Trichotomy and topological equivalence for evolution families

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Abstract

We present a topological classification of some trichotomic evolution families, in the sense of Sacker-Sell ([10]), Elaydi-Hajek ([3]), and the one introduced in ([7]). We hope our work will contribute to the general theory of exponential dichotomies and trichotomies.

Introduction

The topological classification of dichotomies is of a significant importance in the theory of differential equations and dynamical systems (see for example the geometric theory in [12]), as it allows the study of just simple equations, also called standard, instead of general, complicated ones. This is due to the fact that topological equivalence, also called conjugacy, as well as kinematic similarity, preserves the inner structures of the solutions of differential equations. The concept of kinematic similarity proved itself to be too much strong for an accurate study of ODE, as it requires the transformations to be linear. As a consequence, similar differential equations might not be kinematically similar.

In [8], Palmer showed that a nonautonomous linear differential system with bounded growth and decay, is exponentially dichotomic if and only if it is topologically equivalent to a standard autonomous system with evolution operator $U(t,s) = e^{-(t-s)}P + e^{t-s}Q$. Here P and Q stand for the dichotomic projections.

This result, valid in $\mathbb{R}^n$, was later generalized on Hilbert spaces in [11], and on Banach spaces in [5].

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In the present paper we extend these results on Banach spaces, in the case of some trichotomic evolution families, using the technique from [5]. As one can easily observe, in the paper [18], the construction of conjugacies generalizes somehow the one detailed in [5].

Another important characterization of exponential dichotomies is that they are *the only structures of ODE which are stable, via topological equivalence, to small linear perturbations,* i.e. they are *structurally stable* (see [9]). Using the constructions in [18] or in the book [16], in [6] it is proved that all the exponentially dichotomic evolution families, even without bounded growth and decay, are structurally stable. Notably, a converse of this result has not been proved yet in the case of Banach spaces.

In Introduction we present the basic notions, notations and constructions used throughout our work.

In the first section we give a topological classification for trichotomic evolution families in the sense of Sacker-Sell ([10]).

In the second section we briefly present the new concept of trichotomy introduced in [7], dual to the one in the sense of Elaydi-Hajek (see [3] and [4]), and in the last section we give a topological classification for these types of trichotomic evolution families.

Like in [7], just to avoid complicated calculations, we restrict our short survey to the study of reversible evolution families $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ on a Banach space E. The reversibility hypothesis means that the operators $U(t,s)$ are defined for all $t,s \in \mathbb{R}$, and therefore they are all invertible. Let $\mathcal{L}(E)$ be the space of the bounded linear operators acting on E

Definition 1. A family $U = \{U(t,s)\}_{t,s\in\mathbb{R}}$ in the space $\mathcal{L}(E)$, is called a reversible evolution family if it satisfies the following conditions:

(i) $U(t,\tau)U(\tau,s) = U(t,s)$, for all $t,\tau,s \in R$;

(ii) $U(t,t) = I$, for all $t \in R$;

(iii) The map $R^2 \ni (t,s) \mapsto U(t,s) \in \mathcal{L}(E)$ is continuous.

Definition 2. The evolution family $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ is called trichotomic if there exist two families of bounded projections on E, $P(t)$ and $Q(t)$, with $P(t)Q(t) = Q(t)P(t) = 0$, $P(t) + Q(t) \neq 0$, and constants $N,\nu > 0$ such that

$$
\begin{array}{lll}
(i) & P(t)U(t,s) = U(t,s)P(s) = U_P(t,s), & \text{for } t,s \in \mathbb{R}; \\
(ii) & Q(t)U(t,s) = U(t,s)Q(s) = U_Q(t,s), & \text{for } t,s \in \mathbb{R}; \\
(iii) & \|U_P(t,s)\| \leq Ne^{-\nu(t-s)}, & \text{for } t \geq s; \\
(iv) & \|U_Q(t,s)\| \leq Ne^{-\nu(s-t)}, & \text{for } s \geq t.
\end{array}
$$

If $P(t) + Q(t) = I$ (identity on E), then we have exponential dichotomy. If $P(t) + Q(t) \neq I$, then particular types of trichotomies might emerge, depending on the behavior of the evolution family $\mathcal{U}$ with respect to the projections $R(t) = I - P(t) - Q(t)$. Notice that the three projection families are uniformly bounded with respect to $t \in \mathbb{R}$:

$$\sup_{t\in\mathbb{R}} \|P(t)\|, \ \sup_{t\in\mathbb{R}} \|Q(t)\|, \ \sup_{t\in\mathbb{R}} \|R(t)\| < \infty.$$

The reader will easily observe that if we add the conditions of bounded growth and decay along the trajectories through $R(t)E$, i.e. if we assume that for some real constants α and $N \geq 1$, $\|U_R(t,s)\| \leq Ne^{\alpha(s-t)}$, for any $t, s \in \mathbb{R}$ ($U_R(t,s) = R(t)U(t,s) = U(t,s)R(s)$), then we step on the uniform version of the trichotomies in [15].

Definition 3. The evolution families $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ and $\mathcal{V} = \{V(t,s)\}_{t,s\in\mathbb{R}}$ are called topologically equivalent iff there exist a continuous function $h : \mathbb{R} \times E \to E$, with the following properties:

(i) For each $t \in \mathbb{R}$, the function $h_t(x) = h(t,x)$ is a homeomorphism;

(ii) $h_t U(t,s) = V(t,s)h_s$, for each $t, s \in \mathbb{R}$;

(iii) There exist two increasing functions $L, L' : \mathbb{R}_+ \to \mathbb{R}_+$, with $L(0) = L'(0) = 0$, continuous at 0, such that

$$\|h_t(x)\| \leq L(\|x\|)$$
$$\|h_t^{-1}(x)\| \leq L'(\|x\|)$$

for each $t \in \mathbb{R}$ and each $x \in E$ (see [11]).

In [8] and [9] Palmer introduced two weaker forms of topological equivalence by using instead of (iii) above either

(iii)′ $\lim_{x\to 0} \|h_t(x)\| = \lim_{x\to 0} \|h_t^{-1}(x)\| = 0$, uniformly with respect to t, or

(iii)″ $\lim_{\|x\|\to\infty} \|h_t(x)\| = \lim_{\|x\|\to\infty} \|h_t^{-1}(x)\| = +\infty$, uniformly with respect to t.

Let us remark that in the particular case of the differentiable evolution families, relation (ii) above shows in fact that if $x(t)$ is a solution of the differential equation $\frac{dx}{dt} = A(t)x$, with evolution operator $U(t,s)$, then $y(t) = h_t(x(t))$ is a solution of the differential equation $\frac{dy}{dt} = B(t)y$, with evolution operator $V(t,s)$. Relations as (iii), (iii)′ or (iii)″ above imply that the asymptotic properties are transmitted from the first equation (or, more general, evolution family) to the second one, therefore *topologically equivalent equations are similar*. Notably, the topological equivalence relation is an equivalence relation in the evolution families class.

Definition 4. We say that the evolution family $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ has bounded growth and decay iff there are constants $K, M > 0$ such that:

$$\|U(t,s)\| \leq Ke^{M|t-s|}, \tag{1}$$

for each $t, s \in \mathbb{R}$.

Suppose that the evolution family $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ is trichotomic. Let us define the following functionals on E, that are norms on the corresponding subspaces $P(t)E$ and $Q(t)E$:

$$\|P(t)x\|_t = \int_t^{\infty} \|U_P(s,t)x\|ds, \tag{2}$$

$$\|Q(t)x\|_t = \int_{-\infty}^{t} \|U_Q(s,t)x\|ds. \tag{3}$$

Notice that if $\mathcal{U}$ has bounded growth and decay, then the above norms are equivalent to the initial one, on $P(t)E$ and $Q(t)E$, respectively (see [5]). For each $t \in \mathbb{R}$ we consider the functions:

$$h_t^P : P(0)E \to P(t)E \text{ and } h_t^Q : Q(0)E \to Q(t)E, \text{ by:}$$

$$h_t^P(x_1) = \begin{cases} \frac{U(t,o)x_1}{\|U(t+\ln\|x_1\|,0)x_1\|_{t+\ln\|x_1\|}}, & \text{if } x_1 \in P(0)E \setminus \{0\}; \\ 0, & \text{if } x_1 = 0, \end{cases} \tag{4}$$

and

$$h_t^Q(x_2) = \begin{cases} \frac{U(t,0)x_2}{\|U(t-\ln\|x_2\|,0)x_2\|_{t-\ln\|x_2\|}}, & \text{if } x_2 \in Q(0)E \setminus \{0\}; \\ 0, & \text{if } x_2 = 0. \end{cases} \tag{5}$$

Using similar arguments as in the proof of Theorem 1 in [5], one can easily get:

Lemma 5. *Suppose that the (reversible) evolution family $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ is trichotomic and has bounded growth and decay (1). Then the following relations hold true:*
(i) For each $t \in \mathbb{R}$, the above maps are homeomorphisms;
(ii) For each $t \in \mathbb{R}$ and each $x \in E$ we have

$$h_t^P(e^{-t}P(0)x) = U(t,o)h_0^P(P(0)x) \tag{6}$$
$$h_t^Q(e^tQ(0)x) = U(t,0)h_0^Q(Q(0)x); \tag{7}$$

(iii) There exist 4 increasing functions $L_i, L_i' : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,2$, continuous at 0, with $L_i(0) = L_i'(0) = 0$, such that

$$\|h_t^P(P(0)x)\| \leq L_1(\|x\|), \quad \|(h_t^P)^{-1}(P(t)x)\| \leq L_1'(\|x\|)$$

and

$$\|h_t^Q(Q(0)x)\| \leq L_2(\|x\|), \quad \|(h_t^Q)^{-1}(Q(t)x)\| \leq L_2'(\|x\|), \tag{8}$$

for each $t \in \mathbb{R}$ and each $x \in E$.

1 Sacker-Sell trichotomy

Definition 6. The evolution family $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ is called Sacker-Sell trichotomic (see [10]), if it satisfies the following inequality, in addition to relations $(i) - (iv)$ from Definition (2):

$$\|U_R(t,s)\| \leq N, \tag{9}$$

for each $t, s \in \mathbb{R}$.

Notice that this type of trichotomy is a particular uniform version of exponential trichotomy in the sense of [15]. Inspired by the constructions in [8], and using excessive computations in our opinion, paper [13] presents a topological classification for this type of trichotomy, in the particular case of the differentiable evolution families, and in the finite dimensional space $\mathbb{R}^n$. Our next theorem shows that the main result in paper [13] holds true for any evolution family, and in any Banach space.

Theorem 7. *If the evolution family $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ is Sacker-Sell trichotomic and has bounded growth and decay, then it is topologically equivalent (using (iii) in Definition 3) to the evolution family generated by the standard autonomous differential equation:*

$$\frac{dz}{dt} = (Q(0) - P(0))z.$$

Proof. For each $t \in \mathbb{R}$ we consider the function:

$$h_t^R : R(0)E \to R(t)E, \;\; h_t^R(x_3) = U(t,0)x_3,$$

for each $x_3 \in R(0)E$. Obviously, it is invertible and

$$(h_t^R)^{-1} : R(t)E \to R(0)E, \;\; (h_t^R)^{-1}(R(t)x) = R(0)U(0,t)x,$$

for each $t \in \mathbb{R}$ and each $x \in E$. According to relations (11) we have:

$$\|h_t^R(R(0)x)\| \le N\|x\| = L_3(\|x\|),$$

$$\|(h_t^R)^{-1}(R(t)x)\| \le N\|x\| = L_3'(\|x\|).$$

Set $h_t : E \to E$, $h_t(x) = h_t^P(P(0)x) \oplus h_t^Q(Q(0)x) \oplus h_t^R(R(0)x)$ (direct sums). From relations (9) and (10), one can easily get that h_t generates a topological equivalence between the evolution family $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ and the evolution family $\mathcal{V} = \{V(t,s)\}_{t,s\in\mathbb{R}}$, where

$$V(t,s) = e^{-(t-s)}P(0) + e^{t-s}Q(0) + R(0). \qquad \blacksquare$$

Corollary 8. *The evolution family $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ satisfies the relation*

$$\|U(t,s)\| \le N, \text{ for each } t,s \in \mathbb{R}$$

(i.e. is bistable, according to [2, p. 113]), if and only if it is topologically equivalent to the evolution family generated by the standard differential equation

$$\frac{dz}{dt} = 0.$$

Proof. The necessity follows directly from the above proof. For the sufficiency, let $V(t,s)$ be the evolution operator of the equation $\frac{dz}{dt} = 0$. Then $V(t,s) = I$ (identity). Since $h_t U(t,s) = V(t,s)h_s = h_s$, then we get $\|U(t,s)x\| = \left\|h_t^{-1}h_s x\right\| \le L'(\|h_s x\|) \le L'(L(\|x\|))$. If $\|x\| \le 1$, then it follows $\|U(t,s)\| \le L'(L(1)) = N$. $\blacksquare$

2 Exponential trichotomy

Definition 9. (see [7]) (i) The evolution family $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ is called α-exponentially trichotomic (in the sense of Elaydi-Hajek, see [3]) if there exist projections $P_+(t)$ and $P_-(t)$, defined on all $\mathbb{R}$, such that $\mathcal{U}$ is exponentially dichotomic on $\mathbb{R}_+$ with projection $P_+(t)$, on $\mathbb{R}_-$ with projection $P_-(t)$, and the following conditions are fulfilled:

$$P_+(t)P_-(t) = P_-(t)P_+(t) = P_-(t), \tag{10}$$

$$\sup_{t\in\mathbb{R}} \|P_+(t) - P_-(t)\| < \infty. \tag{11}$$

(ii) The evolution family $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ is called β-exponentially trichotomic (see [7]) if it satisfies (i) above, except the relation (10) which is replaced by

$$P_+(t)P_-(t) = P_-(t)P_+(t) = P_+(t). \tag{12}$$

Notice that if $\mathcal{U}$ is reversible, then it suffices to consider instead of relations (10) and (12), the weaker conditions $P_+(0)P_-(0) = P_-(0)P_+(0) = P_-(0)$ and $P_+(0)P_-(0) = P_-(0)P_+(0) = P_+(0)$, respectively. Notably, in this particular situation we do not even need to assume the existence of $P_+(t)$ and $P_-(t)$ on all $\mathbb{R}$. This is a consequence the fact that $P_+(t) = U(t,0)P_+(0)U(0,t)$, for any real t, etc. For the sake of generality, we prefer relations (10) and (12).The notion of α-exponential trichotomy was introduced in [3], for differentiable evolution families generated by the differential equations $\frac{dx}{dt} = A(t)x$, in finite dimensional space, and under the restrictive hypothesis

$$\sup_{t\in\mathbb{R}} \|A(t)\| < \infty.$$

Observe that we added to the definition of α-exponential trichotomy in [3], the relation (11), which is necessary for the next considerations.

The following Lemma extends for evolution families, on Banach spaces, Lemma 1.2 from [3](see also [7]):

Lemma 10. *The statements below are pairwise equivalent:*

(i) The evolution family $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ is α-exponentially trichotomic;

(ii) There exist two bounded projections $P(t)$ and $Q(t)$, with $P(t)Q(t) = Q(t)P(t)$, $P(t) + Q(t) - P(t)Q(t) = I$ and constants $N \geq 1$, $\nu > 0$ such that

$$\|U_P(t,s)\| \leq Ne^{-\nu(t-s)}, \text{ for } t \geq s \geq 0;$$

$$\|U_{I-P}(t,s)\| \leq Ne^{-\nu(s-t)}, \text{ for } t \leq s \geq 0;$$

$$\|U_Q(t,s)\| \leq Ne^{-\nu(s-t)}, \text{ for } t \leq s \leq 0;$$

$$\|U_{I-Q}(t,s)\| \leq Ne^{-\nu(t-s)}, \text{ for } 0 \geq s \leq t;$$

$$\sup_{t\leq 0} \|P(t)\| < \infty \text{ and } \sup_{t\geq 0} \|Q(t)\| < \infty;$$

(iii) *There exist three bounded projections, $P_i(t)$, $i = 1, 2, 3$, with $P_1(t) + P_2(t) + P_3(t) = I$ and $P_i(t)P_j(t) = 0$ for $i \neq j$, such that*

$$\|U_{P_1}(t,s)\| \leq Ne^{-\nu(t-s)}, \text{ for } t \geq s;$$

$$\|U_{P_2}(t,s)\| \leq Ne^{-\nu(s-t)}, \text{ for } s \geq t;$$

$$\begin{aligned} \|U_{P_3}(t,s)\| &\leq Ne^{-\nu(t-s)}, \text{ for } 0 \leq s \leq t; \\ &\leq Ne^{-\nu(s-t)}, \text{ for } t \leq s \leq 0; \end{aligned}$$

We give a similar characterization for β-exponential trichotomy (see [7] for more details):

Lemma 11. *The following statements are pairwise equivalent:*

(i) *The evolution family $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ is β-exponentially trichotomic;*

(ii) *All the conditions in (ii) from the previous Lemma hold true, except that $P(t)Q(t) = Q(t)P(t) = 0$, instead of $P(t)Q(t) = Q(t)P(t)$ and $P(t) + Q(t) - P(t)Q(t) = I$;*

(iii) *All conditions in (iii) from the previous Lemma hold true, except the last two inequalities that become*

$$\begin{aligned} \|U_{P_3}(t,s)\| &\leq Ne^{-\nu(s-t)}, \text{ for } s \geq t \geq 0; \\ &\leq Ne^{-\nu(t-s)}, \text{ for } 0 \geq t \geq s. \end{aligned}$$

Both Lemmas can be easily proved by using similar arguments as in the proof of Lemma 1.2 in [3]. Notice that we have a clear connection between exponential dichotomy on $\mathbb{R}_+$, $\mathbb{R}_-$, and the existence of three projections, i.e. trichotomy.

3 Exponential trichotomy and topological equivalence

In [5] it is proved that any exponentially dichotomic equation, with structural projections P, Q, and with bounded growth and decay, is topologically equivalent to the standard equation with the evolution operator

$$V(t,s) = e^{-(t-s)}P + e^{t-s}Q.$$

In this section we show that all the β-exponentially trichotomic evolution families are topologically equivalent to a standard equation with evolution operator

$$U(t,s) = e^{-(t-s)}P(0) + e^{t-s}Q(0) + e^{|t|-|s|}R(0).$$

This property still holds for α-exponential trichotomy, and for the equations with evolution operator

$$W(t,s) = e^{-(t-s)}P(0) + e^{t-s}Q(0) + e^{|s|-|t|}R(0).$$

In this section we restrict our study to evolutions families $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ with bounded growth and decay, that satisfy βexponential trichotomy. Let $P_1(t)$, $P_2(t)$, and $P_3(t)$ be the structural projections.

If we set $\alpha : \mathbb{R} \to \mathbb{R}$,

$$\alpha(t) = \begin{cases} -2t\delta + \delta^2 & \text{for } t > \delta; \\ -t^2 & \text{for } -\delta \leq t \leq \delta; \\ 2t\delta + \delta^2 & \text{for } t < -\delta. \end{cases}$$

then α is continuously differentiable, even, and Lipschitz:

$$|\alpha(t) - \alpha(s)| \leq 2\delta|t-s|.$$

Put $\delta = \frac{M+\nu}{2}$ and suppose that

$$U(t,0)R(0) = U(-t,0)R(0), \text{ for each } t \in \mathbb{R}. \tag{13}$$

This last supposition is crucial for our considerations.

Using $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$, we construct the evolution families $\mathcal{U}^+ = \{U^+(t,s)\}_{t,s\in\mathbb{R}}$ and $\mathcal{U}^- = \{U^-(t,s)\}_{t,s\in\mathbb{R}}$, defined as follows:

$$U^+(t,0) = \begin{cases} U(t,0), & \text{if } t \geq 0; \\ U(t,0)P(0) + U(t,0)Q(0) + e^{\alpha(t)}U(t,0)R(0), & \text{if } t < 0. \end{cases}$$

Put $U^+(t,s) = U^+(t,0)(U^+(s,0))^{-1}$, for each $t,s \in \mathbb{R}$, and

$$U^-(t,0) = \begin{cases} U(t,0)P(0) + U(t,0)Q(0) + e^{\alpha(t)}U(t,0)R(0), & \text{if } t \geq 0; \\ U(t,0), & \text{if } t < 0. \end{cases}$$

Set $U^-(t,s) = U^-(t,0)(U^-(s,0))^{-1}$ for each $t,s \in \mathbb{R}$.

Lemma 12. *(i) The evolution families $\mathcal{U}^+$ and $\mathcal{U}^-$ above are exponentially dichotomic on $\mathbb{R}$, with structural projections $P(t), Q(t) + R(t)$, and $P(t) + R(t), Q(t)$, respectively;*
(ii) They both have bounded growth and decay;
(iii) $U^-(t,0)R(0) = U^+(-t,0)R(0)$, for each $t \in \mathbb{R}$.

Proof. (i) For $s \leq t \leq -\delta$ we have:

$$\begin{aligned} \|U_R^+(s,t)\| &\leq e^{\alpha(s)-\alpha(t)}NKe^{M(t-s)} \\ &= NKe^{-(2\delta-M)(t-s)} \\ &= \widetilde{N}e^{-\nu(t-s)}, \end{aligned}$$

and for $t \geq s \geq \delta$:

$$\begin{aligned} \|U_R^-(t,s)\| &\leq e^{\alpha(t)-\alpha(s)}NKe^{M(t-s)} \\ &= NKe^{-(2\delta-M)(t-s)} \\ &= \widetilde{N}e^{-\nu(t-s)}. \end{aligned}$$

The βtrichotomy of the family $\mathcal{U}$, together with the definition of operators $U^{+}(t,s)$ and $U^{-}(t,s)$ lead us to the conclusion in (i).

(ii) For $t,s \leq 0$ we have:

$$\begin{aligned}
\|U^{+}(t,s)\| &\leq \|U_P^{+}(t,s)\| + \|U_Q^{+}(t,s)\| + +\|U_R^{+}(t,s)\| \\
&\leq NKe^{M|t-s|} + NKe^{M|t-s|} + NKe^{(M+2\delta)|t-s|} \\
&\leq 3NKe^{(M+2\delta)|t-s|} \\
&= \widetilde{K}e^{\widetilde{M}|t-s|}.
\end{aligned}$$

(ii) comes from the bounded growth and decay hypothesis for the family $\mathcal{U}$, and (iii) follows directly from relation (13). ∎

Using the operators $U^{+}(t,s)$ and $U^{-}(t,s)$, we define the following functionals:

$$\|x\|_t^{+} = \int_t^{\infty} \|U_P^{+}(s,t)x\|ds + \int_{-\infty}^{t} \|U_Q^{+}(s,t)x\|ds + \int_{-\infty}^{t} \|U_R^{+}(s,t)x\|ds,$$

and

$$\|x\|_t^{-} = \int_t^{\infty} \|U_P^{-}(s,t)x\|ds + \int_{-\infty}^{t} \|U_Q^{-}(s,t)x\|ds + \int_t^{\infty} \|U_R^{-}(s,t)x\|ds.$$

Let us observe that:

$$\|P^{+}(t)x\|_t^{+} = \|P^{-}(t)x\|_t^{-} = \|P(t)x\|_t,$$

and

$$\|Q^{+}(t)x\|_t^{+} = \|Q^{-}(t)x\|_t^{-} = \|Q(t)x\|$$

(see relations (5)).

With the homeomorphisms

$$h_t^{R,+}, h_t^{R,-} : R(0)E \to R(t)E,$$

$$h_t^{R,+}(R(0)x) = \begin{cases} \dfrac{U^{+}(t,0)R(0)x}{\|U^{+}(t-\ln\|R(0)x\|,0)R(0)x\|_{t-\ln\|R(0)x\|}^{+}} & \text{for } R(0)x \neq 0; \\ 0 & \text{for } R(0)x = 0, \end{cases}$$

and

$$h_t^{R,-}(R(0)x) = \begin{cases} \dfrac{U^{-}(t,0)R(0)x}{\|U_{-}(t+\ln\|R(0)x\|,0)R(0)x\|_{t+\ln\|R(0)x\|}^{-}} & \text{for } R(0)x \neq 0; \\ 0 & \text{for } R(0)x = 0, \end{cases}$$

we construct the homeomorphisms $h_t^{+}, h_t^{-} : E \to E$,by

$$\begin{aligned}
h_t^{+}(x) &= h_t^{P}(P(0)x) + h_t^{Q}(Q(0)x) + h_t^{R,+}(R(0)x), \\
h_t^{-}(x) &= h_t^{P}(P(0)x) + h_t^{Q}(Q(0)x) + h_t^{R,-}(R(0)x).
\end{aligned} \tag{14}$$

(see (6) and (8)).

Similar arguments as in Theorem 1 in [5] prove that the homeomorphisms h_t^+ and h_t^- yield a topological equivalence between the evolution family $\mathcal{U}^+$ and the evolution operator generated by the differential equation

$$\frac{dy}{dt} = (Q(0) + R(0) - P(0))y,$$

respectively, between $\mathcal{U}^-$ and the evolution operator generated by

$$\frac{dz}{dt} = (Q(0) - P(0) - R(0))z.$$

The main result of this section is exposed in:

Theorem 13. *Suppose that the evolution family $\mathcal{U}$ is exponentially trichotomic, has bounded growth and decay, and verify in addition relation (13).*

(i) If $\mathcal{U}$ is α-exponentially trichotomic, then it is topologically equivalent to the following standard evolution family $\mathcal{V}_1$:

$$V_1(t,s) = e^{-(t-s)}P(0) + e^{t-s}Q(0) + e^{|s|-|t|}R(0).$$

(ii) If $\mathcal{U}$ is β-exponentially trichotomic, then it is topologically equivalent to the standard evolution family $\mathcal{V}_2$:

$$V_2(t,s) = e^{-(t-s)}P(0) + e^{t-s}Q(0) + e^{|t||-|s|}R(0).$$

Proof. (ii) It suffices to prove that $h_0^+ = h_0^-$ (see the relations (14)). Let us notice that

$$h_0^{R,+}(R(0)x) = \frac{R(0)x}{\|U^+(-\ln\|R(0)x\|,0)R(0)x\|^+_{-\ln\|R(0)x\|}}$$

and

$$h_0^{R,-}(R(0)x) = \frac{R(0)x}{\|U^-(\ln\|R(0)x\|,0)R(0)x\|^-_{\ln\|R(0)x\|}}.$$

Since the denominators in the right sides are

$$\|U^+(-\ln\|R(0)x\|,0)R(0)x\|^+_{-\ln\|R(0)x\|} = \int_{-\infty}^{-\ln\|R(0)x\|} \|U^+(s,0)R(0)x\|ds$$

and

$$\|U^-(\ln\|R(0)x\|,0)R(0)x\|^-_{\ln\|R(0)x\|} = \int_{\ln\|R(0)x\|}^{\infty} \|U^-(s,0)R(0)x\|ds,$$

since the maps

$$\varphi(t) = \int_{-\infty}^{-t} \|U^+(s,0)R(0)x\|ds$$

and

$$\psi(t) = \int_{t}^{\infty} \|U^-(s,0)R(0)x\|ds$$

are equal (see relation (iii) in Lemma 12), then $h_0^{R,+} = h_0^{R,-}$, and finally: $h_0^+ = h_0^-$. If we set

$$h_t(x) = \begin{cases} h_t^+(x), & \text{for } t \geq 0; \\ h_t^-(x), & \text{for } t < 0, \end{cases}$$

then h_t gives the topological equivalence required in (ii). For (i) the arguments are similar. ∎

Definition 14. We say that the evolution families $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ and $\mathcal{V} = \{V(t,s)\}_{t,s\in\mathbb{R}}$ are kinematically similar if there exists an invertible operator function $\Phi : J \to \mathcal{L}(E)$, Φ and Φ^{-1} being uniformly bounded and continuously differentiable, such that

$$\Phi(t)V(t,s) = U(t,s)\Phi(s), \text{ for each } t,s \in \mathbb{R}.$$

Obviously this concept is much stronger than topological equivalence, since it requires the transformations $\Phi(t)$ to be linear.

We recall the definition of the Bohl exponents of the evolution family $\mathcal{U}$:

Definition 15.

$$K_B = \sup_{t>s} \frac{\ln \|U(t,s)\|}{t-s},$$

and

$$K_B' = \sup_{s>t} \frac{\ln \|U(t,s)\|}{s-t},$$

are called the upper, respectively the lower Bohl exponents of $\mathcal{U}$.

Notice that the finiteness of the Bohl exponents is equivalent to bounded growth, respectively bounded decay.

Theorem 2.1 in [2, p.159] states that kinematically similar equations have the same Bohl exponents. Notably, topological equivalence does not preserve these exponents, it only preserves their finiteness, as revealed in the following theorem.

Theorem 16. *If the evolution families $\mathcal{U} = \{U(t,s)\}_{t,s\in\mathbb{R}}$ and $\mathcal{V} = \{V(t,s)\}_{t,s\in\mathbb{R}}$ are topologically equivalent, then either they both have finite Bohl exponents, or they both have infinite Bohl exponents.*

Proof. (i) Suppose that the topological equivalence is given by the weaker form (iii)′ in Definition 3, and the upper Bohl exponent of the family $\mathcal{V}$ is finite. According to Theorem 4.2 in [2, p.119], we have:

$$\sup_{0\leq t-s\leq 1} \|V(t,s)\| = K < \infty.$$

For each $\epsilon > 0$, there exist δ_1, δ_2, such that for each $t \in \mathbb{R}$

$$\|h_t(x)\| < \epsilon, \text{ for } \|x\| < \delta_1,$$

and

$$\|h_t^{-1}(x)\| < \epsilon, \text{ for } \|x\| < \delta_2.$$

Choose $\delta > 0$ such that

$$\|x\| \leq \delta \Rightarrow \|h_s(x)\| < \frac{\delta_2}{K}, \text{for each } s \in \mathbb{R}.$$

It follows that

$$\|V(t,s)x\| < K\frac{\delta_2}{K} = \delta_2,$$

and then

$$\|h_t^{-1}V(t,s)h_s(x)\| < \epsilon, \text{ for all } \|x\| \leq \delta.$$

Since $h_t U(t,s) = V(t,s)h_s$, then we have

$$\|U(t,s)x\| < \epsilon, \text{ for } 0 \leq t-s \leq 1 \text{ and } \|x\| \leq \delta,$$

and finally:

$$\sup_{0 \leq t-s \leq 1} \|U(t,s)\| \leq \frac{\epsilon}{\delta} < \infty,$$

which proves that the upper Bohl exponent of the family $\mathcal{U}$ is finite.

(ii) Suppose that the topological equivalence relation is given by the weaker form (iii)$''$ in Definition 3, and the upper Bohl exponent of the family $\mathcal{V}$ is infinite. As

$$\sup_{0 \leq t-s \leq 1} \|V(t),s)\| = +\infty,$$

then we obtain the sequences $t_n, s_n \in \mathbb{R}$, with $0 \leq t_n - s_n \leq 1$, $x_n \in E$, $\|x_n\| = 1$, such that

$$\lim_{n \to \infty} \|V(t_n, s_n)x_n\| = +\infty.$$

If we set $y_n = h_{s_n}^{-1}(x_n)$, then (y_n) is bounded:

$$\|y_n\| \leq K, \text{ for some constant } K > 0.$$

We also have that:

$$\|U(t_n, s_n)\frac{y_n}{K}\| = \frac{1}{K}\|h_{t_n}^{-1}V(t_n, s_n)x_n\| \to \infty,$$

therefore

$$\sup_{0 \leq t-s \leq 1} \|U(t,s)\| = +\infty. \qquad \blacksquare$$

This theorem shows the necessity of assuming the hypothesis of bounded growth and decay in Theorem 1 in [5], or in Theorem 13 in the present paper. To illustrate the necessity of these conditions, let us consider the exponentially dichotomic evolution operator $U(t,s) = U(t)U^{-1}(s)$, where

$$U(t) = \begin{cases} e^{t^2} I & , \text{for } t \geq 0 \\ e^{-t^2} I & , \text{for } t < 0 \end{cases}.$$

It follows that $\mathcal{U}$ cannot be topologically equivalent to a standard one as in Theorem 1 in [5], since it has not bounded growth and decay. Furthermore, if we set $V(t,s) = e^{s^2-t^2}I$ and $W(t) = e^{t^2-s^2}I$, we notice that $\mathcal{V}$ and $\mathcal{W}$ are α, respectively β-exponentially trichotomic, but they are not topologically equivalent to the standard evolution operators $V_1(t,s) = e^{|s|-|t|}I$, respectively $W_1(t) = e^{|t|-|s|}I$. This occurs because they do not have bounded growth and decay (or finite Bohl exponents). To illustrate the necessity of conditions (13), let us consider $U(t,s) = U(t)U^{-1}(s)$, where $U(t) = \begin{cases} e^{at}I & \text{for } t \geq 0; \\ e^{-bt}I & \text{for } t < 0 \end{cases}$ with $a, b > 0$. Obviously, $\mathcal{U}$ is β-exponentially trichotomic. For $a \neq b$, $\mathcal{U}$ is not topologically equivalent to $\mathcal{V}$, for which $V(t,s) = e^{|t|-|s|}I$, since

$$\sup_{t\in\mathbb{R}} \|U(t,-t)\| = +\infty, \text{ and } V(t,-t) = I.$$

We present a generalization of Theorem 2 in [5] for exponentially trichotomic evolution families:

Theorem 17. *Suppose that the evolution families $\mathcal{U}$ and $\mathcal{V}$ are exponentially trichotomic (either α or β), they both have bounded growth and decay, and satisfy in addition*

$$U(t,0)R(0) = U(-t,0)R(0), \quad V(t,0)R_1(0) = V(-t,0)R_1(0), \textit{ for each } t \in \mathbb{R}.$$

($P(t), Q(t), R(t)$, respectively $P_1(t), Q_1(t), R_1(t)$ are the trichotomic structural projections).

(i) If the triplets above are topologically similar (i.e. subspaces $P(0)E$ and $P_1(0)E$, respectively $Q(0)E$ and $Q_1(0)E$, respectively $R(0)E$ and $R_1(0)E$ are homeomorphic), then $\mathcal{U}$ and $\mathcal{V}$ are topologically equivalent.

(ii) If E is finite dimensional and $\mathcal{U}$ and $\mathcal{V}$ are topologically equivalent, then the triplets above are topologically similar.

Being an immediate consequence of Theorem 2 in [5], we omit it's proof..

Since the condition (13) seems to be too much restrictive, we need to identify larger classes of trichotomic families for which Theorem 13 is applicable, i.e. to extend the class of evolution families that are able to be classified.

Lemma 18. *Suppose that the evolution family $\mathcal{U}$ is exponentially trichotomic (either α or β), has bounded growth and decay, and satisfies in addition:*

$$\sup_{t\in J} \|R(t) - U(-t,t)\| \leq \delta < 1,$$

where $J = (-\infty, 0]$ or $J = [0, \infty)$.

(i) Then $\mathcal{U}$ is kinematically similar (therefore topologically equivalent) to an evolution family $\mathcal{V}$ verifying relation (13).

(ii) If $\mathcal{U}$ is generated by a differential equation $\frac{dx}{dt} = A(t)x$ and $A(0)R = 0$, then the statement in (i) remains valid, moreover $\mathcal{V}$ is generated by some differential equation $\frac{dy}{dt} = B(t)y$.

Proof. (ii) Suppose that $J = (-\infty, 0]$. Let us define the operator $V(t)$ as follows:

$$V(t) = \begin{cases} U(t), & \text{for } t \geq 0; \\ U(t)P + U(t)Q + U(-t)R, & \text{for } t < 0. \end{cases}$$

Then for any $t \geq 0$:

$$\frac{dV}{dt} = A(t)V.$$

If $t < 0$ then

$$\frac{dV}{dt} = A(t)U(t)P + A(t)U(t)Q - A(-t)U(-t)R = \widetilde{B}(t)V.$$

We have that

$$\widetilde{B}(t)\widetilde{\Phi}(t) = \widetilde{A}(t), \text{ where } \widetilde{\Phi}(t) = P(t) + Q(t) + U(-t)RU^{-1}(t),$$

and

$$\widetilde{A}(t) = A(t)P(t) + A(t)Q(t) - A(-t)U(-t)RU^{-1}(t).$$

Since the condition (ii) implies

$$\|\widetilde{\Phi}(t) - I\| \leq \delta < 1, \text{ for each } t \in J,$$

then we have that $\widetilde{\Phi}(t)$ is invertible. This proves the existence of the operator function $\widetilde{B}(t)$. If we put

$$B(t) = \begin{cases} A(t) & , \text{ for } t \geq 0; \\ \widetilde{A}(t)\widetilde{\Phi}^{-1}(t) & , \text{ for } t < 0. \end{cases}$$

then $\frac{dV}{dt} = B(t)V$. As $A(0)R = 0$, then $B(t)$ is continuous at 0. If we set

$$\widetilde{\Phi}(t) = I + L(t),$$

where

$$L(t) = U(-t)RU^{-1}(t) - R(t),$$

then

$$\sup_{t<0} \|L(t)\| \leq \delta < 1.$$

According to Lemma 4.2 from [12, p.60], the operator function $\widetilde{\Phi}^{-1}(t)$ is uniformly bounded

$$\sup_{t<0} \|\widetilde{\Phi}^{-1}(t)\| \leq \frac{1}{1-\delta}.$$

If we put

$$\Phi(t) = \begin{cases} I & , \text{ for } t \geq 0; \\ \widetilde{\Phi}(t) & , \text{ for } t < 0, \end{cases}$$

then it follows that $\Phi(t) = V(t)U^{-1}(t)$, and therefore $\Phi(t)$ gives the required kinematic similarity.

If $J = [0, \infty)$, then we set

$$V(t) = \begin{cases} U(t)P + U(t)Q + U(-t)R & , \text{ for } t \geq 0; \\ U(t) & , \text{ for } t < 0, \end{cases}$$

and the arguments are similar.

(i) If $\mathcal{U}$ is not differentiable and $J = (-\infty, 0]$, then it suffices to define $\mathcal{V}$ similarly: $V(t, s) = V(t, 0)V^{-1}(s, 0)$, where

$$V(t,0) = \begin{cases} U(t,0) & \text{for } t \geq 0; \\ U(t,0)P(0) + U(t,0)Q(0) + U(-t,0)R(0) & \text{for } t < 0. \end{cases}$$

Then $\widetilde{\Phi}(t) = P(t) + Q(t) + U_R(-t, t)$, etc. ■

Lemma 19. *If we assume all the conditions in previous Lemma, except that* $\sup_{t \in J} \|R(t) - U(-t, t)\| \leq \delta < 1$,*, which is replaced by*

$$\|R(t) - U_R(-t, t)\| \leq \delta < 2,$$

for each $t \in \mathbb{R}$*,*

Then the evolution family $\mathcal{U}$ *is kinematically similar to an evolution family* $\mathcal{W}$*, verifying* $W(t, 0)R(0) = W(-t, 0)R(0)$*, for each* $t \in \mathbb{R}$*.*

Proof. Let us consider the following operator function:

$$W(t,0) = U(t,0)P(0) + U(t,0)Q(0) + \frac{1}{2}\left[U(t,0)R(0) + U(-t,0)R(0)\right]$$

Then $\Phi(t) = I + \frac{1}{2}\left[U_R(-t, t) - R(t)\right]$ furnishes the required kinematic similarity. ■

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