Lagrangian submanifolds in 3-dimensional complex space forms with isotropic cubic tensor

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Abstract

In this paper, we introduce the notion of manifolds with isotropic cubic tensor. We give a complete classification of the Lagrangian submanifolds in 3-dimensional complex space forms with isotropic cubic tensor.

1 Introduction

Let $\phi : M^n \to \bar{M}^n$ be an isometric immersion from an n-dimensional Riemannian manifold into a complex n-dimensional Kähler manifold $\bar{M}^n$. M^n is called a *Lagrangian submanifold* if the almost complex structure J of $\bar{M}^n$ carries each tangent space of M^n into its corresponding normal space.

In this paper, we study Lagrangian submanifolds of complex space forms $\bar{M}^n(4c)$ with constant holomorphic sectional curvature $4c$. In particular we are interested in Lagrangian submanifolds of

(i) $\bar{M}^n(4c) = \mathbb{C}^n$, when $c = 0$,

(ii) $\bar{M}^n(4c) = \mathbb{CP}^n(4c)$, when $c > 0$,

(iii) $\bar{M}^n(4c) = \mathbb{CH}^n(4c)$, when $c < 0$.

*Supported by Tsinghua University–K.U.Leuven Bilateral scientific cooperation Fund.
†Supported by NSFC grant No. 10701007
‡Supported by NSFC grant No. 10971110
Received by the editors July 2010.
Communicated by L. Vanhecke.
2000 *Mathematics Subject Classification* : Primary 53C20; Secondary 53C42.
Key words and phrases : Lagrangian submanifolds, complex space forms, isotropic cubic tensor, isotropic submanifolds.

From the basic existence and uniqueness theorem it follows that such Lagrangian submanifolds are completely determined by the metric and the (totally symmetric) cubic form $\langle h(X,Y), JZ\rangle$. Here h denotes the second fundamental form of the immersion. In that respect, it is natural to look for Lagrangian submanifolds for which this cubic form (or the underlying second fundamental form) satisfies some geometric properties. One of the most natural properties in this viewpoint is the notion of isotropic submanifold introduced by O'Neill ([15]). A Lagrangian submanifold is called *isotropic* if and only if there exists a constant $\Lambda(p)$ such that for every unit vector $v \in T_pM^n$, $\|h(v,v)\| = \Lambda(p)$. If moreover Λ is independent of the point p, then M^n is called *constant isotropic*. Note that a 2-dimensional minimal Lagrangian surface is always isotropic. In higher dimensions, isotropic Lagrangian submanifolds were studied and completely classified in [10], [13], [14] and [18]. Studying these classifications it follows immediately that such submanifolds necessarily need to have parallel second fundamental form or are H-umbilical in the sense of [7] and [8].

Here in this paper, we study Lagrangian submanifolds M^n in complex space forms *with isotropic cubic tensor*, i.e. there exists a real function Υ on M^n such that for any unit tangent vector v at a point p we have that

$$\|(\nabla h)(v,v,v)\| = \Upsilon(p).$$

Here the *cubic tensor* means the derivative of the second fundamental form, i.e., $(\nabla h)(X,Y,Z)$, which is different from the *cubic form* $\langle h(X,Y), JZ\rangle$. Note that all parallel Lagrangian submanifolds provide trivial examples of such submanifolds.

In [17], L. Su showed that Lagrangian surfaces in 2-dimensional complex space forms with isotropic cubic tensor either have parallel mean curvature vector or are congruent to one of the Whitney spheres (or their analogs in complex hyperbolic space).

In this paper, we deal with the higher dimensional case and give a complete classification of the Lagrangian submanifolds in 3-dimensional complex space forms with isotropic cubic tensor. We will give the explicit constructions of all such examples in section 3. Our classification theorem implies in particular that some of the Lagrangian submanifolds in 3-dimensional complex space forms with isotropic cubic tensor are also isotropic submanifolds (in the sense of O'Neill). We also prove the converse, namely that any n-dimensional isotropic Lagrangian submanifold in complex space forms with $n \geq 3$ has isotropic cubic tensor. More precisely, we show the following results:

Theorem 1.1. *Let M^3 be a Lagrangian submanifold with isotropic cubic tensor of a complex space form. Assume that M^3 is nowhere parallel, then either*

(i) M^3 is congruent with a Whitney sphere in $\mathbb{C}^3$ (see (3.4)*) and $\mathbb{C}P^3$ (see* (3.5)*), or their analogs in $\mathbb{C}H^3$ (see* (3.6), (3.7) *and* (3.8)*), or*

(ii) M^3 is congruent with one of the Examples 6-10 with $n = 3$ defined in section 3 (see (3.10)-(3.14)*).*

Theorem 1.2. *The Whitney sphere in $\mathbb{C}^n$ (see* (3.4)*) and $\mathbb{C}P^n$ (see* (3.5)*), and their analogs in $\mathbb{C}H^n$ (see* (3.6), (3.7) *and* (3.8)*) are Lagrangian immersions with isotropic cubic tensor.*

Theorem 1.3. *Any n-dimensional ($n \geq 3$) H-umbilical isotropic Lagrangian submanifold of a complex space form has isotropic cubic tensor.*

The paper is organized as follows. In Section 2 we will recall the basic formulas for Lagrangian submanifolds of complex space forms. In Section 3 we will recall the construction of some examples, which are fundamental for our paper. We will show in arbitrary dimensions that all these examples have isotropic cubic tensor. In the final two sections we will deal with the converse problem. In Section 4 we will determine all possible isotropic cubic tensors in dimension 3. Finally, in Section 5, in dimension 3, we complete the proof of Theorem 1.1.

2 Preliminaries

In this section, M^n will always denote an n-dimensional Lagrangian submanifold of $\bar{M}^n(4c)$ which is an n-dimensional complex space form with constant holomorphic sectional curvature $4c$. We denote the Levi-Civita connections on M^n, $\bar{M}^n(4c)$ and the normal bundle by ∇, D and $\nabla^\perp_X$ respectively. The formulas of Gauss and Weingarten are given by (see [2], [3], [4], [5], [6])

$$D_X Y = \nabla_X Y + h(X,Y), \; D_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{2.1}$$

where X and Y are tangent vector fields and ξ is a normal vector field on M^n.

The Lagrangian condition implies that (see [6], [7], [8], [18])

$$\nabla^\perp_X JY = J\nabla_X Y, \; A_{JX} Y = -Jh(X,Y) = A_{JY} X, \tag{2.2}$$

where h is the second fundamental form and A denotes the shape operator.

We denote the curvature tensors of ∇ and $\nabla^\perp_X$ by R and $R^\perp$ respectively. The first and second covariant derivatives of h are defined by

$$\begin{aligned} (\nabla h)(X,Y,Z) &= \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y), \\ (\nabla^2 h)(X,Y,Z,W) &= \nabla^\perp_X((\nabla h)(Y,Z,W)) - (\nabla h)(\nabla_X Y, Z, W) \\ &\quad - (\nabla h)(\nabla_X Z, Y, W) - (\nabla h)(\nabla_X W, Y, Z), \end{aligned} \tag{2.3}$$

where X, Y, Z and W are tangent vector fields.

The equations of Gauss, Codazzi and Ricci for a Lagrangian submanifold of $\bar{M}^n(4c)$ are given by (see [3], [4], [6], [9])

$$\begin{aligned} \langle R(X,Y)Z,W\rangle &= \langle h(Y,Z), h(X,W)\rangle - \langle h(X,Z), h(Y,W)\rangle \\ &\quad + c\,(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle), \end{aligned} \tag{2.4}$$

$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z), \tag{2.5}$$

$$\begin{aligned} \langle R^\perp(X,Y)JZ, JW\rangle &= \langle [A_{JZ}, A_{JW}]X, Y\rangle \\ &\quad + c\,(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle), \end{aligned} \tag{2.6}$$

where X, Y Z and W are tangent vector fields. Note that for a Lagrangian submanifold the equations of Gauss and Ricci are mutually equivalent.

We have the following Ricci identity (see [14]):

$$\begin{aligned}(\nabla^2 h)(X,Y,Z,W) = (\nabla^2 h)(Y,X,Z,W) + JR(X,Y)A_{JZ}W \\ - h(R(X,Y)Z,W) - h(R(X,Y)W,Z),\end{aligned} \tag{2.7}$$

where X, Y, Z and W are tangent vector fields.

The Lagrangian condition implies that

$$\langle R^{\perp}(X,Y)JZ,JW\rangle = \langle R(X,Y)Z,W\rangle, \tag{2.8}$$

$$\langle h(X,Y),JZ\rangle = \langle h(X,Z),JY\rangle, \tag{2.9}$$

for tangent vector fields X, Y, Z and W. From (2.3) and (2.9), we also have

$$\langle(\nabla h)(W,X,Y),JZ\rangle = \langle(\nabla h)(W,X,Z),JY\rangle, \tag{2.10}$$

for tangent vector fields X, Y, Z and W.

From now on, we will also assume that M^n has an *isotropic cubic tensor*, i.e. in each point p of M^n, $\|\nabla h(v,v,v)\|$ is independent of the unit vector $v \in T_pM^n$. Hence, we obtain a function Υ on M^n by

$$\Upsilon(p) = \|\nabla h(v,v,v)\|, \tag{2.11}$$

where $v \in T_pM^n$ with $\|v\| = 1$.

We note that M^n is called an *isotropic submanifold* if at each point p of M^n, $\|h(v,v)\|$ is independent of the unit vector $v \in T_pM^n$ (see [10], [13], [15] and [18]).

3 Basic examples and the proof of Theorem 1.2, 1.3

In this section we will recall some basic examples of Lagrangian submanifolds in complex space forms. All of these examples are H-umbilical. Following [7] and [8] a Lagrangian submanifold is called *H-umbilical* if and only if there exists a local orthonormal frame $\{E_1,\ldots,E_n\}$ and differentiable functions λ and μ such that

$$h(E_1,E_1) = \lambda JE_1, \qquad h(E_1,E_i) = \mu E_i, \qquad h(E_i,E_j) = \delta_{ij}\mu E_1, \tag{3.1}$$

where $i,j > 1$. In case that the mean curvature vector does not vanish, we see that the Lagrangian submanifold is H-umbilical if and only if we can write:

$$\begin{aligned}h(X,Y) = \alpha\langle JX,H\rangle\langle JY,H\rangle H \\ + \beta\langle H,H\rangle\{\langle X,Y\rangle H + \langle JX,H\rangle JY + \langle JY,H\rangle JX\},\end{aligned} \tag{3.2}$$

for tangent vectors X, Y, Z with

$$\alpha = \frac{\lambda - 3\mu}{\gamma^3},\ \beta = \frac{\mu}{\gamma^3},\ \gamma = \frac{\lambda + (n-1)\mu}{n}.$$

Moreover, from [7] and [8], when $n \geq 3$, we have that

$$\nabla_{E_1}E_1 = 0,\ \nabla_{E_j}E_1 = \frac{E_1(\mu)}{\lambda - 2\mu}E_j,\ E_j(\lambda) = E_j(\mu) = 0,\ j \geq 2. \tag{3.3}$$

It is known, see [4] and [12], that $\lambda = 3\mu$, characterizes the Whitney spheres (or their analogs in complex hyperbolic space). They are given by:

Example 1. *Whitney sphere* in $\mathbb{C}^n$ (see [1], [2], [5], [16]). It is defined as the Lagrangian immersion of the unit sphere $\mathbb{S}^n$, centered at the origin of $\mathbb{R}^{n+1}$, in $\mathbb{C}^n$, given by

$$\phi : \mathbb{S}^n \to \mathbb{C}^n : \phi(x_1, x_2, \dots, x_n, x_{n+1}) = \frac{1 + ix_{n+1}}{1 + x_{n+1}^2}(x_1, \dots, x_n). \tag{3.4}$$

Example 2. *Whitney spheres* in $\mathbb{CP}^n$ (see [4], [12]). They are a one-parameter family of Lagrangian spheres in $\mathbb{CP}^n$, given by

$$\bar{\phi}_\theta : \mathbb{S}^n \to \mathbb{CP}^n(4) :$$

$$\bar{\phi}_\theta(x_1, x_2, \dots, x_n, x_{n+1}) = \pi \circ \Big(\frac{(x_1, \dots, x_n)}{c_\theta + is_\theta x_{n+1}}; \frac{s_\theta c_\theta(1 + x_{n+1}^2) + ix_{n+1}}{c_\theta^2 + s_\theta^2 x_{n+1}^2} \Big), \tag{3.5}$$

where $\theta > 0$, $c_\theta = \cosh\theta$, $s_\theta = \sinh\theta$, $\pi : \mathbb{S}^{2n+1}(1) \to \mathbb{CP}^n(4)$ is the Hopf fibration.

Example 3. *Whitney spheres* in $\mathbb{CH}^n$ (see [4], [12]). They are a one-parameter family of Lagrangian spheres in $\mathbb{CH}^n$, given by

$$\bar{\phi}_\theta : \mathbb{S}^n \to \mathbb{CH}^n(-4) :$$

$$\bar{\phi}_\theta(x_1, x_2, \dots, x_n, x_{n+1}) = \pi \circ \Big(\frac{(x_1, \dots, x_n)}{s_\theta + ic_\theta x_{n+1}}; \frac{s_\theta c_\theta(1 + x_{n+1}^2) - ix_{n+1}}{s_\theta^2 + c_\theta^2 x_{n+1}^2} \Big), \tag{3.6}$$

where $\theta > 0$, $c_\theta = \cosh\theta$, $s_\theta = \sinh\theta$, $\pi : \mathbb{H}_1^{2n+1}(-1) \to \mathbb{CH}^n(4)$ is the Hopf fibration.

Example 4. If $\mathbb{RH}^{n-1} = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n : y_1^2 + \dots + y_{n-1}^2 - y_n^2 = -1\}$ denotes the (n-1)-dimensional real hyperbolic space, following [4] (cf. [12]), we define a one-parameter family of Lagrangian embeddings

$$\bar{\psi}_\beta : \mathbb{S}^1 \times \mathbb{RH}^{n-1} \to \mathbb{CH}^n(-4),\ \beta \in (0, \frac{\pi}{4}],$$

given by

$$\bar{\psi}_\beta(e^{it}, y) = \pi \circ \Big(\frac{1}{\sin\beta\cos t + i\cos\beta\sin t}(\cos\beta\cos t - i\sin\beta\sin t; y) \Big), \tag{3.7}$$

where $\pi : \mathbb{H}_1^{2n+1}(-1) \to \mathbb{CH}^n(4)$ is the Hopf fibration.

Example 5. Following [4] (cf. [12]), we define a one-parameter family of Lagrangian embeddings

$$\bar{\psi}_\nu : \mathbb{R}^n = \mathbb{R}^1 \times \mathbb{R}^{n-1} \to \mathbb{CH}^n(-4),\ \nu > 0,$$

given by

$$\bar{\psi}_\nu(t, x) = \pi \circ \Big(\frac{1}{\nu + it}(\frac{2}{\nu}x, \frac{2}{\nu}e_1 - (\frac{\nu(\nu^2 + t^2)}{2} + \frac{2\|x\|^2}{\nu} + i\nu^2 t)e_2) \Big), \tag{3.8}$$

where $e_1 = \frac{1}{2}(0, \dots, 1, -1)$, $e_2 = \frac{1}{2}(0, \dots, 1, 1)$, $\pi : \mathbb{H}_1^{2n+1}(-1) \to \mathbb{CH}^n(4)$ is the Hopf fibration.

In literature, several characterizations of the Whitney spheres exist. We recall the following, which follows by combining the Main Theorem and Lemma 3.4 of [12].

Theorem 3.1. *Let M^n be as described in* (3.4)-(3.8). *Then there exists a local function κ on M^n such that the covariant derivative of the second fundamental form satisfies*

$$(\nabla h)(X,Y,Z) = \kappa(\langle Y,Z\rangle JX + \langle X,Z\rangle JY + \langle X,Y\rangle JZ). \tag{3.9}$$

Conversely any non-parallel Lagrangian submanifold satisfying the above property is congruent with a Whitney sphere.

We will now show that they always have isotropic cubic tensor.

From (3.9), for any unit vector $v \in T_pM^n$ we obtain that

$$\langle(\nabla h)(v,v,v),(\nabla h)(v,v,v)\rangle = 9\kappa^2,$$

which is independent of the choice of the unit vector v. Hence M^n has isotropic cubic tensor. This completes the proof of Theorem 1.2.

Another important class of H-umbilical Lagrangian immersions are the ones with isotropic second fundamental form. They correspond with $\lambda = -\mu$. Also in that case a complete classification has been obtained. Those which are not totally geodesic are locally described by the following examples with $n \geq 3$.

Example 6. Following [2], [13] (cf. [8]), we define a Lagrangian immersion in $\mathbb{C}^n$, given by

$$\Phi : \mathbb{S}^n \setminus \{(0,\ldots,\pm1)\} \to \mathbb{C}^n : \phi(x_1,x_2,\ldots,x_n,x_{n+1}) = \frac{1+ix_{n+1}}{1-x_{n+1}^2}(x_1,\ldots,x_n). \tag{3.10}$$

Example 7. Following [13] (cf. [7], [18]), we define a one-parameter family of Lagrangian embeddings in $\mathbb{CP}^n$, given by $\bar{\Phi} = \pi \circ \Phi$, where $\pi : \mathbb{S}^{2n+1}(1) \to \mathbb{CP}^n(4)$ is the Hopf fibration and Φ is given by the following immersion:

$$\Phi : I \times \mathbb{S}^{n-1} \to \mathbb{S}^{2n+1}(1) :$$

$$\Phi(t,y_1,\ldots,y_n) = (z_1(t), z_2(t)y_1,\ldots,z_2(t)y_n), \tag{3.11}$$

where

$$z_1 = \frac{\sinh[y(t)]\exp\{i\int_0^t \coth[y(s)]\sqrt{1-y'(s)^2}ds\}}{\cosh[y(t)]\exp\{i\int_0^t \tanh[y(s)]\sqrt{1-y'(s)^2}ds\}},$$

$$z_2 = \frac{1}{\cosh[y(t)]\exp\{i\int_0^t \tanh[y(s)]\sqrt{1-y'(s)^2}ds\}},$$

with $y(t)$ determined by

$$-2(2\cosh^2 y(t) - 3)(y'(t)^2 - 1) + y''(t)\sinh 2y(t) = 0.$$

Example 8. Following [13] (cf. [7]), we define a one-parameter family of Lagrangian embeddings in $\mathbb{CH}^n$, given by $\bar{\Phi} = \pi \circ \Phi$, where $\pi : \mathbb{H}_1^{2n+1}(-1) \to \mathbb{CH}^n(-4)$ is the Hopf fibration and Φ is given by the following immersion:

$$\Phi : I \times \mathbb{S}^{n-1} \to \mathbb{H}_1^{2n+1}(-1) :$$
$$\Phi(t, y_1, \ldots, y_n) = (z_1(t), z_2(t)y_1, \ldots, z_2(t)y_n), \tag{3.12}$$

where

$$z_1 = \frac{\cosh[y(t)] \exp\{i \int_0^t \tanh[y(s)]\sqrt{1 - y'(s)^2} ds\}}{\sinh[y(t)] \exp\{i \int_0^t \coth[y(s)]\sqrt{1 - y'(s)^2} ds\}},$$
$$z_2 = \frac{1}{\sinh[y(t)] \exp\{i \int_0^t \coth[y(s)]\sqrt{1 - y'(s)^2} ds\}},$$

with $y(t)$ determined by

$$-2(1 + 2\cosh^2 y(t))(y'(t)^2 - 1) + y''(t) \sinh 2y(t) = 0.$$

Example 9. Following [13] (cf. [7]), we define a one-parameter family of Lagrangian embeddings in $\mathbb{CH}^n$, given by $\bar{\Phi} = \pi \circ \Phi$, where $\pi : \mathbb{H}_1^{2n+1}(-1) \to \mathbb{CH}^n(-4)$ is the Hopf fibration and Φ is given by the following immersion:

$$\Phi : I \times \mathbb{H}^{n-1} \to \mathbb{H}_1^{2n+1}(-1) :$$
$$\Phi(t, y_1, \ldots, y_n) = (z_1(t)y_1, \ldots, z_1(t)y_n, z_2(t)), \tag{3.13}$$

where

$$z_1 = \frac{1}{\cos[y(t)] \exp\{-i \int_0^t \tan[y(s)]\sqrt{1 - y'(s)^2} ds\}} \,,$$
$$z_2 = \frac{\sin[y(t)] \exp\{i \int_0^t \cot[y(s)]\sqrt{1 - y'(s)^2} ds\}}{\cos[y(t)] \exp\{-i \int_0^t \tan[y(s)]\sqrt{1 - y'(s)^2} ds\}} \,,$$

with $y(t)$ determined by

$$2(2 - \cos 2y(t))(y'(t)^2 - 1) + y''(t) \sin 2y(t) = 0.$$

Example 10. Following [13] (cf. [7]), we define a one-parameter family of Lagrangian embeddings in $\mathbb{CH}^n$, given by $\bar{\Phi} = \pi \circ \Phi$, where $\pi : \mathbb{H}_1^{2n+1}(-1) \to \mathbb{CH}^n(-4)$ is the Hopf fibration and Φ is given by the following immersion:

$$\Phi : I \times \mathbb{R}^{n-1} \to \mathbb{H}_1^{2n+1}(-1) :$$

$$\begin{aligned} &\phi(t, y_2, \ldots, y_n) \\ &= \exp[\int_0^t (i\mu + k)dx]\Big(1 + \frac{1}{2}\sum_{j=2}^n y_j^2 - \int_0^t (i\mu + k)\exp(-2\int_0^x kds)dx, \\ &[i\mu(0) - k(0)][\frac{1}{2}\sum_{j=2}^n y_j^2 - \int_0^t (i\mu + k)\exp(-2\int_0^x kds)dx], y_2, \ldots, y_n\Big), \end{aligned} \tag{3.14}$$

where $\mu(t) = \frac{1}{\cosh(c \pm 3t)}$ and $k(t) = \mp \tanh(c \pm 3t)$ for some constant c.

If M^n is one of the examples 6-10, then following [7], [8], [13] and [18], we get that the second fundamental form of M^n takes the form in (3.1) with $\mu = -\lambda$, hence by (3.2) we find for arbitrary local vector fields that

$$\begin{aligned} h(U,V) = 4\lambda\langle U,E_1\rangle\langle V,E_1\rangle JE_1 - \lambda\langle U,V\rangle JE_1 - \\ \lambda\langle U,E_1\rangle JV - \lambda\langle V,E_1\rangle JU, \end{aligned} \tag{3.15}$$

where λ is a locally defined function on M^n.

Now let $p \in M^n$. We call $E_1(p) = e_1$ and let $w,\ u,\ v \in T_pM^n$ where w is orthogonal to e_1. Then, using normal coordinates, we can extend u and v to local vector fields U and V defined in the neighborhood of the point p and satisfying $\nabla_w U = \nabla_w V = 0$ at point p. By (2.3) and (3.3) we calculate that

$$\begin{aligned} (\nabla h)(w,u,v) = {} & (-\frac{4}{3}e_1(\lambda)\langle u,w\rangle\langle v,e_1\rangle - \frac{4}{3}e_1(\lambda)\langle v,w\rangle\langle u,e_1\rangle)Je_1 \\ & + \frac{1}{3}e_1(\lambda)\langle u,w\rangle Jv + \frac{1}{3}e_1(\lambda)\langle v,w\rangle Ju \\ & + (\frac{1}{3}e_1(\lambda)\langle u,v\rangle - \frac{4}{3}e_1(\lambda)\langle u,e_1\rangle\langle v,e_1\rangle)Jw, \end{aligned} \tag{3.16}$$

which gives

$$\begin{cases} (\nabla h)(u,u,u) = e_1(\lambda)Ju,\ (\nabla h)(u,u,e_1) = -e_1(\lambda)Je_1, \\ (\nabla h)(u,e_1,e_1) = -e_1(\lambda)Ju,\ u \perp e_1, \|u\| = 1,\ u \in T_pM. \end{cases} \tag{3.17}$$

By (3.15) we have $h(E_1,E_1) = \lambda JE_1$, hence by (3.3) we can calculate that

$$(\nabla h)(e_1,e_1,e_1) = e_1(\lambda)Je_1. \tag{3.18}$$

Let v be an arbitrary unit vector in T_pM^n, if v is neither parallel nor orthogonal with e_1, then there exists a unit vector u which is orthogonal to e_1 such that $v = \cos\theta e_1 + \sin\theta u$, hence by (3.17) and (3.18) we can calculate that

$$(\nabla h)(v,v,v) = e_1(\lambda)(\cos 3\theta Je_1 - \sin 3\theta Ju). \tag{3.19}$$

(3.17), (3.18) and (3.19) imply that M^n has isotropic cubic tensor. Hence we have shown that an H-umbilical isotropic Lagrangian submanifold of a complex space form also has isotropic cubic tensor. This completes the proof of Theorem 1.3.

Remark 3.2. In [18], L. Vrancken gave a complete classification of the isotropic Lagrangian submanifolds in complex projective space $\mathbb{CP}^n$ (see also [10]), $n \geq 3$. Similarly the isotropic Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{CH}^n$ with $n \geq 3$, were classified by H. Li and X. Wang in [13] (see also [10]). Combining these results, we get that if M^n is an isotropic Lagrangian submanifold of $\bar{M}^n(4c)$ with $n \geq 3$ then M^n is totally geodesic or has parallel second fundamental form or is H-umbilical and therefore congruent to one of the examples 6-10. In all cases it follows that M^n also must have isotropic cubic tensor. This shows the following theorem.

Theorem 3.3. *Let M^n be an n-dimensional ($n \geq 3$) Lagrangian submanifold of a complex space form. If M^n has isotropic second fundamental form, then M^n also has isotropic cubic tensor.*

4 Some lemmas

From now on we will always assume that M^n is a Lagrangian submanifold of $\bar{M}^n(4c)$ with isotropic cubic tensor, where $\bar{M}^n(4c)$ is an n-dimensional complex space form with constant holomorphic sectional curvature $4c$. We will also assume that M^n is not a parallel Lagrangian submanifold, i.e. we will assume that $\Upsilon \neq 0$. We have the following lemma.

Lemma 4.1. *Let M^n be a Lagrangian submanifold of $\bar{M}^n(4c)$ with isotropic cubic tensor. Let $f(v) = \langle \nabla h(v,v,v), Jv \rangle$ be a function on the unit tangent bundle,i.e. on $UM_p^n = \{v \in T_pM^n \mid \|v\| = 1\}$. Let e_1 denote a vector where f attains its maximum with $f(e_1) = a_1$. Then $|a_1| = \Upsilon(p)$ and for any u, a unit vector which is orthogonal to e_1, we have*

(i) $\langle (\nabla h)(e_1, e_1, e_1), Ju \rangle = 0.$

(ii) $-a_1 + 3\langle (\nabla h)(e_1, e_1, u), Ju \rangle \leq 0$. *Moreover, if the equality holds we must have* $\langle (\nabla h)(u,u,u), Je_1 \rangle = 0.$

Proof. Let

$g(t) = \langle (\nabla h)(e_1 \cos t + u \sin t, e_1 \cos t + u \sin t, e_1 \cos t + u \sin t), J(e_1 \cos t + u \sin t) \rangle.$

As f attains its maximum at the vector e_1, with $f(e_1) = a_1$, we see that $g(t)$ attains its maximum value at $t = 0$, which implies that

$$g'(0) = 4\langle (\nabla h)(e_1, e_1, e_1), Ju \rangle = 0, \tag{4.1}$$

$$g''(0) = 4(-\langle (\nabla h)(e_1, e_1, e_1), Je_1 \rangle + 3\langle (\nabla h)(e_1, e_1, u), Ju \rangle) \leq 0. \tag{4.2}$$

(4.1) implies (i). Moreover, if the equality holds in (4.2), we must have $g'''(0) = 0$. Using (4.1), we obtain $g'''(0) = 24\langle (\nabla h)(u,u,u), Je_1 \rangle$, from which (ii) follows.

From (4.1) we have that $(\nabla h)(e_1, e_1, e_1)$ and Je_1 are parallel. Since $f(e_1) = a_1$, which implies that $(\nabla h)(e_1, e_1, e_1) = a_1 Je_1$, we obtain that $\Upsilon(p) = |a_1|$. ∎

We now define a linear operator A on T_pM^n by defining

$$A(v) \triangleq -J(\nabla h)(e_1, e_1, v).$$

Note that from (2.10) it follows that A is a symmetric operator. Also, we know that e_1 is an eigenvector of A with eigenvalue a_1. Therefore, we can choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM^n which diagonalizes A, i.e. $A(e_i) = a_i e_i$, which means $(\nabla h)(e_1, e_1, e_i) = a_i Je_i$.

Let $G(t) = \|(\nabla h)(e_1 \cos t + e_i \sin t, e_1 \cos t + e_i \sin t, e_1 \cos t + e_i \sin t)\|^2$, $i > 1$, we get

$$\begin{aligned} G(t) =& \|(\nabla h)(e_1, e_1, e_1) \cos^3 t + 3(\nabla h)(e_1, e_1, e_i) \cos^2 t \sin t \\ &+ 3(\nabla h)(e_1, e_i, e_i) \sin^2 t \cos t + (\nabla h)(e_i, e_i, e_i) \sin^3 t\|^2 \\ =& a_1^2 \cos^6 t + a_1^2 \sin^6 t + (9a_i^2 + 6a_1 a_i) \cos^4 t \sin^2 t \\ &+ (2a_1 + 18a_i)\langle (\nabla h)(e_i, e_i, e_i), Je_1 \rangle \cos^3 t \sin^3 t \\ &+ (9\|(\nabla h)(e_1, e_i, e_i)\|^2 + 6a_i f(e_i)) \cos^2 t \sin^4 t \\ &+ 6\langle (\nabla h)(e_1, e_i, e_i), (\nabla h)(e_i, e_i, e_i) \rangle \cos t \sin^5 t. \end{aligned} \tag{4.3}$$

On the other hand, since M^n has isotropic cubic tensor we have

$$\begin{aligned} G(t) \equiv & a_1^2 \equiv a_1^2(\cos^2 t + \sin^2 t)^3 \\ \equiv & a_1^2(\cos^6 t + 3\cos^4 t \sin^2 t + 3\cos^2 t \sin^4 t + \sin^6 t) \end{aligned} \tag{4.4}$$

By comparing the coefficients in (4.3) and (4.4), we get

$$a_1^2 = 3a_i^2 + 2a_1 a_i, \tag{4.5}$$

$$(a_1 + 9a_i)\langle(\nabla h)(e_i, e_i, e_i), Je_1\rangle = 0, \tag{4.6}$$

$$a_1^2 = 3\|(\nabla h)(e_1, e_i, e_i)\|^2 + 2a_i f(e_i), \tag{4.7}$$

$$\langle(\nabla h)(e_1, e_i, e_i), (\nabla h)(e_i, e_i, e_i)\rangle = 0. \tag{4.8}$$

From (4.5) we have a_i is either $\frac{1}{3}a_1$ or $-a_1$, hence by use of (4.6) we obtain that $a_1 = 0$ or

$$\langle(\nabla h)(e_i, e_i, e_i), Je_1\rangle = 0, \ i > 1. \tag{4.9}$$

From now on, we will assume that $n = 3$.

Proposition 4.2. *Let M^3 be a 3-dimensional Lagrangian submanifold of $\bar{M}^3(4c)$ with isotropic cubic tensor. Let $p \in M^3$, then either*

(i) ∇h vanishes identically at the point p, or

(ii) ∇h takes the following form

$$\begin{aligned} (\nabla h)(u, v, w) =& \tfrac{1}{3}a_1(\langle v, w\rangle Ju + \langle u, w\rangle Jv + \langle u, v\rangle Jw), \\ & \forall\, u, v, w \in T_pM^3, \end{aligned} \tag{4.10}$$

or

(iii) there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of T_pM^3 such that ∇h takes the following form

$$\begin{cases} (\nabla h)(e_i, e_i, e_i) = a_1 Je_i, \ (\nabla h)(e_1, e_1, e_j) = -a_1 Je_j, \\ (\nabla h)(e_1, e_j, e_j) = -a_1 Je_1, \ (\nabla h)(e_1, e_2, e_3) = 0, \\ (\nabla h)(e_j, e_j, e_k) = \dfrac{1}{3}a_1 Je_k, \ i = 1, 2, 3, \ 2 \leq j \neq k \leq 3, \ a_1 \neq 0. \end{cases} \tag{4.11}$$

Proof. When $a_1 = 0$, by the choice of e_1 and the fact that $\langle(\nabla h)(X, Y, Z), JW\rangle$ is totally symmetric, we immediately get that M^3 has parallel second fundamental form.

When $a_1 \neq 0$, we have a_i is either $\frac{1}{3}a_1$ or $-a_1$, so we need to consider three cases.

Case (i): $a_2 = a_3 = \frac{1}{3}a_1$.
Let $L = span\{e_2, e_3\}$, by (4.9) we have

$$\langle(\nabla h)(e_i, e_i, e_i), Je_1\rangle = 0, \ i = 2, 3,$$

so by linearization using the symmetry of ∇h we have

$$\langle(\nabla h)(u,v,w), Je_1\rangle = 0,\ \forall\ u,\ v,\ w \in L,$$

which implies

$$\langle(\nabla h)(e_1,e_i,e_i), Je_j\rangle = 0,\ i,j = 2,3.$$

We also have $\langle(\nabla h)(e_1,e_i,e_i), Je_1\rangle = a_i$, hence we get

$$(\nabla h)(e_1,e_i,e_i) = \frac{1}{3}a_1 Je_1,\ i = 2,3. \tag{4.12}$$

By (4.7) and (4.12) we have

$$(\nabla h)(e_i,e_i,e_i) = a_1 Je_i,\ i = 2,3. \tag{4.13}$$

Since L is the eigenspace of A with eigenvalue $\frac{1}{3}a_1$, we have that the previous formulas are not only valid for e_2, e_3 but for any unit vector v in L. So we get

$$(\nabla h)(e_1,v,v) = \frac{1}{3}a_1\|v\|^2 Je_1,\ \forall\ v \in L, \tag{4.14}$$

and

$$(\nabla h)(v,v,v) = a_1\|v\|^2 Jv,\ \forall\ v \in L. \tag{4.15}$$

Let $v = \frac{1}{\sqrt{2}}(e_2 \pm e_3)$, we get

$$\begin{aligned}(\nabla h)(v,v,v) = \frac{1}{2\sqrt{2}}&((\nabla h)(e_2,e_2,e_2) \pm (\nabla h)(e_3,e_3,e_3)\\ &\pm 3(\nabla h)(e_2,e_2,e_3) + 3(\nabla h)(e_2,e_3,e_3)).\end{aligned} \tag{4.16}$$

(4.15) and (4.16) imply that

$$(\nabla h)(e_j,e_j,e_k) = \frac{1}{3}a_1 Je_k,\ 2 \le j \ne k \le 3. \tag{4.17}$$

Combining all the formulas above, we get

$$\begin{cases}(\nabla h)(e_i,e_i,e_i) = a_1 Je_i,\\ (\nabla h)(e_i,e_i,e_j) = \frac{1}{3}a_1 Je_j,\\ (\nabla h)(e_1,e_2,e_3) = 0,\ 1 \le i \ne j \le 3.\end{cases} \tag{4.18}$$

Using (4.18), we see that for $u,v,w \in \{e_1,e_2,e_3\}$ we get

$$(\nabla h)(u,v,w) = \tfrac{1}{3}a_1(\langle u,v\rangle Jw + \langle v,w\rangle Ju + \langle u,w\rangle Jv\rangle).$$

As two tensors which coincide on a basis, coincide for any tangent vectors, we complete the proof in this case.

Case (ii): $a_2 = \frac{1}{3}a_1,\ a_3 = -a_1$. Since a_1 is the maximum value of f, by (4.2) we have $-a_1 \le -3a_3 = 3a_1$, which implies that $a_1 \ge 0$. As we assume that $a_1 \ne 0$, we get $a_1 > 0$. In this case, we have

$$\begin{cases}(\nabla h)(e_1,e_1,e_2) = \frac{1}{3}a_1 Je_2,\\ (\nabla h)(e_1,e_1,e_3) = -a_1 Je_3,\end{cases} \tag{4.19}$$

(4.9), (4.19) and lemma 4.1 imply

$$\begin{cases} \langle(\nabla h)(e_1,e_2,e_2),Je_1\rangle = \dfrac{1}{3}a_1, \\ \langle(\nabla h)(e_1,e_j,e_j),Je_j\rangle = 0,\ j=2,3, \\ \langle(\nabla h)(e_1,e_3,e_3),Je_1\rangle = -a_1. \end{cases} \tag{4.20}$$

We assume that

$$\begin{cases} \langle(\nabla h)(e_1,e_2,e_2),Je_3\rangle = \alpha, \\ \langle(\nabla h)(e_1,e_3,e_3),Je_2\rangle = \beta, \end{cases} \tag{4.21}$$

by (4.7) we have

$$\begin{cases} f(e_2) = (1-\dfrac{9\alpha^2}{2a_1^2})a_1, \\ f(e_3) = (1+\dfrac{3\beta^2}{2a_1^2})a_1 \le a_1, \end{cases} \tag{4.22}$$

so we get $\beta = 0$ and $f(e_3) = a_1$, which means that the function also attains a maximum at e_3. Consequently we also obtain that

$$\begin{cases} \langle(\nabla h)(e_3,e_3,e_3),Je_i\rangle = 0, \\ \langle(\nabla h)(e_i,e_i,e_i),Je_3\rangle = 0,\ i=1,2. \end{cases} \tag{4.23}$$

From (4.21) and (4.23) we get e_2 is an eigenvector of the operator $-J(\nabla h)(e_3,e_3,-)$, so we can assume

$$(\nabla h)(e_3,e_3,e_2) = bJe_2, \tag{4.24}$$

with $b=\frac{1}{3}a_1$ or $b=-a_1$ since e_3 is also a maximum vector.

(4.20) and (4.23) imply that $(\nabla h)(e_2,e_2,e_2)$ is parallel with Je_2, so we get

$$(\nabla h)(e_2,e_2,e_2) = \pm a_1 Je_2. \tag{4.25}$$

(4.22) and (4.25) imply that $\alpha = 0$ or $\alpha = \pm\frac{2}{3}a_1$, noting that we can always change the direction of e_3 to make $\alpha \ge 0$ without changing the other components of ∇h, so we have to consider four subcases.

Case (ii-1): $\alpha = 0,\ b = \frac{1}{3}a_1$.

In this case, e_1, e_2 and e_3 are all maximum directions of f and ∇h takes the following form

$$\begin{cases} (\nabla h)(e_i,e_i,e_i) = a_1Je_i,\ (\nabla h)(e_1,e_2,e_3) = 0, \\ (\nabla h)(e_1,e_1,e_2) = \dfrac{1}{3}a_1Je_2,\ (\nabla h)(e_1,e_1,e_3) = -a_1Je_3, \\ (\nabla h)(e_1,e_2,e_2) = \dfrac{1}{3}a_1Je_1,\ (\nabla h)(e_1,e_3,e_3) = -a_1Je_1, \\ (\nabla h)(e_j,e_j,e_k) = \dfrac{1}{3}a_1Je_k,\ i=1,2,3,\ 2\le j\ne k\le 3. \end{cases} \tag{4.26}$$

Let $y = y_1e_1+y_2e_2+y_3e_3$, $y_1^2+y_2^2+y_3^2=1$, by using (4.26) we have

$$\|(\nabla h)(y,y,y)\|^2 = a_1^2(y_1^6+3y_1^4(y_2^2+y_3^2)+(y_2^2+y_3^2)^3+3y_1^2(\ y_2^4+18y_2^2y_3^2+y_3^4)), \tag{4.27}$$

we also have

$$\|(\nabla h)(y,y,y)\|^2 \equiv a_1^2(y_1^2+y_2^2+y_3^2)^3. \tag{4.28}$$

By comparing the coefficients of $y_1^2y_2^2y_3^2$ in (4.27) and (4.28), we get a contradiction, so this case can't happen.

Case (ii-2): $\alpha = 0,\ b = -a_1$.

In this case, e_1, e_2 and e_3 are all maximum directions of f and ∇h takes the following form

$$\begin{cases} (\nabla h)(e_i,e_i,e_i) = a_1Je_i,\ (\nabla h)(e_1,e_2,e_3) = 0, \\ (\nabla h)(e_3,e_3,e_j) = -a_1Je_j,\ (\nabla h)(e_3,e_j,e_j) = -a_1Je_3, \\ (\nabla h)(e_j,e_j,e_k) = \dfrac{1}{3}a_1Je_k,\ i=1,2,3,\ 1\le j\ne k\le 2. \end{cases} \tag{4.29}$$

If we take $\tilde e_1 = e_3,\ \tilde e_2 = e_2,\ \tilde e_3 = e_1,\ \tilde a_1 = a_1$, then we get ∇h in case (ii-2) takes the same form with (4.11).

Case (ii-3): $\alpha = \frac{2}{3}a_1,\ b = \frac{1}{3}a_1$.

In this case, e_1 and e_3 are both maximum directions of f, e_2 is a minimal direction of f and ∇h takes the following form

$$\begin{cases} (\nabla h)(e_i,e_i,e_i) = (-1)^{i+1}a_1Je_1,\ i=1,2,3;\ (\nabla h)(e_1,e_2,e_3) = \dfrac{2}{3}a_1Je_2, \\ (\nabla h)(e_1,e_1,e_2) = \dfrac{1}{3}a_1Je_2,\ (\nabla h)(e_1,e_1,e_3) = -a_1Je_3, \\ (\nabla h)(e_1,e_2,e_2) = \dfrac{1}{3}a_1Je_1 + \dfrac{2}{3}a_1Je_3,\ (\nabla h)(e_1,e_3,e_3) = -a_1Je_1, \\ (\nabla h)(e_2,e_2,e_3) = \dfrac{1}{3}a_1Je_3 + \dfrac{2}{3}a_1Je_1,\ (\nabla h)(e_2,e_3,e_3) = \dfrac{1}{3}a_1Je_2. \end{cases} \tag{4.30}$$

If we take $\tilde e_1 = (e_1+e_3)/\sqrt{2},\ \tilde e_2 = e_2,\ \tilde e_3 = (e_1-e_3)/\sqrt{2},\ \tilde a_1 = -a_1$, we obtain that ∇h in case (ii-3) takes the same form with (4.11).

Case (ii-4): $\alpha = \frac{2}{3}a_1,\ b = -a_1$.

In this case, we get e_2 is a minimal direction, by an analogous argument as for e_1 we can obtain that the eigenvalue for the operator $-J(\nabla h)(e_2,e_2,-)$ is either a_1 or $-\frac{1}{3}a_1$.

However, using (4.19), (4.21) and (4.24), we have the following equations

$$\begin{cases} (\nabla h)(e_2,e_2,e_1) = \dfrac{1}{3}a_1Je_1 + \dfrac{2}{3}a_1Je_3, \\ (\nabla h)(e_2,e_2,e_3) = -a_1Je_3 + \dfrac{2}{3}a_1Je_1, \end{cases} \tag{4.31}$$

which implies the eigenvalue of the operator $-J(\nabla h)(e_2,e_2,-)$ is neither a_1 nor $-\frac{1}{3}a_1$. So this case can't happen.

Case (iii): $a_2 = a_3 = -a_1$.

In this case, by a analogous argument with case (ii) we get e_1, e_2 and e_3 are all maximum directions of f and

$$\begin{cases} (\nabla h)(e_i,e_i,e_i) = a_1 Je_i, \ (\nabla h)(e_1,e_2,e_3) = 0, \\ (\nabla h)(e_1,e_1,e_j) = -a_1 Je_j, \ (\nabla h)(e_1,e_j,e_j) = -a_1 Je_1, \\ (\nabla h)(e_j,e_j,e_k) = cJe_k, \ i = 1,2,3, \ 2 \leq j \neq k \leq 3, \end{cases} \tag{4.32}$$

with $c = -a_1$ or $c = \frac{1}{3}a_1$.

If $c = -a_1$, we can easily get contradiction after a similar argument with case (ii-1).

If $c = \frac{1}{3}a_1$, we deduce that ∇h takes the form in (4.11). This completes the proof of Proposition 4.2. ∎

5 Proof of Theorem 1.1

By Proposition 4.2 and the assumption that there are no points for which (∇h) vanishes identically, a continuity argument using $\|(\nabla h)\|$ implies that we deduce that either (4.10) holds everywhere on M^3 or ∇h takes the form as (4.11) everywhere on M^3. In the first case, by the results of [12], see also Theorem 3.1 we obtain that M^3 is locally isometric with a Whitney sphere (or their analogs in complex hyperbolic space).

Hence we may assume that (4.11) holds everywhere on M^3. Let $\{F_1, F_2, F_3\}$ be an arbitrary orthonormal frame defined in a neighborhood of the point p and let $\{f_1, f_2, f_3\}$ denote the corresponding orthonormal basis at the point p. Then, if we denote by $\vec{H}$ the mean curvature vector, it follows that at the point p we have

$$\begin{aligned} \nabla^{\perp}_v \vec{H} &= \tfrac{1}{3}\sum_{i=1}^{3} \nabla^{\perp}_v h(F_i, F_i) \\ &= \tfrac{1}{3}\sum_{i=1}^{3}(\nabla h)(v, F_i, F_i) + \tfrac{2}{3}\sum_{i,j=1}^{3} \langle \nabla_v F_i, F_j\rangle h(F_j, F_i) \\ &= \tfrac{1}{3}\sum_{i=1}^{3}(\nabla h)(v, f_i, f_i). \end{aligned}$$

From this and (4.11) we see that e_1 is characterized as belonging to a 1-dimensional eigenspace of $-J\nabla^{\perp}\vec{H}$ with eigenvalue $-\frac{1}{3}a_1$. We also see that the eigenspaces of this operator have constant dimensions. Hence using the classical theorem of Kobayashi and Nomizu (see page 38 of [11]), we see that e_1 can be extended to a differentiable vector field E_1 on a neighborhood of p such that at each point the function f attains a maximum at e_1. Note that in order to have the form (4.11) on a neighborhood it is now sufficient to take local orthonormal vector fields E_2 and E_3 orthogonal to E_1. Note that we still have some rotational freedom. Indeed rotating the vector fields E_2 and E_3 over an angle θ (where θ is a local function) preserves the expression (4.11). So we have

$$\begin{cases} (\nabla h)(E_i,E_i,E_i) = a_1 JE_i, (\nabla h)(E_1,E_1,E_j) = -a_1 JE_j, \\ (\nabla h)(E_1,E_j,E_j) = -a_1 JE_1, (\nabla h)(E_1,E_2,E_3) = 0, \\ (\nabla h)(E_j,E_j,E_k) = \dfrac{1}{3}a_1 JE_k, \ i = 1,2,3, \ 2 \leq j \neq k \leq 3, \ a_1 \neq 0. \end{cases} \tag{5.1}$$

We now write

$$\begin{cases} h(E_1,E_1) = x_1JE_1 + x_2JE_2 + x_3JE_3, \\ h(E_1,E_2) = x_2JE_1 + y_1JE_2 + y_3JE_3, \\ h(E_1,E_3) = x_3JE_1 + y_3JE_2 + y_2JE_3, \\ h(E_2,E_2) = y_1JE_1 + z_1JE_2 + z_2JE_3, \\ h(E_2,E_3) = y_3JE_1 + z_2JE_2 + z_3JE_3, \\ h(E_3,E_3) = y_2JE_1 + z_3JE_2 + z_4JE_3. \end{cases} \tag{5.2}$$

$$\begin{cases} \nabla_{E_1}E_1 = z_{112}E_2 + z_{113}E_3,\ \nabla_{E_1}E_2 = -z_{112}E_1 + z_{123}E_3, \\ \nabla_{E_1}E_3 = -z_{113}E_1 - z_{123}E_2,\ \nabla_{E_2}E_1 = z_{212}E_2 + z_{213}E_3, \\ \nabla_{E_2}E_2 = -z_{212}E_1 + z_{223}E_3,\ \nabla_{E_2}E_3 = -z_{213}E_1 - z_{223}E_2, \\ \nabla_{E_3}E_1 = z_{312}E_2 + z_{313}E_3,\ \nabla_{E_3}E_2 = -z_{312}E_1 + z_{323}E_3, \\ \nabla_{E_3}E_3 = -z_{313}E_1 - z_{323}E_2. \end{cases} \tag{5.3}$$

$$\begin{cases} \langle R(E_1,E_2)E_1,E_2\rangle = b_1,\ \langle R(E_1,E_2)E_1,E_3\rangle = b_2, \\ \langle R(E_1,E_3)E_1,E_3\rangle = b_3,\ \langle R(E_1,E_2)E_2,E_3\rangle = b_4, \\ \langle R(E_1,E_3)E_3,E_2\rangle = b_5,\ \langle R(E_2,E_3)E_2,E_3\rangle = b_6. \end{cases} \tag{5.4}$$

As mentioned before, from (5.1), if we choose another basis $\{\tilde{E}_2,\tilde{E}_3\}$ by rotating $\{E_2,E_3\}$, we preserve the form of ∇h. By making such a rotation we may always assume that $y_3 = 0$. Note that if moreover, $y_1 = y_2$ on an open set, we recover again the same rotation freedom. So in that case we may assume that $x_2 = 0$. From now on we will always assume that we have made the appropriate rotations. We define

$$\begin{aligned} S_{ijkl} \triangleq &(\nabla^2 h)(E_i,E_j,E_k,E_l) - (\nabla^2 h)(E_j,E_i,E_k,E_l) - JR(E_i,E_j)A_{JE_k}E_l \\ &+ h(R(E_i,E_j)E_k,E_l) + h(R(E_i,E_j)E_l,E_k). \end{aligned} \tag{5.5}$$

By (2.7) we have $S_{ijkl} \equiv 0$. Let $i = k = 1,\ j = l = 2$ in (5.5), from (5.1)-(5.4) we get

$$z_{113} = \frac{-3(b_4(y_1-y_2) + b_1(x_3-z_2) + b_2(x_2-z_3))}{4a_1}. \tag{5.6}$$

Let $i = 1,\ j = k = l = 2$ in (5.5), we get

$$z_{212} = \frac{3b_1y_1 - 3b_4z_2 - E_1(a_1)}{4a_1}. \tag{5.7}$$

Let $i = 1,\ j = 2,\ k = l = 3$ in (5.5), we get

$$z_{213} = \frac{3(b_2y_2 + b_4z_3)}{4a_1}. \tag{5.8}$$

Let $i = k = 1,\ j = l = 3$ in (5.5), we get

$$z_{112} = \frac{3(b_5(y_1-y_2) + b_2(-x_3+z_2) + b_3(-x_2+z_3))}{4a_1}. \tag{5.9}$$

Let $i=1,\ j=3,\ k=l=2$ in (5.5), we get

$$z_{312}=\frac{3(b_2y_1+b_5z_2)}{4a_1}. \tag{5.10}$$

Let $i=1,\ j=k=l=3$ in (5.5), we get

$$z_{313}=\frac{3b_3y_2-3b_5z_3-E_1(a_1)}{4a_1}. \tag{5.11}$$

Let $i=2,\ j=3,\ k=l=1$ in (5.5), we get

$$E_2(a_1)=-b_6x_2+b_5(x_1-2y_2)+2b_4y_3,\ E_3(a_1)=-b_6x_3+b_4(x_1-2y_1)+2b_5y_3. \tag{5.12}$$

By using (5.6)-(5.12), we obtain $S_{ijkl}\equiv 0$ is equivalent to the following 22 equations:

$$\begin{cases} eq1:=(3b_1-3b_3+b_6)x_2-b_5(x_1-3y_1+y_2)+3b_2z_2+3b_3z_3=0,\\ eq2:=-b_1(x_1+y_1)+b_4(x_3+3z_2)=0,\\ eq3:=b_2(x_1+y_2)+b_4(x_2+3z_3)=0,\\ eq4:=3b_3x_2-b_6x_2+3b_2x_3+b_5(x_1-3y_1+y_2)\\ \qquad +b_1(-2x_2+z_1)-2b_2z_2-3b_3z_3=0,\\ eq5:=-b_6x_2+b_5(x_1-y_1-y_2)+b_3(x_2-z_3)+b_1z_3-b_2(x_3+z_2-z_4)=0,\\ eq6:=-b_2y_1+b_2y_2-b_4z_1+3b_4z_3=0,\\ eq7:=b_1(y_1-y_2)+b_4(-3z_2+z_4)=0,\\ eq8:=-3b_1x_3+3b_3x_3+b_6x_3-b_4(x_1+y_1-3y_2)+3b_1z_2+3b_2z_3=0,\\ eq9:=b_2(x_1+y_1)+b_5(x_3+3z_2)=0,\\ eq10:=-b_3(x_1+y_2)+b_5(x_2+3z_3)=0,\\ eq11:=b_1x_3-b_6x_3+b_4(x_1-y_1-y_2)-b_1z_2+b_3z_2-b_2(x_2-z_1+z_3)=0,\\ eq12:=3b_2x_2+3b_1x_3-2b_3x_3-b_6x_3+b_4(x_1+y_1-3y_2)\\ \qquad -3b_1z_2-2b_2z_3+b_3z_4=0,\\ eq13:=b_3(-y_1+y_2)+b_5(z_1-3z_3)=0,\\ eq14:=b_2y_1-b_2y_2+3b_5z_2-b_5z_4=0,\\ eq15:=b_2(-y_1+y_2)-b_5(x_3+z_2)+b_4(x_2+z_3)=0,\\ eq16:=3b_2y_1-b_2y_2+2b_5z_2+b_4(-2x_2+z_1-z_3)=0,\\ eq17:=-(b_1+b_6)y_1+(b_3+b_6)y_2-b_4(x_3-2z_2)+b_5(x_2-2z_3)=0,\\ eq18:=b_2(y_1-3y_2)-2b_4z_3+b_5(2x_3+z_2-z_4)=0,\\ eq19:=-b_4(x_1+y_1)+b_6(x_3+3z_2)=0,\\ eq20:=b_5(x_1+3y_1-2y_2)-b_6(x_2+3z_1-6z_3)=0,\\ eq21:=-b_4(x_1-2y_1+3y_2)+b_6(x_3-6z_2+3z_4)=0,\\ eq22:=b_5(x_1+y_2)-b_6(x_2+3z_3)=0. \end{cases} \tag{5.13}$$

By the Gauss equation (2.4) we get moreover that

$$\begin{cases} b_1 = -c + x_2^2 - x_1 y_1 + y_1^2 - x_2 z_1 - x_3 z_2, \\ b_2 = x_2(x_3 - z_2) - x_3 z_3, \\ b_3 = -c + x_3^2 - x_1 y_2 + y_2^2 - x_2 z_3 - x_3 z_4, \\ b_4 = x_3 y_1 + (-y_1 + y_2) z_2, \\ b_5 = x_2 y_2 + (y_1 - y_2) z_3, \\ b_6 = -c - y_1 y_2 + z_2^2 - z_1 z_3 + z_3^2 - z_2 z_4. \end{cases} \tag{5.14}$$

First we look at the case that $b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 = 0$, i.e. $b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = 0$. Assume that this is true in a neighborhood of the point p. Hence M^3 is flat, i.e. $\langle R(E_i, E_j)E_k, E_l \rangle = 0$, $\forall i,\ j,\ k,\ l = 1,\ 2,\ 3$. By (2.4) we get

$$\begin{aligned} &\langle h(E_i, E_k), h(E_j, E_l) \rangle - \langle h(E_i, E_l), h(E_j, E_k) \rangle \\ &+ c(\langle E_i, E_k \rangle \langle E_j, E_l \rangle - \langle E_i, E_l \rangle \langle E_j, E_k \rangle) = 0,\ \forall i,\ j,\ k,\ l = 1,\ 2,\ 3, \end{aligned} \tag{5.15}$$

which implies

$$\begin{aligned} &\langle \nabla h(E_i, E_j, E_k), h(E_l, E_m) \rangle + \langle \nabla h(E_i, E_l, E_m), h(E_j, E_k) \rangle \\ &- \langle \nabla h(E_i, E_k, E_l), h(E_j, E_m) \rangle - \langle \nabla h(E_i, E_j, E_m), h(E_k, E_l) \rangle = 0, \\ &\forall\ i,\ j,\ k,\ l,\ m = 1,\ 2,\ 3. \end{aligned} \tag{5.16}$$

We insert (5.1) and (5.2) into (5.16) and obtain $x_1 = x_2 = x_3 = y_1 = y_2 = z_1 = z_2 = z_3 = z_4 = 0$, in a neighborhood of p. Hence M^3 is totally geodesic, and therefore also has parallel second fundamental form. This is a contradiction with $a_1 \neq 0$.

When $b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 \neq 0$, we can look at the system (5.13) as a system of linear equations in the variables $x_1,\ x_2,\ x_3,\ y_1,\ y_2,\ z_1,\ z_2,\ z_3, z_4$ under the condition (5.14). Note that if $x_1 = x_2 = x_3 = y_1 = y_2 = z_1 = z_2 = z_3 = z_4 = 0$, M^3 is totally geodesic which is a contradiction. Henceforth we are only interested in nontrivial solutions. Note also that if $y_1 = y_2$ (on an open set), by the choice of frame we also must have that $x_2 = 0$.

Using the Reduce command of Mathematica, more precisely using

$$\begin{aligned} &Reduce[\{eq2 == 0, eq3 == 0, eq6 == 0, eq7 == 0, eq19 == 0, eq22 == 0, \\ &eq20 == 0, eq21 == 0, eq9 == 0, eq10 == 0, eq13 == 0, eq14 == 0, \\ &eq15 == 0, eq18 == 0, eq17 == 0, eq16 == 0, eq8 == 0, eq11 == 0, \\ &eq12 == 0, eq4 == 0, eq5 == 0, eq1 == 0\}, \{x1, y1, y2, x2, x3, z1, z2, z3, z4\}] \end{aligned}$$

which is particularly adapted for solving a system of linear equations with parameters, we find that the system (5.13) only has nontrivial real solutions in the following cases:

(i) $b_5 = b_4 = b_2 = 0$, $b_1 = b_3 = b_6 \neq 0$, $y_1 = y_2 = -x_1$, $x_2 = 0$, $z_1 = z_3 = 0$, $z_2 = -\frac{1}{3}x_3$ and $z_4 = -x_3$,

(ii) $b_5 = b_4 = b_3 = b_2 = b_1 = 0$, $b_6 \neq 0$, $y_1 = y_2$, $x_2 = x_3 = 0$, $z_1 = z_2 = z_3 = z_4 = 0$,

(iii) $b_6 = b_5 = b_4 = b_2 = b_1 = 0$, $b_3 \neq 0$, $y_1 = y_2 = x_1 = 0$, $x_2 = x_3 = 0$, $z_2 = z_3 = z_4 = 0$,

(iv) $b_6 = b_5 = b_4 = b_3 = b_2 = 0$, $b_1 \neq 0$, $y_1 = y_2 = x_1 = 0$, $x_2 = 0$, $z_1 = z_3 = 0$, $z_2 = x_3$,

(v) $b_5 = b_3 = b_2 = 0$, $b_1 b_6 = b_4^2$, $b_6 \neq 0 \neq b_4$, $y_2 = \frac{b_1^2 x_1 - 3b_1^2 y_1 + b_4^2 y_1}{b_4^2}$, $x_2 = 0$, $x_3 = \frac{b_1 x_1 - 2b_1 y_1}{b_4}$, $z_1 = z_3 = 0$, $z_2 = \frac{b_1 x_3 + b_4 y_1 - b_4 y_2}{b_1}$ and $z_4 = \frac{2b_1 y_1 + b_1 y_2}{b_4}$,

(vi) $b_5 = b_4 = b_2 = 0$, $b_1 = b_3 = -b_6 \neq 0$, $y_1 = y_2 = -x_1$,$x_2 = x_3 = 0$, $z_1 = z_2 = z_3 = z_4 = 0$,

(vii) $b_4 = b_5 = b_6 = 0$, $b_1 b_3 = b_2^2 \neq 0$, $x_1 = y_1 = y_2 = x_2 = x_3 = 0$, $z_2 = -\frac{b_1}{b_2} z_1$, $z_3 = -\frac{b_1}{b_2} z_2$, $z_4 = -\frac{b_1}{b_2} z_3$.

(viii) $b_4 = b_2 = b_1 = 0$, $3b_5^2 = b_6^2$, $3b_3 = b_6$, $y_1 = \frac{1}{3} x_1$, $x_2 = \frac{b_5 x_1 - 2b_5 y_2}{b_6}$, $x_3 = z_2 = z_4 = 0$, $z_1 = \frac{b_6 x_1 + 6b_6 y_2}{9b_5}$, $z_3 = \frac{1}{8}(-x_2 + 3z_1)$,

(ix) $b_4 = b_2 = b_1 = 0$, $b_3 b_6 = b_5^2$, $b_6 \neq 0$, $3b_3 - b_6 \neq 0$, $y_2 = \frac{b_3 x_1 - b_6 y_1}{3b_3 - b_6}$, $x_2 = \frac{b_5}{b_6}(x_1 - 2y_2)$, $x_3 = z_2 = z_4 = 0$, $z_1 = \frac{2b_3 x_1 + 3b_3 y_1 - 2b_6 y_1 + 2b_6 y_2}{3b_5}$ and $z_3 = \frac{(b_3 x_2 - b_5 y_1 + b_5 y_2)}{b_3}$,

(x) $b_5 = b_4 = b_2 = 0$, $b_1 = b_3 \neq 0$, $b_3^2 \neq b_6^2$, $y_1 = y_2 = -x_1$, $x_2 = x_3 = 0$, $z_1 = z_2 = z_3 = z_4 = 0$,

(xi) $b_6 = b_5 = b_4 = b_2 = 0$, $2b_1 = b_3 \neq 0$, $y_1 = y_2 = x_1 = 0$, $x_2 = 0$, $z_1 = z_3 = 0$, $z_2 = \frac{b_1 - b_3}{b_1} x_3$ and $z_4 = -x_3$.

We now look at the above solutions in more detail, taking also into account (5.14) and the fact that M^3 has isotropic cubic tensor. If necessary by restricting to an open dense subset of M^3, we may assume that a solution remains valid on an open set. We get:

(i) $b_5 = b_4 = b_2 = 0$, $b_1 = b_3 = b_6 \neq 0$. From $b_1 = b_6$ and (5.14), we immediately obtain that also $3x_1^2 + \frac{5x_3^2}{9} = 0$. This implies that all components of the second fundamental form vanish, which is a contradiction.

(ii) $b_1 = b_2 = b_3 = b_4 = b_5 = 0$, $b_6 \neq 0$, in this case, using also (5.14), we get $x_1 = y_1 - \frac{c}{y_1}$, $x_2 = x_3 = 0$, $y_2 = y_1$, $z_1 = z_2 = z_3 = z_4 = 0$. Moreover, $b_6 = -c - y_1^2 \neq 0$. So by (5.9) and (5.11) we get $z_{112} = z_{113} = 0$. Next from (2.3), (4.11), (5.2) and (5.3) we get

$$\begin{cases} \nabla h(E_1, E_1, E_1) = (1 + \dfrac{c}{y_1^2}) E_1(y_1) J E_1 = a_1 J E_1, \\ \nabla h(E_1, E_1, E_2) = E_1(y_1) J E_2 = -a_1 J E_2, \end{cases}$$

So, as $a_1 \neq 0$, we get that $c = -2y_1^2$. However, this implies that y_1 is constant and hence a_1 vanishes. This is a contradiction.

(iii) $b_1 = b_2 = b_4 = b_5 = b_6 = 0,\ b_3 \neq 0$, in this case a contradiction follows from (5.14).

(iv) $b_2 = b_3 = b_4 = b_5 = b_6 = 0,\ b_1 \neq 0$, in this case we get $x_1 = x_2 = y_1 = y_2 = 0,\ z_1 = 0,\ z_2 = x_3,\ z_3 = 0,\ z_4 = x_3 - \frac{c}{x_3},\ b_1 = -c - x_3^2 \neq 0$. We proceed again as for case (ii) in order to obtain a contradiction.

(vii) $b_4 = b_5 = b_6 = 0,\ b_1 b_3 = b_2^2 \neq 0,\ x_1 = y_1 = y_2 = x_2 = x_3 = 0,$ $z_2 = -\frac{b_1}{b_2} z_1, z_2 = \frac{b_1^2}{b_2^2} z_1, z_2 = -\frac{b_1^3}{b_2^3} z_1$. Using (5.14) we find that $b_2 = 0$ which leads to a contradiction.

(viii) and (ix) In both cases we have $b_4 = b_2 = b_1 = 0, b_5^2 = b_3 b_6 \neq 0$ and the solution can be rewritten as $x_1 = \frac{b_5}{b_3}(x_2 + 2z_3),\ \ x_3 = 0,\ \ y_1 = \frac{b_5}{b_3} z_3 + \frac{b_3}{b_5}(x_2 - z_3),\ \ y_2 = \frac{b_5}{b_3} z_3,\ y_3 = 0,\ z_1 = \frac{b_3^2}{b_5^2}(x_2 - z_3) + 3z_3,\ z_2 = 0,\ z_4 = 0$. After a direct calculation, we get $z_{112} = z_{113} = z_{213} = z_{312} = 0,\ z_{212} = z_{313} = -\frac{E_1(a_1)}{4a_1}$. Hence by the definition of the curvature $\langle R(E_i, E_j)E_k, E_l\rangle$, we can calculate that

$$\langle R(E_2, E_1)E_2, E_1\rangle = \langle R(E_3, E_1)E_3, E_1\rangle = \frac{5(E_1(a_1))^2 - 4a_1 E_1(E_1(a_1))}{16a_1^2},$$

which is a contradiction with $b_1 = 0,\ b_3 \neq 0$.

(v) $b_5 = b_3 = b_2 = 0,\ b_1 b_6 = b_4^2,\ b_6 \neq 0 \neq b_4,\ y_2 = \frac{b_1^2 x_1 - 3b_1^2 y_1 + b_4^2 y_1}{b_4^2},\ x_2 = 0,$ $x_3 = \frac{b_1 x_1 - 2b_1 y_1}{b_4}, z_1 = z_3 = 0, z_2 = \frac{b_1 x_3 + b_4 y_1 - b_4 y_2}{b_1}$ and $z_4 = \frac{2b_1 y_1 + b_1 y_2}{b_4}$. Changing the roles of E_2 and E_3 this case reduces to the previous one and hence a contradiction follows in the same way.

(vi) and (x) In both cases, we get $x_2 = x_3 = 0,\ y_1 = y_2 = -x_1,\ y_3 = 0,\ z_1 = z_2 = z_3 = z_4 = 0$ and $b_2 = b_4 = b_5 = 0,\ b_1 = b_3 = -c + 2x_1^2 \neq 0,\ b_6 = -c - x_1^2,$ and the second fundamental form of M^3 takes the following form

$$\begin{cases} h(E_1, E_1) = x_1 JE_1,\ h(E_2, E_2) = h(E_3, E_3) = -x_1 JE_1, \\ h(E_1, E_2) = -x_1 JE_2,\ h(E_1, E_2) = -x_1 JE_2,\ h(E_2, E_3) = 0. \end{cases}$$

So by (5.9) and (5.11) we get $z_{112} = z_{113} = 0$. Next from (2.3), (4.11), (5.2) and (5.3) we get

$$\nabla h(E_1, E_1, E_1) = E_1(x_1) JE_1 = a_1 JE_1.$$

In case (vi), from $b_6 = -b_1$ we get $x_1^2 = -2c$ hence $E_1(x_1) = 0$, which together with the previous formula imply a_1 vanishes. So we get a contradiction.

In case (x), following [7], [8] (cf. [13], [18]), we obtain that M^3 is isotropic and H-umbilical and therefore locally congruent to one of the examples 6-10 with $n = 3$.

(xi) $b_2 = b_4 = b_5 = b_6 = 0,\ b_3 = 2b_1 \neq 0$. From (5.14) it follows that $c = 0$ and $b_3 = 2x_3^2$. From (5.6) - (5.11) we get $z_{113} = -\frac{3x_3^3}{2a_1},\ z_{112} = z_{312} = z_{213} = 0,\ z_{212} = z_{313} = -\frac{E_1(a_1)}{4a_1}$. It follows from (2.3), (4.11), (5.2) and (5.3) that

$$\begin{cases} \langle \nabla h(E_1, E_1, E_3), JE_3\rangle = -\dfrac{9x_3^4}{2a_1} = -a_1, \\ \langle \nabla h(E_1, E_1, E_2), JE_2\rangle = -\dfrac{3x_3^4}{2a_1} = -a_1, \end{cases}$$

so we get $a_1 = 0$, which is a contradiction.

Hence, we have completed the proof of Theorem 1.1.

Acknowledgements: The authors would like to express their thanks to the referee for some useful comments.

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