Large algebraic structures inside the set of surjective functions

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Abstract

In this note we study large linear structures inside the set of Jones functions, which is a *highly pathological class* of surjective functions. We show that there exists an infinite dimensional linear space inside this set of functions. Moreover, we show that this linear space is isomorphic to $\mathbb{R}^{\mathbb{R}}$, that is, it has the *biggest* possible dimension. The result presented in this note is an improvement of several recent results in the topic of *lineability*.

1 Preliminaries

This note is a contribution to an ongoing search for what are often large linear spaces of functions on $\mathbb{R}$ which have special properties. Take a function with some special, weird or *unbelievably pathological* property. For example, a differentiable nowhere monotone function (see, e.g., [2, 5]). Coming up with a concrete example of such a function can be difficult. In fact, it may seem so difficult that if you succeed, you may think that there cannot be too many functions of that kind. Probably one cannot find infinite dimensional vector spaces or infinitely generated algebras of such functions. This is, however, exactly what has happened. The search for large algebraic structures of functions with pathological properties has lately become somewhat of a new trend in mathematics (see, for instance, [1–3, 5, 6]). Let us recall that, given certain "special" property, we say that the subset M of $\mathbb{R}$ that satisfies it is *κ-lineable* if $M \cup \{0\}$ contains a vector

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space of dimension κ (finite or infinite). If M contains an infinite-dimensional vector space, it will be called *lineable* for short ([2]).

As we have just said, in the last years many authors have been studying the *lineability* problem for many subsets of $\mathbb{R}^{\mathbb{R}}$. The technique for this series of studies on lineability has always been sort of standard, namely an infinitely independent family of functions in $\mathbb{R}$ every one of which enjoys this property is constructed, and then we take the linear span of that family to generate a infinite dimensional vector space, which is usually the candidate to obtain lineability.

One of the properties that has been studied thoroughly in the last years is related to the many different degrees of *surjectivity* in $\mathbb{R}^{\mathbb{R}}$. Let us recall some definitions that will be useful in this note:

Definition 1.1 (see, e.g., [5,6]). *Let $f \in \mathbb{R}^{\mathbb{R}}$. We say that:*

1. *$f \in \mathrm{ES}(\mathbb{R})$ (f is everywhere surjective) if $f(I) = \mathbb{R}$ for every non-trivial interval I.*
2. *$f \in \mathrm{SES}(\mathbb{R})$ (f is strongly everywhere surjective) if f takes all values $\mathfrak{c}$ times on any non-trivial interval.*
3. *$f \in \mathrm{PES}(\mathbb{R})$ (f is perfectly everywhere surjective) if for every perfect set P, $f(P) = \mathbb{R}$.*
4. *$f \in J(\mathbb{R})$ (f is a Jones function) if for every closed set $K \subset \mathbb{R}^2$ with uncountable projection on the x-axis, we have $f \cap K \neq \varnothing$ (see [7]).*

Moreover, if $S(\mathbb{R}) \subset \mathbb{R}^{\mathbb{R}}$ denotes the set of surjective functions, we have ([5,6]) that

$$J(\mathbb{R}) \subsetneq \mathrm{PES}(\mathbb{R}) \subsetneq \mathrm{SES}(\mathbb{R}) \subsetneq \mathrm{ES}(\mathbb{R}) \subsetneq S(\mathbb{R}).$$

All the previous different degrees of surjectivity have been thoroughly studied in the last years. In [2] the authors proved that $\mathrm{ES}(\mathbb{R})$ is $2^{\mathfrak{c}}$-lineable, where $\mathfrak{c}$ denotes the continuum. Later, in [5] the authors proved that $\mathrm{PES}(\mathbb{R})$ (and therefore $\mathrm{SES}(\mathbb{R})$) is $2^{\mathfrak{c}}$-lineable as well, which improves the result from [2] since the class $\mathrm{PES}(\mathbb{R})$ is strictly contained in $\mathrm{ES}(\mathbb{R})$. Very recently, in [6], the authors proved that the set of Jones functions, $J(\mathbb{R})$, is $e_{\mathfrak{c}}$-lineable, where

$$e_{\mathfrak{c}} = \min\{\, \mathrm{card}\, F \,:\, F \subset \mathbb{R}^{\mathbb{R}},\ (\forall \varphi \in \mathbb{R}^{\mathbb{R}})(\exists f \in F)(\mathrm{card}(f \cap \varphi) < \mathfrak{c}) \,\},$$

where $\mathfrak{c}^+ \leq e_{\mathfrak{c}} \leq 2^{\mathfrak{c}}$. It is known that $e_{\mathfrak{c}}$ can be any regular cardinal between $\mathfrak{c}^+$ and $2^{\mathfrak{c}}$, depending on the set theoretic system we are working on. In any case, it is clear that under ZFC+GCH it must be $e_{\mathfrak{c}} = 2^{\mathfrak{c}}$, and in this case the lineability problem is solved in the optimal way, meaning that we obtain the biggest possible dimension of a linear space inside $J(\mathbb{R}) \cup \{0\}$. In this note we answer the following question in the negative:

> *Is the Generalized Continuum Hypothesis (GCH) necessary in order to obtain the $2^{\mathfrak{c}}$-lineability of $J(\mathbb{R})$?*

As we shall see in the next section GCH is not needed to obtain the $2^{\mathfrak{c}}$-lineability of $J(\mathbb{R})$. Set theoretical considerations, cardinal theory, and classical linear algebra techniques will be used.

2 Main result and conclusions

Firstly, let us give the construction of a Jones function. Recall that a set S is a Bernstein set if neither S nor $\mathbb{R} \setminus S$ contain a perfect set. (See, for example, [4, (3.11)].) It is obvious that the intersection of a Bernstein set with any perfect set is not empty. Actually, this intersection has cardinality $\mathfrak{c}$, due to the fact that any perfect set contains $\mathfrak{c}$ many pairwise disjoint perfect sets. As a consequence, since any uncountable Borel (or even analytic) set contains a perfect set, the intersection of a Bernstein set with any uncountable analytic set has cardinality $\mathfrak{c}$. (The intersection with a Bernstein set has cardinality $\mathfrak{c}$ also for sets in other rather wide perfect-saturated classes, such as Lebesgue measurable sets with positive measure or non-meagre sets with the property of Baire.)

Lemma 2.1. *Let $B \subset \mathbb{R}$ a Bernstein set. There exists $f \in \mathbb{R}^{\mathbb{R}}$ such that $f|_B \cap K \neq \varnothing$ for every closed set $K \in \mathbb{R}^2$ with uncountable projection on the x-axis.*

Proof. Let

$$\mathcal{K} = \{\, K \subset \mathbb{R}^2 \,:\, K \text{ is closed and } \pi_x(K) \text{ is uncountable} \,\}.$$

Obviously, $\operatorname{card} \mathcal{K} = \mathfrak{c}$. Therefore, we can write $\mathcal{K} = \{K_\alpha : \alpha < \mathfrak{c}\}$. We shall build f using transfinite induction. For every $\alpha < \mathfrak{c}$, we shall define $x_\alpha \in B$ and $f(x_\alpha)$, in such way that $(x_\alpha, f(x_\alpha)) \in K_\alpha$. Let $\beta < \mathfrak{c}$ and suppose that we have defined x_α and $f(x_\alpha)$ for every $\alpha < \beta$. Notice that $\pi_x(K_\beta)$ is an uncountable F_σ set and, as B is a Bernstein set, it must be that $\operatorname{card}(\pi_x(K_\beta) \cap B) = \mathfrak{c}$. Since $\operatorname{card}\{x_\alpha : \alpha < \beta\} < \mathfrak{c}$, we can find an $x_\beta \in (\pi_x(K_\beta) \cap B) \setminus \{x_\alpha : \alpha < \beta\}$. Choose now y_β to hold $(x_\beta, y_\beta) \in K_\beta$ and define $f(x_\beta) = y_\beta$. This transfinite process has defined f on a set $\{x_\alpha : \alpha < \mathfrak{c}\} \subset B$. Set now $f(x) = 0$ if $x \in \mathbb{R} \setminus \{x_\alpha : \alpha < \mathfrak{c}\}$, and the function is completely defined. ∎

Now we are ready to state and prove the main result of this note:

Theorem 2.2. *$J(\mathbb{R})$ is $2^{\mathfrak{c}}$-lineable.*

Proof. It is a widely known fact that $\mathbb{R}$ can be decomposed into $\mathfrak{c}$ many pairwise disjoint Bernstein sets B_α, $\alpha \in \mathbb{R}$. (Consider, for example, the level sets of a function in $\mathrm{PES}(\mathbb{R})$.) For every $\alpha \in \mathbb{R}$, define a Jones function f_α like that of the Lemma, i.e., such that $f|_{B_\alpha} \cap K \neq \varnothing$ for every closed set $K \in \mathbb{R}^2$ with uncountable projection on the x-axis.

For every $\varphi \in \mathbb{R}^{\mathbb{R}}$, define $f_\varphi(x) = \varphi(\alpha) f_\alpha(x)$, if $x \in B_\alpha$. The set $V = \{\, f_\varphi \mid \varphi \in \mathbb{R}^{\mathbb{R}} \,\}$ is a $2^{\mathfrak{c}}$-dimensional vector space, since the map $\varphi \mapsto f_\varphi$ is a linear isomorphism from $\mathbb{R}^{\mathbb{R}}$ onto V. So, it will be enough to prove that $V \subset J(\mathbb{R}) \cup \{0\}$.

Let $\varphi \in \mathbb{R}^{\mathbb{R}}$, and assume that $f_\varphi \not\equiv 0$. It must be $\varphi(\alpha) \neq 0$ for some $\alpha \in \mathbb{R}$. Let $K \subset \mathbb{R}^2$ a closed set with uncountable projection, and let $K_\alpha = \{\, (x,y) \mid (x, \varphi(\alpha) y) \in K \,\}$. As K_α is also closed with uncountable projection, $f|_{B_\alpha} \cap K_\alpha \neq \varnothing$. Thus, there exists $x \in B_\alpha$ such that $(x, f_\alpha(x)) \in K_\alpha$, i.e., $(x, f(x)) = (x, \varphi(\alpha) f_\alpha(x)) \in K$. That is, $f \cap K \neq \varnothing$. ∎

Of course, the previous Theorem is a clear improvement of [6, Theorem 3.16] and it gives the optimal solution for the lineability problem for this class of surjective functions without the need of the Generalized Continuum Hypothesis. Notice, also, that Theorem 2.2 improves as well [2, Theorem 4.3] and [5, Theorem 2.6], since the $2^{\mathfrak{c}}$-lineability of $J(\mathbb{R})$ automatically provides the $2^{\mathfrak{c}}$-lineability of the sets $\mathrm{PES}(\mathbb{R})$, $\mathrm{SES}(\mathbb{R})$, $\mathrm{ES}(\mathbb{R})$, and $S(\mathbb{R})$.

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