Yosida-Hewitt type decompositions for order-weakly compact operators

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Abstract

Let E be an ideal of L^0 over a σ-finite measure space (Ω, Σ, μ). For a real Banach space $(X, \|\cdot\|_X)$ let $E(X)$ be the subspace of the space $L^0(X)$ of μ-equivalence classes of strongly Σ-measurable functions $f : \Omega \to X$ consisting of all those $f \in L^0(X)$ for which the scalar function $\|f(\cdot)\|_X$ belongs to E. For a real Banach space Y a linear operator $T : E(X) \to Y$ is said to be order-weakly compact whenever for each $u \in E^+$ the set $T(\{f \in E(X) : \|f(\cdot)\|_X \leq u\})$ is relatively weakly compact in Y. In this paper we derive Yosida-Hewitt type decompositions for order-weakly compact operators $T : E(X) \to Y$. In particular, it is shown that if X is an Asplund space, then an order-weakly compact operator $T : E(X) \to Y$ can be uniquely decomposed as $T = T_1 + T_2$, where T_1, T_2 are order-weakly compact operators, T_1 is smooth and T_2 is weakly singular.

1 Introduction and preliminaries

The problem of Yosida-Hewitt type decompositions of linear mappings from vector lattices to vector lattices (Banach spaces) has been considered in [E], [S], [AB$_1$], [KM], [BBuY], [BBu]. In particular, Basile, Bukhvalov and Yakubson ([BBuY], [BBu]) have derived Yosida-Hewitt type decompositions for order-weakly compact operators from vector lattices to Banach spaces. Recall here that a linear operator T from a vector lattice E to a Banach space Y is said to be order-weakly compact if the set $T([-u, u])$ is relatively weakly compact in Y for every $u \in E^+$

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(see [D], [AB_2, §18]). In [N_7] we obtained Yosida-Hewitt type decompositions for weakly compact operators from Köthe-Bochner function spaces $E(X)$ to Banach spaces. The purpose of this paper is to derive Yosida-Hewitt type decompositions for order-weakly compact operators acting from more general function spaces $E(X)$ to Banach spaces (see Theorems 3.3, 3.4 and 3.6 below).

We denote by $\sigma(L,K)$ and $\tau(L,K)$ the weak topology and the Mackey topology on L with respect to a dual pair $\langle L,K\rangle$. For terminology concerning vector-lattices and function spaces we refer to [AB_2], [KA].

Throughout the paper we assume that (Ω,Σ,μ) is a complete σ-finite measure space. Let L^0 denote the space of μ-equivalence classes of all Σ-measurable real valued functions defined on Ω. Let E be an ideal of L^0 with $\operatorname{supp} E = \Omega$, and let E' stand for the Köthe dual of E. We will assume that $\operatorname{supp} E' = \Omega$. Let $E^\sim$, $E^\sim_n$ and $E^\sim_s$ stand for the order dual, the order continuous dual and the singular dual of E respectively. Then $E^\sim_n$ separates the points of E and it can be identified with E' through the mapping: $E' \ni v \mapsto \varphi_v \in E^\sim_n$, where $\varphi_v(u) = \int_\Omega u(\omega)v(\omega)d\mu$ for all $u \in E$.

From now on we assume that $(X,\|\cdot\|_X)$, $X \neq \{0\}$ and $(Y,\|\cdot\|_Y)$, $Y \neq \{0\}$ are real Banach spaces and let X^* and Y^* stand for their Banach duals. Let S_X stand for the unit sphere in X. By $L^0(X)$ we denote the set of μ-equivalence classes of all strongly Σ-measurable functions $f:\Omega \to X$. For $f \in L^0(X)$ let us set $\widetilde{f}(\omega) := \|f(\omega)\|_X$ for $\omega \in \Omega$. Let

$$E(X) = \{\, f \in L^0(X) : \widetilde{f} \in E \,\}.$$

Basic concepts of the theory of vector-valued spaces $E(X)$ can be found in monographs: [CM], [DU], [L]. Recall that the algebraic tensor product $E \otimes X$ is the subspace of $E(X)$ spanned by the functions of the form $u \otimes x$, $(u\otimes x)(\omega) = u(\omega)x$, where $u \in E$, $x \in X$. For each $u \in E^+$ the set $D_u = \{f \in E(X) : \widetilde{f} \leq u\}$ will be called an *order interval* in $E(X)$ (see [BuL]).

Following [D], [N_4], [N_5] we are now ready to define two classes of linear operators.

Definition 1.1. A linear operator $T: E(X) \to Y$ is said to be *order-weakly compact* (resp. *order-bounded*) whenever for each $u \in E^+$ the set $T(D_u)$ is relatively-weakly compact (resp. norm bounded) in Y.

Clearly each order-weakly compact operator $T: E(X) \to Y$ is order-bounded.

2 Duality of vector-valued function spaces

In this section we establish terminology and prove some results concerning duality of vector-valued function spaces $E(X)$ (see [BuL], [N_1], [N_2], [N_3], [N_4]).

For an order-bounded functional F on $E(X)$ let us put

$$|F|(f) := \sup\{|F(h)| : h \in E(X),\ \widetilde{h} \leq \widetilde{f}\} \quad \text{for } f \in E(X).$$

Clearly $|F(f)| \le |F|(f)$ for each $f \in E(X)$ and $|F|(f_1) \le |F|(f_2)$ whenever $\widetilde{f_1} \le \widetilde{f_2}$. One can check that the mapping $f \mapsto |F|(f)$ is a seminorm on $E(X)$.

The set

$$E(X)^{\sim} = \{F \in E(X)^{\#} : |F|(f) < \infty \text{ for all } f \in E(X)\}$$

will be called the *order dual* of $E(X)$ (here $E(X)^{\#}$ denotes the algebraic dual of $E(X)$). It is known that a linear operator $T : E(X) \to Y$ is order bounded if and only if T is $(\tau(E(X), E(X)^{\sim}), \|\cdot\|_Y)$-continuous (see [$N_4$, Theorem 2.3]).

Let $F \in E(X)^{\sim}$ and $x_0 \in S_X$ be fixed. For $u \in E^+$ let us set

$$\varphi_F(u) := |F|(u \otimes x_0) = \sup\{|F(h)| : h \in E(X), \widetilde{h} \le u\}.$$

Note that $\varphi_F(u)$ does not depend on $x_0 \in S_X$. Then $\varphi_F : E^+ \to \mathbb{R}^+$ is an additive mapping and φ_F has a unique positive extension to a linear mapping from E to $\mathbb{R}$ (denoted by φ_F again) and given by

$$\varphi_F(u) := \varphi_F(u^+) - \varphi_F(u^-) \quad \text{for all} \quad u \in E$$

(see [BuL, §3, Lemma 7]). Clearly $\varphi_F \in E^{\sim}$ and for $f \in E(X)$ we have

$$\varphi_F(\widetilde{f}) = |F|(f) \quad \text{for all } f \in E(X).$$

Now we recall the concept of solidness in $E(X)^{\sim}$ (see [N_1, §2], [N_2]). For $F_1, F_2 \in E(X)^{\sim}$ we will write $|F_1| \le |F_2|$ whenever $|F_1|(f) \le |F_2|(f)$ for all $f \in E(X)$. A subset A of $E(X)^{\sim}$ is said to be *solid* whenever $|F_1| \le |F_2|$ with $F_1 \in E(X)^{\sim}$ and $F_2 \in A$ imply $F_1 \in A$. A linear subspace I of $E(X)^{\sim}$ will be called an *ideal* of $E(X)^{\sim}$ whenever I is solid.

An order bounded linear functional F on $E(X)$ is said to be *smooth* whenever for a net (f_α) in $E(X)$, $\widetilde{f_\alpha} \xrightarrow{(o)} 0$ in E implies $F(f_\alpha) \to 0$ (see [BuL, § 3, Definition 2], [N_1], [N_2]). (Note that Bukhvalov and Lozanovskii [BuL] use the term "integral" and in [N_1], [N_2] we use the term "order continuous"). The set consisting of all smooth functionals on $E(X)$ will be denoted by $E(X)^{\sim}_n$. Note that $E(X)^{\sim}_n$ separates the points of $E(X)$ because we assume that $\operatorname{supp} E' = \Omega$.

A subset H of $E(X)$ is said to be *solid* whenever $\widetilde{f_1} \le \widetilde{f_2}$ and $f_1 \in E(X)$, $f_2 \in H$ imply $f_1 \in H$. A linear topology τ on $E(X)$ is said to be *locally solid* if it has a local base at zero consisting of solid sets. A locally solid topology τ on $E(X)$ is said to be a *Lebesgue topology* whenever for a net (f_α) in $E(X)$, $\widetilde{f_\alpha} \xrightarrow{(o)} 0$ in E implies $f_\alpha \to 0$ for τ (see [N_3, Definition 2.2]).

It is known that a Banach space X is an Asplund space if and only if X^* has the Radon-Nikodym property (see [DU, p. 213]).

The following theorem will be of importance (see [N_6, Theorems 1.2 and 4.1]):

Theorem 2.1. *Assume that X is an Asplund space. Then the Mackey topology $\tau(E(X), E(X)^{\sim}_n)$ is a locally convex-solid Lebesgue topology.*

Recall that a functional $F \in E(X)^\sim$ is said to be *singular* if there exists an ideal M of E with $\operatorname{supp} M = \Omega$ and such that $F(f) = 0$ for all $f \in M(X)$. The set consisting of all singular functionals on $E(X)$ will be denoted by $E(X)_s^\sim$ and called the *singular dual* of $E(X)$ (see [BuL, § 3, Definition 2]).

It is known that $E(X)_n^\sim$ and $E(X)_s^\sim$ are ideals of $E(X)^\sim$ (see [N$_1$]).

Due to Bukhvalov and Lozanovski (see [BuL, §3, Theorem 2]) we have the following Yosida-Hewitt type decomposition of $E(X)^\sim$.

Theorem 2.2. *The following decomposition of* $E(X)^\sim$ *holds:*

$$E(X)^\sim = E(X)_n^\sim \oplus E(X)_s^\sim \tag{1.1}$$

and $\varphi_F = \varphi_{F_1} + \varphi_{F_2}$ *whenever* $F = F_1 + F_2$ *with* $F_1 \in E(X)_n^\sim$, $F_2 \in E(X)_s^\sim$. *Moreover,* $\varphi_{F_1} \in E_n^\sim$ *and* $\varphi_{F_2} \in E_s^\sim$.

One can note that $E(X)_n^\sim = E(X)^\sim$ if and only if $E_n^\sim = E^\sim$.

In view of (1.1) we have linear projections $P_k : E(X)^\sim \to E(X)^\sim$ $(k = 1, 2)$ defined by $P_k(F) = F_k$. Note that for $F \in E(X)^\sim$ and every $f \in E(X)$ we have:

$$|P_k(F)|(f) = |F_k|(f) = \varphi_{F_k}(\widetilde{f}) \le \varphi_F(\widetilde{f}) = |F|(f),$$

i.e., $|P_k(F)| \le |F|$.

Proposition 2.3. *For a linear operator* $T : E(X) \to Y$ *the following statements are equivalent:*

(i) $y^* \circ T \in E(X)_n^\sim$ *for every* $y^* \in Y^*$.

(ii) T *is* $(\sigma(E(X), E(X)_n^\sim), \sigma(Y, Y^*))$*-continuous.*

(iii) T *is* $(\tau(E(X), E(X)_n^\sim), \|\cdot\|_Y)$*-continuous.*

Proof. (i)$\Longleftrightarrow$(ii) See [AB$_2$, Theorem 9.26]; (ii)$\Longleftrightarrow$(iii) See [W, Corollary 11-1-3, Corollary 11-2-6]. ■

Following [BBuY] we define smooth and singular operators on $E(X)$.

Definition 2.1. (i) An order bounded linear operator $T : E(X) \to Y$ is said to be *smooth* if for a net (f_α) in $E(X)$, $\widetilde{f_\alpha} \xrightarrow{(o)} 0$ in E implies $\|T(f_\alpha)\|_Y \to 0$.

(ii) An order bounded linear operator $T : E(X) \to Y$ is said to be *singular* if there exists an ideal M of E with $\operatorname{supp} M = \Omega$ such that $T(f) = 0$ for all $f \in M(X)$.

(iii) An order bounded linear operator $T : E(X) \to Y$ is said to be *weakly singular* if $y^* \circ T \in E(X)_s^\sim$ for every $y^* \in Y^*$.

The following theorem gives a characterization of smooth operators $T : E(X) \to Y$ when X is an Asplund space.

Theorem 2.4. *Assume that X is an Asplund space. Then for a linear operator $T : E(X) \to Y$ the following statements are equivalent:*

(i) *T is smooth.*

(ii) *$y^* \circ T \in E(X)_n^\sim$ for every $y^* \in Y^*$.*

(iii) *T is $(\tau(E(X), E(X)_n^\sim), \|\cdot\|_Y)$-continuous.*

Proof. (i)$\Longrightarrow$(ii) It is obvious. (ii)$\Longleftrightarrow$(iii) See Proposition 2.3.

(iii)$\Longrightarrow$(i) Clearly, because $\tau(E(X), E(X)_n^\sim)$ is a Lebesgue topology (see Theorem 2.1). ■

We will need the following lemma.

Lemma 2.5. *Assume that $|F| \leq |G|$, where $F, G \in E(X)^\sim$. Then $|P_k(F)| \leq |P_k(G)|$ for $k = 1, 2$.*

Proof. We have $F = F_1 + F_2$, $G = G_1 + G_2$, where $F_1, G_1 \in E(X)_n^\sim$, $F_2, G_2 \in E(X)_s^\sim$ and $\varphi_F = \varphi_{F_1} + \varphi_{F_2}$, $\varphi_G = \varphi_{G_1} + \varphi_{G_2}$, where $\varphi_{F_1}, \varphi_{G_1} \in E_n^\sim$ and $\varphi_{F_2}, \varphi_{G_2} \in E_s^\sim$ (see (1.1)). Let $u \in E^+$ and $x_0 \in S_X$ be fixed. Then

$$\varphi_F(u) = |F|(u \otimes x_0) \leq |G|(u \otimes x_0) = \varphi_G(u).$$

Since the order projections of $E^\sim$ onto $E_n^\sim$ and $E_s^\sim$ are positive operators, for $f \in E(X)$ we have

$$\begin{aligned} |P_k(F)|(f) &= |F_k|(f) = \varphi_{F_k}(\widetilde{f}) \\ &\leq \varphi_{G_k}(\widetilde{f}) = |G_k|(f) = |P_k(G)|(f). \end{aligned}$$

■

For a linear functional V on $E(X)^\sim$ let us put:

$$|V|(F) = \sup\{|V(G)| : G \in E(X)^\sim,\ |G| \leq |F|\} \text{ for } F \in E(X)^\sim.$$

The set

$$(E(X)^\sim)^\sim = \{V \in (E(X)^\sim)^\# : |V|(F) < \infty \quad \text{for all} \quad F \in E(X)^\sim\}$$

will be a called the *order dual* of $E(X)^\sim$ (see [N_2]) (here $(E(X)^\sim)^\#$ denotes the algebraic dual of $E(X)^\sim$).

For $V_1, V_2 \in (E(X)^\sim)^\sim$ we will write $|V_1| \leq |V_2|$ whenever $|V_1|(F) \leq |V_2|(F)$ for all $F \in E(X)^\sim$. A subset K of $(E(X)^\sim)^\sim$ is said to be *solid* whenever $|V_1| \leq |V_2|$ with $V_1 \in (E(X)^\sim)^\sim$, $V_2 \in K$ imply $V_1 \in K$. A linear subspace L of $(E(X)^\sim)^\sim$ is called an *ideal* if L is a solid subset of $(E(X)^\sim)^\sim$.

For each $f \in E(X)$ let us put

$$\pi_f(F) = F(f) \quad \text{for all } F \in E(X)^\sim.$$

One can show (see [N_2, §1]) that for $f \in E(X)$,

$$|\pi_f|(F) = |F|(f) \quad \text{for} \quad F \in E(X)^\sim \quad \text{and that} \quad \pi_f \in (E(X)^\sim)^\sim.$$

Thus we have a natural embedding $\pi : E(X) \ni f \mapsto \pi_f \in (E(X)^\sim)^\sim$.

Denote by $E(X)_0$ the ideal of $(E(X)^\sim)^\sim$ generated by the set $\pi(E(X))$, i.e., $E(X)_0$ is the smallest ideal of $(E(X)^\sim)^\sim$ containing $\pi(E(X))$. One can show that (see [N_2, Theorem 3.2]):

$$E(X)_0 = \{V \in (E(X)^\sim)^\sim : |V| \le |\pi_f| \quad \text{for some} \quad f \in E(X)\}.$$

Let

$$P_k^\sim : (E(X)^\sim)^\# \to (E(X)^\sim)^\#$$

stand for the conjugate of P_k $(k = 1, 2)$ defined by

$$P_k^\sim(V)(F) = V(P_k(F)) \;\; \text{for} \;\; V \in (E(X)^\sim)^\# \;\; \text{and} \;\; F \in E(X)^\sim.$$

Observe that

$$P_k^\sim((E(X)^\sim)^\sim) \subset (E(X)^\sim)^\sim.$$

Indeed, let $V \in (E(X)^\sim)^\sim$. Then by making use of Lemma 2.5 we have for $F \in E(X)^\sim$,

$$\begin{aligned} |P_k^\sim(V)|(F) &= \sup\{|P_k^\sim(V)(G)| : G \in E(X)^\sim,\ |G| \le |F|\} \\ &= \sup\{|V(P_k(G)| : G \in E(X)^\sim,\ |G| \le |F|\} \\ &\le \sup\{|V|(P_k(G)) : G \in E(X)^\sim,\ |G| \le |F|\} \\ &\le |V|(P_k(F)) \le |V|(F) < \infty. \end{aligned}$$

Hence, in particular, we get:

Corollary 2.6. *Let $f \in E(X)$. Then for every $F \in E(X)^\sim$ we have*

$$|P_k^\sim(\pi_f)|(F) \le |\pi_f|(P_k(F)) \le |\pi_f|(F),$$

and hence $P_k^\sim(\pi_f) \in E(X)_0$ $(k = 1, 2)$.

3 A Yosida-Hewitt type decomposition for order-weakly compact operators

In this section we derive Yosida-Hewitt type decompositions for order-weakly compact operators $T : E(X) \to Y$.

Assume now that $T : E(X) \to Y$ is an order bounded operator, i.e., T is $(\tau(E(X), E(X)^\sim), \|\cdot\|_Y)$-continuous. It follows that $y^* \circ T \in E(X)^\sim$ for every $y^* \in Y^*$. Then we can consider the linear mappings (see [N_6]):

$$T^\sim : Y^* \to E(X)^\sim$$

defined by

$$T^{\sim}(y^*)(f) = y^*(T(f)) \quad \text{for } y^* \in Y \text{ and all } f \in E(X),$$

and

$$T^{\sim\sim} : E(X)_0 \to Y^{**}$$

defined by

$$T^{\sim\sim}(V)(y^*) = V(T^{\sim}(y^*)) \text{ for } V \in E(X)_0 \text{ and all } y^* \in Y^*.$$

The map $T^{\sim\sim}$ is $(\sigma(E(X)_0, E(X)^{\sim}), \sigma(Y^{**}, Y^*))$-continuous.

Let $i : Y \ni y \mapsto i_y \in Y^{**}$ stand for the canonical isometry, i.e., $i_y(y^*) = y^*(y)$ for $y^* \in Y^*$. Moreover, let $j : i(Y) \to Y$ stand for the left inverse of i, i.e., $j \circ i = id_Y$. Then $T^{\sim\sim} \circ \pi = i \circ T$.

The following characterization of order-weakly compact operators $T\colon E(X) \to Y$ will be of importance.

Theorem 3.1 (see [N_5, Theorem 2.3]). *For an order-bounded operator $T : E(X) \to Y$ the following statements are equivalent:*

(i) *T is order-weakly compact.*

(ii) *$T^{\sim\sim}(E(X)_0) \subset i(Y)$.*

For $f \in E(X)$ let us set

$$I_f = \{V \in E(X)_0 : |V| \leq |\pi_f|\}.$$

The following property of I_f will be needed.
Theorem 3.2. *For $f \in E(X)$ the set I_f is $\sigma(E(X)_0, E(X)^{\sim})$-compact in $E(X)_0$.*

Proof. Clearly $\sigma(E(X)_0, E(X)^{\sim}) = \sigma((E(X)^{\sim})^{\#}, E(X)^{\sim})|_{E(X)_0}$. We shall show that I_f is a totally bounded and closed set in $((E(X)^{\sim})^{\#}, \sigma((E(X)^{\sim})^{\#}, E(X)^{\sim}))$. In fact, let $F \in E(X)^{\sim}$. Then for each $V \in I_f$ we have

$$|V(F)| \leq |V|(F) \leq |\pi_f|(F) = |F|(f) < \infty.$$

This means that I_f is bounded for $\sigma((E(X)^{\sim})^{\#}, E(X)^{\sim})$, so by [KA, Lemma 3.3.5] it is totally bounded in $(E(X)^{\sim})^{\#}, \sigma((E(X)^{\sim})^{\#}, E(X)^{\sim}))$.
To see that I_f is closed in $((E(X)^{\sim})^{\#}, \sigma((E(X)^{\sim})^{\#}, E(X)^{\sim}))$, assume that $V_\alpha \to V$ for $\sigma((E(X)^{\sim}))^{\#}, E(X)^{\sim})$, where (V_α) is a net in I_f and $V \in (E(X)^{\sim})^{\#}$. It is enough to show that $|V| \leq |\pi_f|$, i.e., $|V|(F) \leq |\pi_f|(F) = |F|(f)$ for each $F \in E(X)^{\sim}$. In fact, let $F \in E(X)^{\sim}$ and $\varepsilon > 0$ be given. Let $G \in E(X)^{\sim}$ and $|G| \leq |F|$. Since $V_\alpha(G) \to V(G)$, there exists α_0 such that for $\alpha \geq \alpha_0$ we get

$$|V(G)| \leq |V_\alpha(G)| + \varepsilon \leq |V_\alpha|(G) + \varepsilon \leq |\pi_f|(G) + \varepsilon \leq |\pi_f|(F) + \varepsilon.$$

It follows that $|V|(F) \leq |\pi_f|(F)$, so $|V| \leq |\pi_f|$, as desired.

Since the space $((E(X)^\sim)^\#, \sigma((E(X)^\sim)^\#, E(X)^\sim))$ is complete (see [KA, Lemma 3.3.4]), the set I_f is complete for $\sigma((E(X)^\sim)^\#, E(X)^\sim)$, so we can conclude that I_f is compact for $\sigma((E(X)^\sim)^\#, E(X)^\sim)$ (see [KA, Theorem 3.1.4]). It follows that I_f is also $\sigma(E(X)_0, E(X)^\sim)$-compact. ∎

Now we are in position to prove our main result.

Theorem 3.3. *Let $T : E(X) \to Y$ be an order-weakly compact operator. Then T can be uniquely decomposed as $T = T_1 + T_2$, where T_1, T_2 are order-weakly compact operators, T_1 is $(\tau(E(X), E(X)_n^\sim), \|\cdot\|_Y)$-continuous and T_2 is weakly singular.*

Proof. In view of Corollary 2.6, $P_k^\sim(\pi_f) \in E(X)_0$ $(k = 1, 2)$. Hence by Theorem 3.1, $T^{\sim\sim}(P_k^\sim(\pi_f)) \in i(Y)$, and we can define linear mappings:

$$T_k = j \circ T^{\sim\sim} \circ P_k^\sim \circ \pi \ : E(X) \to Y.$$

Then for $y^* \in Y^*$ and $f \in E(X)$ we have

$$\begin{aligned} y^*(T_k(f)) &= y^*(j((T^{\sim\sim} \circ P_k^\sim \circ \pi)(f))) \\ &= (T^{\sim\sim} \circ P_k^\sim \circ \pi(f))(y^*) \\ &= (T^{\sim\sim}(\pi_f \circ P_k))(y^*) \\ &= (\pi_f \circ P_k)(T^\sim(y^*)) \\ &= (\pi_f \circ P_k)(y^* \circ T) \\ &= \pi_f(P_k(y^* \circ T)) \\ &= P_k(y^* \circ T)(f), \end{aligned}$$

i.e., $y^* \circ T_1 = P_1(y^* \circ T) \in E(X)_n^\sim$ and $y^* \circ T_2 = P_2(y^* \circ T) \in E(X)_s^\sim$, and this means that T_1 is $(\tau(E(X), E(X)_n^\sim), \|\cdot\|_Y)$-continuous (see Proposition 2.2) and T_2 is weakly singular (see Definition 2.1). Moreover, for every $y^* \in Y^*$ and $f \in E(X)$ we have

$$y^*(T_1(f) + T_2(f)) = P_1(y^* \circ T)(f) + P_2(y^* \circ T)(f) = y^*(T(f)),$$

so $T(f) = T_1(f) + T_2(f)$. The uniqueness of the decomposition $T = T_1 + T_2$ follows from the uniqueness of the decomposition $y^* \circ T = y^* \circ T_1 + y^* \circ T_2$ for each $y^* \in Y^*$ (see (1.1)).

Now we shall show that $T_k : E(X) \to Y$ are order-weakly compact operators. Indeed, let $u \in E^+$ and $D_u = \{h \in E(X) : \widetilde{h} \leq u\}$. In view of Corollary 2.6 for $h \in D_u$ and a fixed $x_0 \in S_X$ we get for $F \in E(X)^\sim$:

$$|P_k^\sim(\pi_h)|(F) \leq |\pi_h|(F) = |F|(h) \leq |F|(u \otimes x_0) = |\pi_{u \otimes x_0}|(F)$$

i.e., $|P_k^\sim(\pi_h)| \leq |\pi_{u\otimes x_0}|$. Then $\{P_k^\sim(\pi_h) : h \in D_u\} \subset I_{u\otimes x_0}$. According to Theorem 3.2 the set $I_{u\otimes x_0}$ is $\sigma(E(X)_0, E(X)^\sim)$-compact in $E(X)_0$, and this means that $\{P_k^\sim(\pi_h) : h \in D_u\}$ is a relatively $\sigma(E(X)_0, E(X)^\sim)$-compact subset of $E(X)_0$.

Since $T^{\sim\sim}(E(X)_0) \subset i(Y) \subset Y^{**}$ and $T^{\sim\sim}$ is $(\sigma(E(X)_0, E(X)^{\sim}),\ \sigma(Y^{**}, Y^*))$-continuous, the set $\{T^{\sim\sim}(P_k^{\sim}(\pi_h)) : h \in D_u\}$ is relatively $\sigma(Y^{**}, Y^*)$-compact in Y^{**}. But the mapping j is $(\sigma(i(Y), Y^*), \sigma(Y, Y^*))$-continuous, so the set $T_k(D_u) = \{j(T^{\sim\sim}(P_k^{\sim}(\pi_h))) : h \in D_u\}$ is relatively $\sigma(Y, Y^*)$-compact in Y. ∎

Using Theorems 2.4 and 3.3 we obtain the following Yosida-Hewitt type decomposition for order-weakly compact operators $T : E(X) \longrightarrow Y$.

Theorem 3.4. *Let $T : E(X) \longrightarrow Y$ be an order weakly compact operator. Assume that X is an Asplund space. Then T can be uniquely decomposed as $T = T_1 + T_2$, where T_1, T_2 are order-weakly compact, T_1 is smooth and T_2 is weakly singular.*

From now on we assume that $(E, \|\cdot\|_E)$ is a Banach function space. Then the space $E(X)$ provided with the norm $\|f\|_{E(X)} := \|\widetilde{f}\|_E$ is a Banach space and is usually called a *Köthe-Bochner function space.* Then the Mackey topology $\tau(E(X), E(X)^{\sim})$ coincides with the $\|\cdot\|_{E(X)}$-norm topology and a linear operator $T : E(X) \to Y$ is order bounded if and only if T is $(\|\cdot\|_{E(X)}, \|\cdot\|_Y)$-continuous (see [N$_4$, Theorem 2.3]). Let

$$E_a = \{u \in E : |u| \geq u_n \downarrow 0 \text{ in } E \text{ implies } \|u_n\|_E \to 0\}.$$

It is well known that E_a is $\|\cdot\|_E$ – closed ideal of E and $E_a = E$ if and only if $\|\cdot\|_E$ is order continuous.

We will need the following useful characterization of singular operators on Köthe-Bochner function spaces (see [N$_7$, Proposition 1.4]).

Proposition 3.5. *Assume that $(E, \|\cdot\|_E)$ is a Banach function space with* $\operatorname{supp} E_a = \Omega$. *Then for a $(\|\cdot\|_{E(X)}, \|\cdot\|_Y)$-continuous linear operator $T : E(X) \to Y$ the following statements are equivalent:*

(i) *T is singular.*

(ii) *T is weakly singular.*

(iii) *$T(f) = 0$ for all $f \in E_a(X)$.*

Combining Theorem 3.4 with Proposition 3.5 we are ready to state a Yosida-Hewitt type decomposition for order-weakly compact operators acting from Köthe-Bochner function spaces $E(X)$ to Banach spaces.

Theorem 3.6. *Assume that $(E, \|\cdot\|_E)$ is a Banach function space with* $\operatorname{supp} E_a = \Omega$ *and X is an Asplund space. Let $T : E(X) \to Y$ be an order-weakly compact operator. Then T can be uniquely decomposed as $T = T_1 + T_2$, where T_1, T_2 are order-weakly compact operators, T_1 is smooth and T_2 is singular.*

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