On the algebraic K-theory of $R[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$*

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Abstract

It is well known after R. Swan that $\tilde{K}_0(R[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1))$ is isomorphic to the integers $\mathbb{Z}$, whenever R is a field of characteristic not two which contains the squared root of -1, see [6, Corollary 10.8] and [7, §7].

First, we give explicit idempotent matrices γ^p of order two, corresponding to the integer p, in the isomorphism above, if R is a field of characteristic zero. Then, we use the algebraic de Rham cohomology of Kähler differentials to define Brouwer degree for polynomial homomorphisms of $R[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$ to itself, and relate the problem of finding hermitian representatives for $R = K(i)$, K a field not containing i, to some unsolved problems of representing Brouwer degrees by polynomial maps [8].

Introduction

It is well known that $\tilde{K}_0$ of the coordinate ring $A = R[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$ of the algebraic 2-sphere $\mathbb{S}^2(R)$ over a field R is isomorphic to the integers $\mathbb{Z}$, whenever R is of characteristic not two, and contains i, the squared root of -1, see [6, Corollary 10.8] and [7, §7]. Regarding the elements of $\tilde{K}_0(A)$ as stable classes of idempotent matrices over the ring A, the generator of $\tilde{K}_0(A)$ is represented by the matrix

$$\gamma^1 = \frac{1}{2}\begin{pmatrix} 1+Z & X+iY \\ X-iY & 1-Z \end{pmatrix},$$

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and so the element $p[\gamma^1]$ is represented by the idempotent matrix of order $2p$,

$$\begin{pmatrix} \gamma^1 & 0 & . & . & 0 \\ 0 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & 0 \\ 0 & . & . & 0 & \gamma^1 \end{pmatrix}.$$

It seems interesting to give representatives of $p[\gamma^1]$ by matrices of order two because of the obvious bijection between rank one idempotent matrices of order two over A and polynomial maps from the two sphere $\mathbb{S}^2(R)$ to itself, given by

$$\frac{1}{2}\begin{pmatrix} 1+R & P+iQ \\ P-iQ & 1-R \end{pmatrix} \leftrightarrows (P,Q,R)$$

for $P,Q,R \in R[X,Y,Z]$.

Observe that even for $R = K(i)$ being the algebraic extension of a field K (not containing i) by i the coefficients of polynomials P and Q belong to R and not necessarily to K, and so the matrices above need not to be hermitian.

Let $A_p(X), B_p(X) \in \mathbb{Z}[\frac{1}{2}, X]$ be the unique polynomials with $\deg A_p(X) = p-1$, $\deg B_p(X) = p-1$ and $A_p(X)(1+X)^p + B_p(X)(1-X)^p = 2$, $p \geq 1$, [1, Proposition 1.7], and consider

$$\gamma^p = \frac{1}{2}\begin{pmatrix} A_p(Z)(1+Z)^p & B_p(Z)(X+iY)^p \\ A_p(Z)(X-iY)^p & B_p(Z)(1-Z)^p \end{pmatrix},$$

$$\gamma^{-p} = \frac{1}{2}\begin{pmatrix} A_p(Z)(1+Z)^p & B_p(Z)(X-iY)^p \\ A_p(Z)(X+iY)^p & B_p(Z)(1-Z)^p \end{pmatrix}$$

the idempotent matrices of order two [1, p. 66].

The matrices above make sense over $R[X,Y,Z]/(X^2+Y^2+Z^2-1)$ and we show in Section 1 that they represent $p[\gamma^1]$ and $-p[\gamma^1]$, respectively, provided the field R is of characteristic zero.

We also have

$$\gamma^p = \frac{1}{2}\begin{pmatrix} 1+ZR_p(Z^2) & P(X,Y,Z)+iQ(X,Y,Z) \\ P(X,Y,Z)-iQ(X,Y,Z) & 1-ZR_p(Z^2) \end{pmatrix},$$

$$\gamma^{-p} = \frac{1}{2}\begin{pmatrix} 1+ZR_p(Z^2) & P(X,-Y,Z)+iQ(X,-Y,Z) \\ P(X,-Y,Z)-iQ(X,-Y,Z) & 1-ZR_p(Z^2) \end{pmatrix},$$

where

$$P = \frac{1}{2}(A_p(Z)(X-iY)^p + B_p(Z)(X+iY)^p),$$

$$Q = \frac{i}{2}(A_p(Z)(X-iY)^p - B_p(Z)(X+iY)^p)$$

and $R_p(Z) \in \mathbb{Z}[\frac{1}{2}, Z]$ is the polynomial of degree $p-1$ given by

$$ZR_p(Z^2) = A_p(Z)(1+Z)^p - 1 = 1 - B_p(Z)(1-Z)^p,$$

see [1, p. 60].

Note that $R_p(X^2)$ is the polynomial $r(X)$ in [9, p. 494]. These $R_p(X)$ were later used in [8] with the notations

$$R_p(X) = \varphi_{p-1}(1-X) = \alpha_{p-1}(1-X).$$

In Section 2, we define Brouwer degree for polynomial self-maps $\mathbb{S}^2(R) \to \mathbb{S}^2(R)$ of the 2-sphere $\mathbb{S}^2(R)$ over R and show that if R contains the field of algebraic real numbers $\mathbb{R}_{alg}$, then the polynomials P and Q of the matrices γ^p, for p an odd integer, can be taken in $\mathbb{R}_{alg}[X,Y,Z]$, so the matrices are hermitian for $R = \bar{\mathbb{Q}}$, the field of algebraic numbers.

In particular, we show that the polynomial maps

$$(P(X,Y,Z), Q(X,Y,Z), ZR_p(Z^2)) : \mathbb{S}^2(R) \to \mathbb{S}^2(R),$$

(resp.

$$(P(X,-Y,Z), Q(X,-Y,Z), ZR_p(Z^2)) : \mathbb{S}^2(R) \to \mathbb{S}^2(R))$$

have Brouwer degree p (resp. $-p$). Further, for p to be odd and $R = \bar{\mathbb{Q}}$, they restricts to polynomial maps

$$\mathbb{S}^2(\mathbb{R}_{alg}) \to \mathbb{S}^2(\mathbb{R}_{alg}).$$

For the reals $\mathbb{R}$, polynomial maps $\mathbb{S}^2(\mathbb{R}) \to \mathbb{S}^2(\mathbb{R})$ of any given odd Brouwer degree were first given by F.J. Turiel [8]. The case of even Brouwer degree remains open.

If R is one of the fields $\mathbb{R}$, $\mathbb{C}$ (the field of complex numbers) or $\mathbb{H}$ (the skew-algebra of quaternions), the Grassmannian of r-planes in R^n, $G_{n,r}(R)$, is canonically diffeomorphic to the manifold of idempotent matrices φ over R, which are hermitians, i.e., $\bar{\varphi}^t = \varphi$, and whose trace equals r. This manifold can be replaced by that of idempotent matrices $\mathrm{Idem}_{n,r}(R)$, of order n with the trace r, because the inclusion $G_{n,r}(R) \subset \mathrm{Idem}_{n,r}(R)$ is a strong deformation retract with retraction map given by $\varphi \to \varphi(\varphi + \bar{\varphi}^t - I_n)^{-1}$, [1, Lemma 1.1]. If we apply this retraction map to our idempotent matrix γ^p, (resp. γ^{-p}) for $p \geq 1$, we obtain the rank one hermitian idempotent matrix

$$\tilde{\gamma}^p = \frac{1}{(1+Z)^p + (1-Z)^p} \begin{pmatrix} (1+Z)^p & (X+iY)^p \\ (X-iY)^p & (1-Z)^p \end{pmatrix},$$

$$\text{(resp. } \tilde{\gamma}^{-p} = \frac{1}{(1+Z)^p + (1-Z)^p} \begin{pmatrix} (1+Z)^p & (X-iY)^p \\ (X+iY)^p & (1-Z)^p \end{pmatrix}).$$

Unfortunately, these idempotent matrices do not make sense in $R[X,Y,Z]/(X^2+Y^2+Z^2-1)$ because of the denominator $(1+Z)^p + (1-Z)^p$.

1 Chern classes for idempotent matrices and $\tilde{K}_0(A)$

It is well known, after R. Swan [5], the equivalence between isomorphisms classes of vector bundles and that of finitely generated projective modules over the ring of continuous functions on the base or, equivalently, the idempotent matrices over the ring of those functions.

Given a smooth complex vector bundle ξ, represented by an idempotent matrix φ of order n, over the ring of smooth functions on the base, the Chern classes of ξ in the standard de Rham cohomology of differential forms, are given as follows:

$$\det(\varphi(d\varphi)^2 + I_n) = 1 + c_1(\varphi(d\varphi)^2) + \cdots + c_n(\varphi(d\varphi)^2),$$

where c_p is the p-th characteristic coefficient for $p = 1, \ldots, n$. Then, $c_p(\varphi(d\varphi)^2)$ is a closed form and $\frac{1}{(-2\pi i)^p} c_p((\varphi(d\varphi)^2)$ represents the p-th Chern class of ξ.

In the same way, for any idempotent matrix φ over a commutative R-algebra A of characteristic zero, we define the p-th Chern class of φ as the class in $H^{2p}_{dR}(A)$ of $c_p(\varphi(d\varphi)^2)$, where c_p denotes the p-th characteristic coefficient. It is well known that these classes depend only on the stable class of φ in $\tilde{K}_0(A)$, and so there are well defined maps $c_p : \tilde{K}_0(A) \to H^{2p}_{dR}(A)$ (see for instance [2]). These classes were first defined for projective modules by H. Ozeki [4].

Here $H^*_{dR}(A)$ denotes the algebraic de Rham cohomology of A which we recall now:

$\Omega^1_R(A) = I/I^2$ is the A-module of Kähler differentials, where I is the kernel of multiplication $A \otimes_R A \to A$, for $a \in A$, da is the class of $a \otimes 1 - 1 \otimes a$ in $\Omega^1_R(A)$, $\Omega^p_R(A)$ is the p-th exterior power of $\Omega^1_R(A)$ and $d : A \to \Omega^1_R(A)$ extends to a differential of degree 1. The cohomology of this complex is the algebraic de Rham cohomology of A denoted by $H^*_{dR}(A)$.

In the case of $A = R[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$, $\Omega^1_R(A)$ is the A-projective module generated by dX, dY, dZ with the relation $XdX + YdY + ZdZ = 0$, $\Omega^2_R(A)$ is the A-free module generated by $\omega = XdYdZ + YdZdX + ZdXdY$, $\Omega^p_R(A) = 0$ for $p \geq 3$ and $H^2_{dR}(A) = R[\omega] \cong R$ [3].

For the sake of completeness, since we are going to use only the first Chern class, we prove in the following lemma (cf. [2, Lemma]) that this class gives a homomorphism $c_1 : \tilde{K}_0(A) \to H^2_{dR}(A)$.

Lemma 1.1. *Let A a commutative ring and suppose that φ and ψ are equivalent idempotent matrices over A. Then* $\operatorname{tr}(\psi(d\psi)^2) - \operatorname{tr}(\varphi(d\varphi)^2)$ *belongs to the image of* $d : \Omega^1_R(A) \to \Omega^2_R(A)$.

Proof. Suppose we have matrices a and b over A such that $ab = \varphi$, $ba = \psi$. Then,

$$\begin{aligned}
\operatorname{tr}(\psi(d\psi)^2 - \varphi(d\varphi)^2) &= \operatorname{tr}(ba(db \cdot a + bda)^2 - ab(da \cdot b + adb)^2) = \\
&\operatorname{tr}(badb.adb \cdot a + babda \cdot bda + badb \cdot abda + babdadb \cdot a \\
&- abda \cdot bda \cdot b - abadb \cdot adb - abda \cdot badb - abadbda \cdot b) = \\
&\operatorname{tr}(2\psi db \cdot \varphi da + \varphi dadb - dbda \cdot \psi).
\end{aligned}$$

But,

$$\begin{aligned}\operatorname{tr}(\psi db\cdot\varphi da) &= -\operatorname{tr}(\varphi da\psi db) = -\operatorname{tr}(\varphi^2 da\psi db) = -\operatorname{tr}(a\psi bda\cdot\psi db) = \\ &-\operatorname{tr}(\psi bda\cdot\psi db\cdot a) = -\operatorname{tr}(\psi bda\cdot\psi d\psi) + \operatorname{tr}(\psi bda\cdot\psi bda) = \\ &-\operatorname{tr}(\psi bda\cdot\psi d\psi) = -\operatorname{tr}(bda\psi d\psi\cdot\psi) = 0,\end{aligned}$$

and so

$$\operatorname{tr}(\psi(d\psi)^2-\varphi(d\varphi)^2) = \operatorname{tr}(\varphi dadb - dbda\cdot\psi).$$

On the other hand,

$$\begin{aligned}d(\operatorname{tr}(abadb)) &= \operatorname{tr}(dabadb + adb\cdot adb + abdadb) = \\ &\operatorname{tr}(-dbda\cdot\psi + \varphi dadb + adb\cdot adb) = \operatorname{tr}(\varphi dadb - dbda\cdot\psi)\end{aligned}$$

and the proof is complete. ■

Write $\sim$ for the congruence on the cocycle module $\operatorname{Ker} d$ (modulo the coboundary module $\operatorname{Im} d$) in $H^2_{dR}(A)$. To compute the first Chern class for the idempotent matrices γ^p, we need:

Lemma 1.2. *For any polynomial $F(Z)$ it holds $F'(Z)\omega \sim (Z^2F(Z))'\omega$. In particular, $(P(Z^2)F(Z))' \sim P(1)F'(Z)\omega$, for polynomials $F(Z), P(Z)$.*

Proof. Observe that

$$\begin{aligned}F'(Z)\omega &= F'(Z)(-XdY+YdX) + F'(Z)ZdXdY = \\ &d(F(Z)(-XdY+YdX)) + (2F(Z)+ZF'(Z))dXdY \sim (Z^2F(Z))'\omega.\end{aligned}$$

■

Proposition 1.3. *Let $A = R[X,Y,Z]/(X^2+Y^2+Z^2-1)$ and R be an integral domain such that the field of rationals $\mathbb{Q} \subseteq R$. Then, $c_1(\gamma^p) = -\frac{i}{2}p[\omega] \in H^2_{dR}(A)$.*

Proof. A straightforward computation gives

$$\operatorname{tr}(\gamma^p(d\gamma^p)^2) = -\frac{i}{2}p(ZR_p(Z^2))'\omega \sim -\frac{i}{2}p(ZR_p(1))'\omega = -\frac{i}{2}p\omega,$$

because of Lemma 1.2, and so $\operatorname{tr}(\gamma_p(d\gamma_p)^2))$ represents the class $-\frac{i}{2}p[\omega]$. ■

Now, we are in a position to state the main result of this section.

Theorem 1.4. *For any integers p and q, $[\gamma^p] = p[\gamma^1]$, $[\gamma^p]+[\gamma^q] = [\gamma^{p+q}]$ and the group homomorphism $2ic_1 : \tilde{K}_0(A) \to H^2_{dR}(A) = R[\omega] \cong R$ factorizes throughout an isomorphism $\tilde{K}_0(A) \to \mathbb{Z}$.*

Proof. For any γ^p, by R. Swan [6, Corollary 10.8] and [7, §7], there exists an integer q such that $[\gamma^p] = q[\gamma^1]$. But, $2ic_1([\gamma^p]) = p[\omega]$ and clearly $2ic_1(q[\gamma^1]) = q[\omega]$. Thus, $p = q$, and $[\gamma^p] = p[\gamma^1]$.

Therefore $[\gamma^p]+[\gamma^q] = (p+q)[\gamma^1] = [\gamma^{p+q}]$ for any integers p, q, and the proof is complete because $\tilde{K}_0(A)$ is free generated by $[\gamma^1]$, and $2ic_1([\gamma^1]) = [\omega]$. ■

Remark 1.5. Theorem 1.4 says in particular that there exist matrices M and N over A and a natural number r such that

$$MN = \begin{pmatrix} \gamma^p & 0 & 0 \\ 0 & \gamma^q & 0 \\ 0 & 0 & I_r \end{pmatrix}$$

and

$$NM = \begin{pmatrix} \gamma^{p+q} & 0 \\ 0 & I_{2+r} \end{pmatrix},$$

where I_s denotes the identity matrix of order s.

We do not know if actually $r = 0$, i.e.,

$$MN = \begin{pmatrix} \gamma^p & 0 \\ 0 & \gamma^q \end{pmatrix}$$

and

$$NM = \begin{pmatrix} \gamma^{p+q} & 0 \\ 0 & 1 \end{pmatrix}.$$

2 Polynomial maps from spheres to spheres, Brouwer degree, and idempotent matrices

Let R be a field of characteristic zero and $\mathbb{S}^2(R) = \{(x,y,z) \in R^3 | x^2 + y^2 + z^2 = 1\}$ the 2-sphere over R. We say that a map $f = (f_0, f_1, f_2) : \mathbb{S}^2(R) \to \mathbb{S}^2(R)$ is *polynomial* if there are polynomials $F_0, F_1, F_2 \in R[X,Y,Z]$ such that $f_j(x,y,z) = F_j(x,y,z)$ for all $(x,y,z) \in \mathbb{S}^2(R)$ and $j = 0,1,2$.

Let $A = R[X,Y,Z]/(X^2 + Y^2 + Z^2 - 1)$. It is known after M. Kong [3], that $H^2_{dR}(A) = R[\omega] \cong R$. This allows us to imitate topology to define *Brouwer degree* $B(f)$ of a polynomial map $f = (f_0, f_1, f_2) : \mathbb{S}^2(R) \to \mathbb{S}^2(R)$ by the formula

$$[f^*(\omega)] = B(f)[\omega].$$

It is clear that $f^*(\omega) = \operatorname{tr}(f^*(\gamma^1)d(f^*(\gamma^1))^2)$, and so $B(f)[\omega] = c_1([f^*(\gamma^1)]) = -\frac{i}{2}p[\omega]$, where p is an integer because of Theorem 1.4. Therefore, $B(f) = p \in \mathbb{Z}$.

For any integer $n \geq 1$ consider now the polynomials $A_n(X), B_n(X)$ in $\mathbb{Z}[\frac{1}{2}, X]$, of degree $p-1$, uniquely determined by $A_n(X)(1+X)^n + B_n(X)(1-X)^n = 2$, [1, Proposition 1.7].

Since $P_n(X) = 1 - B_n(X)(1-X)^n = -(1 - A_n(X))(1+X)^n = -P_n(-X)$, there exists a unique polynomial $R_n(X)$ of degree $\leq n-1$ with $P_n(X) = 1 - B_n(X)(1-X)^n = XR_n(X^2)$. Moreover, from $1 - XR_n(X^2) = B_n(X)(1-X)^n$, we deduce as in [1] that

$$(*) \quad R_n(X) = \sum_{k=0}^{n-1} \frac{1}{2^{2k}} \binom{2k}{k} (1-X)^k$$

which leads to the recursive formula

$$(**) \quad R_{n+1}(X) - R_n(X) = \frac{1}{2^{2n}} \binom{2n}{n} (1-X)^n.$$

From this also follows that

$$(***)\ \ P_{n+1}(X)-P_n(X)=X(R_{n+1}(X^2)-R_n(X^2))=\frac{1}{2^{2n}}\binom{2n}{n}X(1-X^2)^n.$$

Observe that polynomials $A_n(X), B_n(X), P_n(X)$ and $R_n(X)$ are in $\mathbb{Z}[\frac{1}{2},X]$. Now, we list some further properties of polynomials $A_n(X), B_n(X), P_n(X)$ and $R_n(X)$.

(1) For all $n \geq 1$, the polynomial $P_n(X)$ coincides with the polynomial $p_n(X)$ given by

$$p_n(x)=\frac{2n-1}{2^{2n-2}}\binom{2n-2}{n-1}\int_0^x(1-t^2)^{n-1}dt,\ \text{for}\ x\geq 0.$$

Notice that the formula $(***)$ also holds for the polynomials $p_n(X)$ because $p_{n+1}(X)$ and $p_n(X)+\frac{1}{2^{2n}}\binom{2n}{n}X(1-X^2)^n$ vanish at 0 and have the same derivative $\frac{2n+1}{2^{2n}}\binom{2n}{n}(1-X^2)^n$. Finally, we get $p_n(X)=P_n(X)$ by using the recursive formula above and the obvious fact that $p_1(X)=P_1(X)=1$.

Remark 2.1. (1) By [1, p. 60], $P_n(1)=1$ and so $P_n(X)$ coincides with the polynomial $p(X)$ defined in [9, p. 494]. In particular, α given there equals to $\frac{2n-1}{2^{2n-2}}\binom{2n-2}{n-1}$.
(2) $R_n(X^2)=r(X)$ for the polynomial $r(X)$ in [9, p. 494].

(2) The polynomials $R_n(X)$ were later used in [8] with the notations

$$\varphi_{n-1}(1-X)=\alpha_{n-1}(1-X)=R_n(X).$$

In fact, by [1, p. 60], $R_n^{(k)}(1)=(-1)^k\frac{(2k)!}{k!}\frac{1}{2^{2k}}$, coincides with $(\frac{1}{\sqrt{x}})^{(k)}(1)$ for $k=0,\ldots,n-1$. Therefore, $R_n(X)$ is the $(n-1)-$th partial sum of the Taylor expansion of the function $\frac{1}{\sqrt{x}}$ at 1.

(3) $R_n(x)>0$ for any real number x, if n is odd.

In fact, by $(*)$, $R_n(x)>0$ if $x\leq 1$ and

$$xR_n(x^2)=P_n(x)=\frac{2n-1}{2^{2n-2}}\binom{2n-2}{n-1}\int_0^x(1-t^2)^{n-1}dt>0$$

for $x>0$ gives the result.

(4) $R_n(X)$ has a unique real zero $\beta_n>1$ of order one, if n is even.
In fact, we already know that $R_n(x)>0$ if $x\leq 1$. If $x>1$ then

$$P_n'(x)=\frac{2n-1}{2^{2n-2}}\binom{2n-2}{n-1}(1-x^2)^{n-1}<0.$$

This shows that $P_n(x)$ is decreasing for $x>1$. But $P_n(1)=1$ and $P_n(+\infty)=-\infty$. Therefore, $P_n(x)$ has a unique real zero >1 and the relation $XR_n(X^2)=P_n(X)$ gives that $R_n(X)$ has a unique real zero $\beta_n>1$ for even n.

To show that β_n has order one, observe that the relation

$$R_n(X) + 2XR'_n(X) = \frac{2n-1}{2^{2n-2}}\binom{2n-2}{n-1}(1-X)^{n-1}$$

determined by **(1)** gives

$$2\beta_n R'_n(\beta_n) = \frac{2n-1}{2^{2n-2}}\binom{2n-2}{n-1}(1-\beta_n)^{n-1}.$$

Consequently, $R'_n(\beta_n) < 0$ and β_n is the root of $R_n(X)$ with order one.

(5) The complex zeros of $R_n(X)$ are two by two conjugates and all have order one.

In fact, the relation

$$R_n(X) + 2XR'_n(X) = \frac{2n-1}{2^{2n-2}}\binom{2n-2}{n-1}(1-X)^{n-1}$$

yields $R'_n(\alpha) \neq 0$ provided $R_n(\alpha) = 0$. This shows that all zeros of $R_n(X)$ have multiplicity one. To finish the proof, decompose the real polynomials $R_n(X)$ as product of primes factors (which are of degrees one or two).

(6) $B_n(x) > 0$ and $A_n(x) > 0$ for any real number x, if n is odd.

In fact, suppose $B_n(\alpha) = 0$. Then, $\alpha \neq 1$ because $B_n(1) = \frac{1}{2^n}\binom{2n}{n} > 0$ and $0 = B_n(\alpha)(1-\alpha)^n = 1 - P_n(\alpha)$. Hence, $P_n(\alpha) = 1$ and this gives $\alpha = 1$, because $P_n(x)$ is increasing. Thus, we arrive to a contradiction.

Finally, $A_n(x) = B_n(-x) > 0$ for any real number x and odd n.

(7) $A_n(X)$ has a unique real zero α_n if n is even, $\alpha_n > 1$ and its multiplicity is one. Similarly, $-\alpha_n$ is the unique real zero of $B_n(X)$ for n even and its multiplicity is one.

In fact, $B_n(X)$, being a real polynomial of odd degree $n-1$, has at least one real zero. On the other hand, $P_n(x)$ is decreasing for $x < -1$ or $x > 1$, and increasing in the interval $(-1,1)$. Therefore, the formula $B_n(X)(1-X)^n = 1 - P_n(X)$ gives $B_n(x)(1-x)^n > 0$ and so $B_n(x) > 0$ for $x \in [-1,1]$, because $P_n(x) < P_n(1) = 1$ for $x \in [-1,1)$ and $B_n(1) > 0$. But also $B_n(x) > 0$ for $x > 1$, because $P_n(1) = 1 < P_n(x)$.

Now, if $B_n(\alpha) = 0$ for some $\alpha < -1$ then $P_n(\alpha) = 1$. But $P_n(x)$ is decreasing for $x < -1$ and so there is a unique $\alpha < -1$ such that $P_n(\alpha) = 1$. This shows that $B_n(X)$ has a unique real zero $-\alpha_n < -1$.

To show that $-\alpha_n$ has order one, observe that $B_n(X)(1-X)^n = 1 - P_n(X)$ gives

$$B'_n(-\alpha_n)(1+\alpha_n)^n = -\frac{2n-1}{2^{2n-2}}\binom{2n-2}{n-1}(1-\alpha_n^2)^{n-1}$$

and so $B_n'(-\alpha_n) \neq 0$, because $\alpha_n \neq \pm 1$.

(8) The complex zeros of $A_n(X)$, resp. $B_n(X)$, are two by two conjugates and have multiplicity one.

In fact, by the formula $A_n(X)(1+X)^n = 1 + P_n(X)$, we get

$$nA_n(X)(1+X)^{n-1} + A_n'(X)(1+X)^n = P_n'(X) = \frac{2n-1}{2^{2n-2}}\binom{2n-2}{n-1}(1-X^2)^{n-1}.$$

Thus, for $A_n(\alpha) = 0$, we have certainly $\alpha \neq \pm 1$ and

$$A_n'(\alpha)(1+\alpha)^n = \frac{2n-1}{2^{2n-2}}\binom{2n-2}{n-1}(1-\alpha^2)^{n-1}$$

shows that $A_n'(\alpha) \neq 0$.

To finish the proof, decompose $A_n(X)$ as product of real prime factors.

(9) Write $n = 2m+1$ or $n = 2m+2$, according to n being odd or even, and let $a_{n1} \pm ib_{n1}, \ldots, a_{nm} \pm ib_{nm}$ be the complex zeros of $A_n(X)$. Then, we have

$$A_n(X) = \begin{cases} \frac{(-1)^{n-1}}{2^{2n-2}}\binom{2n-2}{n-1}\prod_{k=1}^m((X-a_{nk})^2+b_{nk}^2) \text{ if } n = 2m+1, \\ \frac{(-1)^{n-1}}{2^{2n-2}}\binom{2n-2}{n-1}(X-\alpha_n)\prod_{k=1}^m((X-a_{nk})^2+b_{nk}^2) \text{ if } n = 2m+2. \end{cases}$$

Hence,

$$B_n(X) = \begin{cases} \frac{(-1)^{n-1}}{2^{2n-2}}\binom{2n-2}{n-1}\prod_{k=1}^m((X+a_{nk})^2+b_{nk}^2) \text{ if } n = 2m+1, \\ -\frac{(-1)^{n-1}}{2^{2n-2}}\binom{2n-2}{n-1}(X+\alpha_n)\prod_{k=1}^m((X+a_{nk})^2+b_{nk}^2) \text{ if } n = 2m+2. \end{cases}$$

Define complex polynomials

$$\Gamma_n(X) = \frac{1}{2^{n-1}}\sqrt{\binom{2n-2}{n-1}}\prod_{k=1}^m(X-a_{nk}+ib_{nk})$$

of degree m, where we have chosen $b_{nk} > 0$, $k = 1, \ldots, m$. Then, we get

$$A_n(X) = \begin{cases} |\Gamma_n(X)|^2 \text{ if } n = 2m+1, \\ (\alpha_n - X)|\Gamma_n(X)|^2 \text{ if } n = 2m+2 \end{cases}$$

and

$$B_n(X) = \begin{cases} |\Gamma_n(-X)|^2 \text{ if } n = 2m+1. \\ (\alpha_n + X)|\Gamma_n(-X)|^2 \text{ if } n = 2m+2. \end{cases}$$

In particular, for p a positive integer, we have

$$\gamma^p = \begin{cases} \frac{1}{2}\begin{pmatrix} 1+ZR_p(Z^2) & |\Gamma_p(-Z)|^2(X+iY)^p \\ |\Gamma_p(Z)|^2(X-iY)^p & 1-ZR_p(Z^2) \end{pmatrix} \text{ if } p = 2m+1, \\ \frac{1}{2}\begin{pmatrix} 1+ZR_p(Z^2) & (\alpha_p - Z)|\Gamma_p(-Z)|^2(X+iY)^p \\ (\alpha_p+Z)|\Gamma_p(Z)|^2(X-iY)^p & 1-ZR_p(Z^2) \end{pmatrix} \text{ if } p = 2m+2. \end{cases}$$

and

$$\gamma^{-p} = \begin{cases} \frac{1}{2}\begin{pmatrix} 1+ZR_p(Z^2) & |\Gamma_p(-Z)|^2(X-iY)^p \\ |\Gamma_p(Z)|^2(X+iY)^p & 1-ZR_p(Z^2) \end{pmatrix} \text{ if } p = 2m+1, \\ \frac{1}{2}\begin{pmatrix} 1+ZR_p(Z^2) & (\alpha_p - Z)|\Gamma_p(-Z)|^2(X-iY)^p \\ (\alpha_p+Z)|\Gamma_p(Z)|^2(X+iY)^p & 1-ZR_p(Z^2) \end{pmatrix} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } p = 2m+2. \end{cases}$$

For an odd positive integer $p = 2m+1$ write

$$\tilde{\gamma}^p = \frac{1}{2}\begin{pmatrix} 1+ZR_p(Z^2) & \Gamma_p(Z)\Gamma_p(-Z)(X+iY)^p \\ \bar{\Gamma}_p(Z)\bar{\Gamma}_p(-Z)(X-iY)^p & 1-ZR_p(Z^2) \end{pmatrix},$$

(resp.

$$\tilde{\gamma}^{-p} = \frac{1}{2}\begin{pmatrix} 1+ZR_p(Z^2) & \Gamma_p(Z)\Gamma_p(-Z)(X-iY)^p \\ \bar{\Gamma}_p(Z)\bar{\Gamma}_p(-Z)(X+iY)^p & 1-ZR_p(Z^2) \end{pmatrix},)$$

and let $P(X,Y,Z), Q(X,Y,Z)$ be real polynomials such that $P + iQ = \Gamma_p(Z)\Gamma_p(-Z)(X+iY)^p$.

Define then, for an odd positive integer

$$\tilde{\gamma}^{2p} = \frac{1}{2}\begin{pmatrix} 1-P^2 & -PQ+iZR_p(Z^2) & -PZR_p(Z^2)-iQ \\ -PQ-iZR_p(Z^2) & 1-Q^2 & -QZR_p(Z^2)+iP \\ -PZR_p(Z^2)+iQ & -QZR_p(Z^2)-iP & 1-Z^2R_p(Z^2)^2 \end{pmatrix}$$

and similarly define $\tilde{\gamma}^{-2p}$ as the matrix obtained from $\tilde{\gamma}^{2p}$ by replacing Y by $-Y$ in the polynomials P and Q.

With the help of polynomials and properties, we have proved above the following:

Proposition 2.2. (1) *For any positive integer the idempotent matrices γ^p, (resp. γ^{-p}) produce polynomial maps*

$$(P(X,Y,Z), Q(X,Y,Z), ZR_p(Z^2)) : \mathbb{S}^2(\mathbb{Q}[i]) \to \mathbb{S}^2(\mathbb{Q}[i])$$

of Brouwer degree p (resp.

$$(P(X,-Y,Z), Q(X,-Y,Z), ZR_p(Z^2)) : \mathbb{S}^2(\mathbb{Q}[i]) \to \mathbb{S}^2(\mathbb{Q}[i])$$

of Brouwer degree $-p$), where P and Q are the polynomials given in Introduction.

(2) *For p an odd integer the class of $\tilde{\gamma}^p$ coincides with that of γ^p in $\tilde{K}_0(A)$ and produce polynomial maps of Brouwer degree p,*

$$(Re(\Gamma_p(Z)\Gamma_p(-Z)(X+iY)^p), Im(\Gamma_p(Z)\Gamma_p(-Z)(X+iY)^p), ZR_p(Z^2)) : \\ \mathbb{S}^2(\mathbb{R}_{alg}) \to \mathbb{S}^2(\mathbb{R}_{alg}),$$

for p positive and

$$(Re(\Gamma_p(Z)\Gamma_p(-Z)(X-iY)^p), Im(\Gamma_p(Z)\Gamma_p(-Z)(X-iY)^p), ZR_p(Z^2)) : \\ \mathbb{S}^2(\mathbb{R}_{alg}) \to \mathbb{S}^2(\mathbb{R}_{alg}),$$

for a negative p.

(3) *The class of $\tilde{\gamma}^{2p}$ coincides with that of γ^{2p} for any odd integer p.* ■

Remark 2.3. The case (1) appears in [1], where it is also extended to produce polynomial maps $\mathbb{S}^n(\mathbb{C}) \to \mathbb{S}^n(\mathbb{C})$ with coefficients in $\mathbb{Z}[\frac{1}{2}, i]$ of any Brouwer degree p and algebraic degree $2|p|-1$.

The matrices

$$\frac{1}{2}\begin{pmatrix} 1+ZR_p(Z^2) & (X+iY)^p \\ A_p(Z)B_p(Z)(X-iY)^p & 1-ZR_p(Z^2) \end{pmatrix},$$

(resp.

$$\frac{1}{2}\begin{pmatrix} 1+ZR_p(Z^2) & (X-iY)^p \\ A_p(Z)B_p(Z)(X+iY)^p & 1-ZR_p(Z^2) \end{pmatrix})$$

are also equivalent to γ^p (resp. γ^{-p}) and produce polynomial maps $\mathbb{S}^2(\mathbb{C}) \to \mathbb{S}^2(\mathbb{C})$, given by Wood [9], with coefficients in $\mathbb{Z}[\frac{1}{2}, i]$ having Brouwer degree p and algebraic degrees $3p-2$.

The case (2) is done by F.J. Turiel [8], where he has observed that for p odd the polynomial $A_p(Z)B_p(Z) = |F_p(Z^2)|^2$ for some complex polynomial F_p.

The case (3) is simply obtained by looking at topology, but the proof is algebraic.

It is not known whether γ^p, for p even admits a hermitian and idempotent representative of order two for $R = K(i)$, where K is a field of characteristic zero not containing i. Or equivalently, whether there are polynomial maps from $\mathbb{S}^2(K) \to \mathbb{S}^2(K)$ with any even Brouwer degree.

We do not know either if hermitian matrices of order three are enough to represent all elements $\alpha \in \tilde{K}_0(A)$ with $2ic_1(\alpha) = 2^r[\omega]$ for some r. But certainly order four suffices, because the class of $\begin{pmatrix} \gamma^{2p} & 0 \\ 0 & 1 \end{pmatrix}$ coincides with that of $\begin{pmatrix} \gamma^{2p-1} & 0 \\ 0 & \gamma^1 \end{pmatrix}$.

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