1959, Hayman [6] proved the following theorem.

Theorem A. *Let f be a transcendental entire function and let $n(\geq 1)$ be an integer. Then $f^n f' = 1$ has infinitely many solutions.*

Corresponding to which, the following result was obtained by Fang and Hua [3] and by Yang and Hua [15] respectively.

Theorem B. *Let f and g be two nonconstant entire functions, $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

Considering k th derivative instead of first derivative, Hennekemper [8], Chen [2] and Wang [13] proved the following theorem which extends Theorem A.

Theorem C. *Let f be a transcendental entire function and n, k be two positive integers with $n \geq k+1$. Then $(f^n)^{(k)} = 1$ has infinitely many solutions.*

Corresponding to Theorem C Fang [4] proved the following theorems.

Theorem D. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1c_2)^n (nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem E. *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n \geq 2k+8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.*

So natural question arises: What can be said if the share value 1 be replaced by a fixed point. It is worth mentioning that in the above area some investigations has already been carried out by Fang - Qiu [5] and Lin - Yi [11].

In 2008, Zhang [18] proved the following theorems.

Theorem F. *Let f and g be two nonconstant entire functions, and n, k be two positive integers with $n > 2k+4$. If $E_{(f^n)^{(k)}} = E_{(g^n)^{(k)}}$, then either*
(i) $k = 1$, $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^n (nc)^2 = -1$ or
(ii) $f \equiv tg$ for a constant t such that $t^n = 1$.

Theorem G. *Let f and g be two nonconstant entire functions, and n, k be two positive integers with $n \geq 2k+6$. If $E_{(f^n(f-1))^{(k)}} = E_{(g^n(g-1))^{(k)}}$, then $f \equiv g$.*

Recently Qi - Dou [12] replace CM sharing value by IM sharing value and proved the following theorems.

Theorem H. *Let f and g be two transcendental entire functions, and let n, k be two positive integers with $n > 5k+7$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z IM, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^n(nc)^2 = -1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem I. *Let f and g be two transcendental entire functions, and let n, k be two positive integers with $n > 5k+11$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share z IM, then $f \equiv g$.*

Naturally one may ask the following questions which are the motivation of the paper.
Question 1. Is it really possible in any way to relax the nature of sharing the fixed point in Theorem F and Theorem G without changing the lower bound of n ?
Question 2. Whether one can deduce a generalized result in which Theorem H and Theorem I will be included ?

In the paper we will concentrate our attention on the above questions and provide an affirmative solution in this direction. To state the main results we need the following definition known as weighted sharing of values introduced by I. Lahiri [9, 10] which measures how close a shared value is to being shared CM or to being shared IM.

Definition 1. *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Again f and g share (z, l) means that $f(z) - z$ and $g(z) - z$ share $(0, l)$ where $l(\geq 0)$ is an integer.

In the paper, we will prove two theorems second of which will not only improve Theorems F and G by relaxing the nature of sharing the fixed point and at the same time improve and generalize Theorems H and I. We now state the main results of the paper.

Theorem 1. *Let f be a transcendental entire function and n, k be two positive integers such that $n \geq k+2$. Let $P(z) = a_m(z)z^m + a_{m-1}(z)z^{m-1} + ... + a_1z + a_0$ or $P(z) = c_0$, where $a_0(\neq 0)$, a_1,...,a_{m-1}, $a_m(\neq 0)$, $c_0(\neq 0)$ are complex constants. Then $(f^nP(f))^{(k)}$ has infinitely many fixed points.*

Theorem 2. *Let f and g be two transcendental entire functions, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 1)$ be three integers. Let $P(z)$ be defined as in Theorem 1. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share (z,l) where $l(\geq 0)$ is an integer, then*
(i) when $P(z) = a_m(z)z^m + a_{m-1}(z)z^{m-1} + ... + a_1z + a_0$, and one of $l \geq 2$ and $n > 2k + m + 4$; $l = 1$ and $n > \frac{5k+3m+9}{2}$; $l = 0$ and $n > 5k + 4m + 7$ holds, either $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = (n + m, ..., n + m - i, ..., n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, ..., m$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$R(x,y) = x^n(a_mx^m + a_{m-1}x^{m-1} + ... + a_0) - y^n(a_my^m + a_{m-1}y^{m-1} + ... + a_0);$$

(ii) when $P(z) = c_0$, and one of $l \geq 2$ and $n > 2k + 4$; $l = 1$ and $n > \frac{5k+9}{2}$; $l = 0$ and $n > 5k + 7$ holds, either $f(z) = c_1/c_0^{\frac{1}{n}}e^{cz^2}$, $g(z) = c_2/c_0^{\frac{1}{n}}e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^n(c)^2 = -1$ or $f = tg$ for a constant t such that $t^n = 1$.

Corollary 1. *Under the same condition of Theorem 2, we set $P(z) = (z - 1)$. Then either $f(z) \equiv g(z)$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(x, y) = x^n(x - 1) - y^n(y - 1)$, provided one of the following holds:*
(i) $l \geq 2$, $n > 2k + 5$;
(ii) $l = 1$, $n > \frac{5k+12}{2}$;
(iii) $l = 0$, $n > 5k + 11$.

Remark 1. *Obviously Corollary 1 is an extension of Theorem G.*

Remark 2. *Clearly Theorem 2 improves Theorem F when $P(z) = c_0 = 1$.*

Remark 3. *Since Theorems H and I can be obtained as the special cases of Theorem 2, clearly Theorem 2 improves and supplements Theorems H and I.*

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[14] Let f be a nonconstant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z)$, ... , $a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, ..., n$. Then*

$$T(r, a_nf^n + a_{n-1}f^{n-1} + ... + a_1f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2. *[19] Let f be a nonconstant meromorphic function, and p, k be positive integers. Then*

$$N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{2.1}$$

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2.2}$$

Lemma 3. *[10] Let f and g be two nonconstant meromorphic functions sharing $(1,2)$. Then one of the following cases holds:*
(i) $T(r) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + S(r)$,
(ii) $f = g$,
(iii) $fg = 1$.

Lemma 4. *[1] Let f and g be two nonconstant meromorphic functions sharing $(1,l)$ and*

$$\frac{f''}{f'} - \frac{2f'}{f-1} \not\equiv \frac{g''}{g'} - \frac{2g'}{g-1}.$$

Now the following hold:
(i) if $l = 1$ *then* $T(r,f) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) + S(r,f) + S(r,g)$;
(ii) if $l = 0$ *then* $T(r,f) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + 2\overline{N}(r,0;f) + \overline{N}(r,0;g) + 2\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g)$.

Lemma 5. *[7, 16] Let f be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two distinct meromorphic functions such that $T(r,a_i(z)) = S(r,f)$, i=1,2. Then*

$$T(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,a_1;f) + \overline{N}(r,a_2;f) + S(r,f).$$

Lemma 6. *Let f and g be two nonconstant entire functions and let n, k be two positive integers. Suppose that $F_1 = (f^nP(f))^{(k)}$ and $G_1 = (g^nP(g))^{(k)}$ where $P(z) = a_m(z)z^m + a_{m-1}(z)z^{m-1} + ... + a_1z + a_0$, $a_0(\neq 0)$, a_1, ... ,a_{m-1}, $a_m(\neq 0)$ are complex constants. If there exist two nonzero constants c_1 and c_2 such that $\overline{N}(r,c_1;F_1) = \overline{N}(r,0;G_1)$ and $\overline{N}(r,c_2;G_1) = \overline{N}(r,0;F_1)$, then $n \leq 2k+m+2$.*

Proof. By the second fundamental theorem of Nevanlinna we have

$$\begin{aligned} T(r,F_1) &\leq \overline{N}(r,0;F_1) + \overline{N}(r,\infty;F_1) + \overline{N}(r,c_1;F_1) + S(r,F_1) \\ &\leq \overline{N}(r,0;F_1) + \overline{N}(r,0;G_1) + S(r,F_1). \end{aligned} \tag{2.3}$$

By (2.1), (2.2), (2.3) and Lemma 1 we obtain

$$\begin{aligned} (n+m)T(r,f) &\leq T(r,F_1) - \overline{N}(r,0;F_1) + N_{k+1}(r,0;f^nP(f)) + S(r,f) \\ &\leq \overline{N}(r,0;G_1) + N_{k+1}(r,0;f^nP(f)) + S(r,f) \\ &\leq N_{k+1}(r,0;f^nP(f)) + N_{k+1}(r,0;g^nP(g)) + S(r,f) + S(r,g) \\ &\leq (k+m+1)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \end{aligned} \tag{2.4}$$

Similarly we obtain

$$(n+m)T(r,g) \leq (k+m+1)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \tag{2.5}$$

Combining (2.4) and (2.5) we get

$$(n-2k-m-2)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

which gives $n \leq 2k+m+2$. This completes the proof of the lemma. ∎

Lemma 7. *Suppose that f is a transcendental meromorphic function with finite number of poles, g is a transcendental entire function, and n, k are two positive integers. Suppose that F_1 and G_1 are given by Lemma 6. If $F_1G_1 = \alpha$, where $\alpha = 1$ or $\alpha = z^2$, then $n \leq k+2$.*

Proof. Suppose that $n > k+2$. From $F_1G_1 = \alpha$, we have

$$(f^nP(f))^{(k)}(g^nP(g))^{(k)} = \alpha.$$

Let z_0 be a zero of f with multiplicity p. Then z_0 is a zero of $(f^nP(f))^{(k)}$ with multiplicity $np-k$. Since g is an entire function and $n > k+2$, z_0 is a zero of α with multiplicity > 2, which is impossible. Thus f has no zeros. We put $f(z) = \frac{e^\beta}{h}$, where β is a nonconstant entire function and h is a polynomial. Now

$$(a_mf^{n+m})^{(k)} = t_m(\beta', \beta'', ..., \beta^{(k)}, h)e^{(n+m)\beta}, \tag{2.6}$$

$$\cdot$$

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$$(a_0f^n)^{(k)} = t_0(\beta', \beta'', ..., \beta^{(k)}, h)e^{n\beta}, \tag{2.7}$$

where $t_i(\beta', \beta'', ..., \beta^{(k)}, h)$ $(i = 0, 1, ..., m)$ are differential polynomials in β', β'', $. \ . \ . \ , \beta^{(k)}$ with coefficients which are rational functions in h or its derivatives. Obviously

$$t_i(\beta', \beta'', ..., \beta^{(k)}, h) \not\equiv 0$$

for $i = 0, 1, 2, ..., m$, and

$$(f^nP(f))^{(k)} \not\equiv 0.$$

From (2.6) and (2.7) we have

$$t_m(\beta', \beta'', ..., \beta^{(k)}, h)e^{m\beta(z)} + ... + t_0(\beta', \beta'', ..., \beta^{(k)}, h) \neq 0. \tag{2.8}$$

Since $\beta(z)$ is an entire function, we obtain $T(r, \beta^{(j)}) = S(r, f)$ for $j = 1, 2, ..., k$.
Hence $T(r, t_i) = S(r, f)$ for $i = 0, 1, 2, ..., m$.
So from (2.8), Lemmas 1 and 5 we obtain

$$\begin{aligned} mT(r,f) &= T(r, t_me^{m\beta} + ... + t_1e^\beta) + S(r,f) \\ &\leq \overline{N}(r, 0; t_me^{m\beta} + ... + t_1e^\beta) + \overline{N}(r, 0; t_me^{m\beta} + ... + t_1e^\beta + t_0) \\ &\quad + S(r,f) \\ &\leq \overline{N}(r, 0; t_me^{(m-1)\beta} + ... + t_1) + S(r,f) \\ &\leq (m-1)T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction. This completes the proof of the lemma. ∎

Following lemma can be proved in the line of Lemma 9 [18].

Lemma 8. *Let f and g be two nonconstant entire functions and let n, k be two positive integers. Suppose that $F_2 = (c_0 f^n)^{(k)}$ and $G_2 = (c_0 g^n)^{(k)}$, where $c_0(\neq 0)$ is a complex constant. If there exist two nonzero constants c_1 and c_2 such that $\overline{N}(r, c_1; F_2) = \overline{N}(r, 0; G_2)$ and $\overline{N}(r, c_2; G_2) = \overline{N}(r, 0; F_2)$, then $n \leq 2k + 2$.*

Note 1. *Though in Lemma 9 [18] authors claim that $n \leq 2k + 4$, it is obvious from the proof of Lemma 6 above that Lemma 8 holds for $n \leq 2k + 2$.*

Lemma 9. *Suppose that F_2 and G_2 are given as in Lemma 8 and let n, k be two positive integers such that $n > 2k$. If $F_2 = G_2$, then $f = tg$ for a constant t such that $t^n = 1$.*

Proof. We omit the proof because it can be carried out that of Lemma 10 [18]. ∎

The following lemma can be proved in the line of the proof of Proposition 1 [5] and Theorem 4 [18].

Lemma 10. *Suppose that F_2 and G_2 are given by Lemma 8 and let n, k be two positive integers such that $n > 2k + 4$. If $F_2 G_2 = z^2$, then $f(z) = c_1 / c_0^{\frac{1}{n}} e^{cz^2}$, $g(z) = c_2 / c_0^{\frac{1}{n}} e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(nc)^2 (c_1 c_2)^n = -1$.*

3 Proof of the Theorem

Proof of Theorem 1. We consider the following two cases.

Case 1. *Let $P(z) = a_m(z)z^m + a_{m-1}(z)z^{m-1} + ... + a_2 z^2 + a_1 z + a_0$, where $a_0(\neq 0)$, $a_1, \ldots, a_{m-1}$, $a_m(\neq 0)$ are complex constants. We consider $F(z) = f^n P(f)$ and $G(z) = g^n P(g)$. Then by Lemma 5 we have*

$$T\left(r, F^{(k)}\right) \leq \overline{N}\left(r, 0; F^{(k)}\right) + \overline{N}\left(r, z; F^{(k)}\right) + S(r, F).$$

Using Lemma 2 and the above inequality we obtain

$$\begin{aligned}(n+m)T(r,f) &\leq T\left(r, F^{(k)}\right) - \overline{N}\left(r, 0; F^{(k)}\right) + N_{k+1}(r, 0; F) + S(r, f) \\ &\leq \overline{N}\left(r, z; F^{(k)}\right) + N_{k+1}(r, 0; F) + S(r, f) \\ &\leq (k + m + 1)T(r, f) + \overline{N}\left(r, z; F^{(k)}\right) + S(r, f).\end{aligned}$$

Since $n \geq k + 2$, from this we can say that $F^{(k)} = (f^n P(f))^{(k)}$ has infinitely many fixed points.

Case 2. *Let $P(z) = c_0$, where c_0 $(\neq 0)$ is a complex constant. We omit the proof as it can be carried out in the line of the proof of Case 1.* ∎

Proof of Theorem 2. Let $P(z) = a_m(z)z^m + a_{m-1}(z)z^{m-1} + ... + a_2 z^2 + a_1 z + a_0$, where $a_0(\neq 0)$, $a_1, \ldots, a_{m-1}$, $a_m(\neq 0)$ are complex constants. We consider $F(z) = \frac{(f^n P(f))^{(k)}}{z}$ and $G(z) = \frac{(g^n P(g))^{(k)}}{z}$. Then $F(z)$ and $G(z)$ are transcendental meromorphic functions that share $(1, l)$. Let

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F - 1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1}\right). \tag{3.1}$$

We assume that $H \not\equiv 0$. Then from Lemma 1 and (2.1) we obtain

$$\begin{aligned} N_2(r,0;F) &\leq N_2\left(r,0;(f^nP(f))^{(k)}\right) + S(r,f) \\ &\leq T\left(r,(f^nP(f))^{(k)}\right) - (n+m)T(r,f) + N_{k+2}(r,0;f^nP(f)) + S(r,f) \\ &\leq T(r,F) - (n+m)T(r,f) + N_{k+2}(r,0;f^nP(f)) + S(r,f). \qquad (3.2)\end{aligned}$$

In a similar way we obtain

$$N_2(r,0;G) \leq T(r,G) - (n+m)T(r,g) + N_{k+2}(r,0;g^nP(g)) + S(r,g). \qquad (3.3)$$

Again by (2.2) we have

$$N_2(r,0;F) \leq N_{k+2}(r,0;f^nP(f)) + S(r,f). \qquad (3.4)$$

$$N_2(r,0;G) \leq N_{k+2}(r,0;g^nP(g)) + S(r,g). \qquad (3.5)$$

From (3.2) and (3.3) we get

$$\begin{aligned} (n+m)\{T(r,f)+T(r,g)\} &\leq T(r,F) + T(r,G) + N_{k+2}(r,0;f^nP(f)) \\ &\quad + N_{k+2}(r,0;g^nP(g)) - N_2(r,0;F) \\ &\quad - N_2(r,0;G) + S(r,f) + S(r,g). \qquad (3.6)\end{aligned}$$

Now we consider the following three cases.
Case I. Let $l \geq 2$. We suppose that (i) of Lemma 3 holds. Then using Lemma 1, (3.4) and (3.5) we obtain from (3.6)

$$\begin{aligned} (n+m)\{T(r,f)+T(r,g)\} &\leq N_2(r,0;F) + N_2(r,0;G) + 2N_2(r,\infty;F) \\ &\quad + 2N_2(r,\infty;G) + N_{k+2}(r,0;f^nP(f)) \\ &\quad + N_{k+2}(r,0;g^nP(g)) + S(r,f) + S(r,g) \\ &\leq 2N_{k+2}(r,0;f^nP(f)) + 2N_{k+2}(r,0;g^nP(g)) \\ &\quad + S(r,f) + S(r,g) \\ &\leq 2(k+m+2)\{T(r,f)+T(r,g)\} \\ &\quad + S(r,f) + S(r,g).\end{aligned}$$

From this we get

$$(n-m-2k-4)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

which leads to a contradiction as $n > 2k+m+4$.
Hence by Lemma 3 we have either $FG = 1$ or $F = G$. If $FG = 1$, then

$$(f^nP(f))^{(k)}(g^nP(g))^{(k)} = z^2,$$

a contradiction by Lemma 7. Hence $F = G$. That is

$$\begin{aligned} [f^n(a_mf^m + a_{m-1}f^{m-1} + ... + a_1f + a_0)]^{(k)} &= [g^n(a_mg^m \\ &\quad + a_{m-1}g^{m-1} + ... + a_1g + a_0)]^{(k)}.\end{aligned}$$

Integrating we get

$$\begin{aligned}[][f^n(a_mf^m+a_{m-1}f^{m-1}+...+a_1f+a_0)]^{(k-1)} &= [g^n(a_mg^m\\ +a_{m-1}g^{m-1}+...+a_1g+a_0)]^{(k-1)}+c_{k-1},\end{aligned}$$

where c_{k-1} is a constant. If $c_{k-1}\neq 0$, from Lemma 6 we obtain $n\leq 2k+m$, a contradiction. Hence $c_{k-1}=0$. Repeating k-times, we obtain

$$\begin{aligned}&f^n(a_mf^m+a_{m-1}f^{m-1}+...+a_1f+a_0)\\ &=g^n(a_mg^m+a_{m-1}g^{m-1}+...+a_1g+a_0).\end{aligned}\tag{3.7}$$

Let $h=\frac{f}{g}$. If h is a constant, by putting $f=gh$ in (3.7) we get

$$a_mg^{n+m}(h^{n+m}-1)+a_{m-1}g^{n+m-1}(h^{n+m-1}-1)+...+a_0g^n(h^n-1)=0,$$

which implies $h^d=1$, where $d=(n+m,...,n+m-i,...,n+1,n)$, $a_{m-i}\neq 0$ for some $i=0,1,...,m$. Thus $f(z)=tg(z)$ for a constant t such that $t^d=1$, $d=(n+m,...,n+m-i,...,n+1,n)$, $a_{m-i}\neq 0$ for some $i=0,1,...,m$.

If h is not a constant, then from (3.7) we can say that f and g satisfy the algebraic equation $R(f,g)=0$, where

$$R(x,y)=x^n(a_mx^m+a_{m-1}x^{m-1}+...+a_0)-y^n(a_my^m+a_{m-1}y^{m-1}+...+a_0).$$

Case II. Let $l=1$. Using Lemma 1, (i) of Lemma 4, (3.4) and (3.5) we obtain from (3.6)

$$\begin{aligned}(n+m)\{T(r,f)+T(r,g)\} &\leq N_2(r,0;F)+N_2(r,0;G)+2N_2(r,\infty;F)\\ &\quad +2N_2(r,\infty;G)+\frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,0;G)\\ &\quad +\frac{1}{2}\overline{N}(r,\infty;F)+\frac{1}{2}\overline{N}(r,\infty;G)\\ &\quad +N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))\\ &\quad +S(r,f)+S(r,g)\\ &\leq 2N_{k+2}(r,0;f^nP(f))+2N_{k+2}(r,0;g^nP(g))\\ &\quad +\frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,0;G)+S(r,f)+S(r,g)\\ &\leq 2N_{k+2}(r,0;f^nP(f))+2N_{k+2}(r,0;g^nP(g))\\ &\quad +\frac{1}{2}N_{k+1}(r,0;f^nP(f))+\frac{1}{2}N_{k+1}(r,0;g^nP(g))\\ &\quad +S(r,f)+S(r,g)\\ &\leq \left(\frac{5k+5m+9}{2}\right)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g).\end{aligned}$$

This gives

$$\left(n-\frac{5k+3m+9}{2}\right)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

which contradicts with our assumption that $n > \frac{5k+3m+9}{2}$.
Case III. Let $l = 0$. Using Lemma 1, (ii) of Lemma 4, (3.4) and (3.5) we obtain from (3.6)

$$\begin{aligned}(n+m)\{T(r,f)+T(r,g)\} \leq\ & N_2(r,0;F)+N_2(r,0;G)+2N_2(r,\infty;F)\\ & +2N_2(r,\infty;G)+3\overline{N}(r,0;F)+3\overline{N}(r,0;G)\\ & +3\overline{N}(r,\infty;F)+3\overline{N}(r,\infty;G)\\ & +N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))\\ & +S(r,f)+S(r,g)\\ \leq\ & 2N_{k+2}(r,0;f^nP(f))+2N_{k+2}(r,0;g^nP(g))\\ & +3N_{k+1}(r,0;f^nP(f))+3N_{k+1}(r,0;g^nP(g))\\ & +S(r,f)+S(r,g)\\ \leq\ & (5k+5m+7)\{T(r,f)+T(r,g)\}\\ & +S(r,f)+S(r,g).\end{aligned}$$

This gives

$$(n-5k-4m-7)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

contradicting with the fact that $n > 5k+4m+7$.
We now assume that $H \equiv 0$. That is

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)=0.$$

Integrating both sides of the above equality we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{3.8}$$

where $A(\neq 0)$ and B are constants. Now we consider the following three subcases.
Subcase (i) Let $B \neq 0$ and $A = B$. Then from (3.8) we get

$$\frac{1}{F-1}=\frac{BG}{G-1}. \tag{3.9}$$

If $B = -1$, then from (3.9) we obtain

$$FG = 1,$$

i.e.,

$$(f^nP(f))^{(k)}(g^nP(g))^{(k)} = z^2,$$

a contradiction by Lemma 7. If $B \neq -1$, from (3.9), we have $G = \frac{-1}{BF-(B+1)}$ and so $\overline{N}(r, \frac{B+1}{B}; F) = \overline{N}(r, G)$. Now from the second fundamental theorem, we get

$$\begin{aligned}T(r,F) &\leq \overline{N}(r,0;F)+\overline{N}\left(r,\frac{B+1}{B};F\right)+S(r,F)\\ &\leq \overline{N}(r,0;F)+S(r,F).\end{aligned}$$

Using (2.1) we obtain from above inequality

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,0;F) + S(r,f) \\ &\leq T(r,F) - (n+m)T(r,f) + N_{k+1}(r,0;f^nP(f)) + S(r,f). \end{aligned}$$

Hence

$$(n+m)T(r,f) \leq (k+m+1)T(r,f) + S(r,f),$$

a contradiction as $n > 2k+m+4$.

Subcase (ii) Let $B \neq 0$ and $A \neq B$. Then from (3.8) we get $G = \frac{(B-A)F+(A-B-1)}{BF-(B+1)}$ and so $\overline{N}(r, \frac{B+1}{B}; F) = \overline{N}(r, G)$. Proceeding as in **Subcase (i)** we obtain a contradiction.

Subcase (iii) Let $B = 0$ and $A \neq 0$. Then from (3.8) $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, we have $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, 0; G)$ and $\overline{N}(r, 1-A; G) = \overline{N}(r, 0; F)$. So by Lemma 6 we have $n \leq 2k+m+2$, a contradiction. Thus $A = 1$ and hence $F = G$. Hence by **Case I** we obtain either $f(z) = tg(z)$ for a constant t such that $t^d = 1$, $d = (n+m, ..., n+m-i, ..., n+1, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$ or f and g satisfy the algebraic equation $R(f,g) = 0$, where

$$R(x,y) = x^n(a_m x^m + a_{m-1}x^{m-1} + ... + a_0) - y^n(a_m y^m + a_{m-1}y^{m-1} + ... + a_0).$$

Now We consider the case when $P(z) = c_0$, where c_0 ($\neq 0$) is a complex constant. Let $F(z) = \frac{(c_0 f^n)^{(k)}}{z}$ and $G(z) = \frac{(c_0 g^n)^{(k)}}{z}$. Then $F(z)$ and $G(z)$ are transcendental meromorphic functions that share $(1,l)$. Using Lemma 8 and proceeding in the like manner as above we obtain either $FG = 1$ or $F = G$.
If $FG = 1$, then

$$(c_0 f^n)^{(k)}(c_0 g^n)^{(k)} = z^2.$$

So by Lemma 10 we obtain $f(z) = c_1/c_0^{\frac{1}{n}} e^{cz^2}$,$g(z) = c_2/c_0^{\frac{1}{n}} e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(nc)^2(c_1c_2)^n = -1$.
If $F = G$, then by Lemma 9 we have $f = tg$ for a constant t such that $t^n = 1$. This completes the proof of the theorem. ∎

Proof of Corollary 1. Proceeding as in Theorem 2 we obtain either $FG = 1$ or $F = G$, where $F = \frac{(f^n(f-1))^{(k)}}{z}$ and $G = \frac{(g^n(g-1))^{(k)}}{z}$.
Suppose that $FG = 1$. Then

$$(f^n(f-1))^{(k)}(g^n(g-1))^{(k)} = z^2,$$

a contradiction by Lemma 7.
Hence $F = G$. That is

$$(f^n(f-1))^{(k)} = (g^n(g-1))^{(k)}.$$

Arguing similarly as the proof of **Case I** in Theorem 2 we obtain

$$f^n(f-1) = g^n(g-1). \tag{3.10}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ in (3.10) we deduce

$$g^{n+1}(h^{n+1} - 1) - g^n(h^n - 1) = 0,$$

which implies $h = 1$. Thus $f(z) \equiv g(z)$.

If h is not a constant, then from (3.10) we can say that f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(x, y) = x^n(x - 1) - y^n(y - 1).$$

This completes the proof of Corollary 1. ∎

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