A Subordination Result with Salagean-Type Certain Analytic Functions of Complex Order

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Abstract

In the present paper, we obtain an interesting subordination relation for Salagean-type certain analytic functions by using subordination theorem.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $\mathcal{C}$ denote the familiar class of functions $f(z) \in \mathcal{A}$ which are convex in $\mathbb{U}$. Salagean [3] has introduced the following operator called the Salagean operator :

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$\vdots$$

$$D^n f(z) = D(D^{n-1} f(z)),\ n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}.$$

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With the help of the Salagean operator D^n, we say that a function $f(z)$ belonging to $\mathcal{A}$ is in the class $H_n(b,M)$ iff $\frac{D^n f(z)}{z} \neq 0$ in $\mathbb{U}$, and

$$\left| \frac{b-1+\frac{D^{n+1}f(z)}{D^n f(z)}}{b} - M \right| < M \quad (z \in \mathbb{U}), \tag{1.2}$$

$M > \frac{1}{2}$ and $b \neq 0$; complex. The class $H_n(b,M)$ was introduced by Aouf et.al. [1]. They showed that

$$f \in H_n(b,M) \quad if\, and\, only\, if \quad D^n f \in H_0(b,M) = F(b,M),$$

the class $F(b,M)$ of bounded starlike functions of complex order was introduced by Nasr and Aouf [2].

Aouf et.al. [1] proved that if the function $f(z)$ defined by (1.1) and

$$\sum_{k=2}^{\infty} \{k-1+|b(1+m)+m(k-1)|\}\, k^n |a_k| \leq |b(1+m)| \tag{1.3}$$

hold then $f(z)$ belongs to $H_n(b,M)$, where $m = 1-\frac{1}{M}$ $(M > \frac{1}{2})$.

Let α_j $(j = 1,2,\cdots,p)$ and β_j $(j = 1,2,\cdots,q)$ be complex numbers with

$$\beta_j \neq 0,-1,-2,\cdots;\ j = 1,2,\cdots,q.$$

The generalized hypergeometric function ${}_pF_q$ is defined by (cf.[4, p.33])

$${}_pF_q(z) = {}_p F_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z)$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k z^k}{(\beta_1)_k \cdots (\beta_q)_k\ k!} \quad (p \leq q+1), \tag{1.4}$$

where $(\mu)_k$ is the Pochhammer symbol defined by

$$(\mu)_k = \begin{cases} 1 & if \quad k = 0 \\ \mu(\mu+1)\cdots(\mu+k-1) & if \quad k \in \mathbb{N} = \{1,2,\cdots\} \end{cases}.$$

We note that the ${}_pF_q$ series in (1.4) converges absolutely for $|z| < \infty$ if $p < q+1$, and for $z \in \mathbb{U}$ if $p = q+1$.

Let $K_n(b,M)$ denote the class of functions $f(z) \in \mathcal{A}$ whose coefficients satisfy the condition (1.3).

We note that

$$K_n(b,M) \subseteq H_n(b,M).$$

We can show that:

Example

i) Let $b \neq 0$; complex and $m = 1-\frac{1}{M}$ $(M > \frac{1}{2};\ M \neq 1)$, then $f_0(z) \in K_n(b,M)$, where

$$D^n f_0(z) = z(1-mz)^{\frac{-b(1+m)}{m}} \qquad (z \in \mathbb{U})$$

which gives

$$f_0(z) = z\left[1 + b(1+m)z\ _{n+2}F_{n+1}(1,\cdots,1,\frac{b(1+m)}{m}+1;2,\cdots,2;mz)\right]\ (z \in \mathbb{U})$$

for real b with $b \neq 0$.

ii) Let $b \neq 0$; complex and $m = 1 - \frac{1}{M}$ $(M > \frac{1}{2})$, the functions

$$f_1(z) = z \pm \frac{(1+m)|b|}{4(1+|m+b(1+m)|)}z^2$$

and

$$f_2(z) = z \pm \frac{(1+m)|b|}{9(2+|2m+b(1+m)|)}z^3$$

are members in the class $K_n(b, M)$.

In this paper, we prove an interesting subordination result for the class $K_n(b, M)$. In our proposed investigation of functions in the class $K_n(b, M)$, we need the following definitions and lemma.

Definition 1. Given two functions $f, g \in \mathcal{A}$ where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

The Hadamard product $f * g$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \quad (z \in \mathbb{U}).$$

Definition 2. (Subordination Principle) For two functions f and g analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ and write $f(z) \prec g(z)$, $z \in \mathbb{U}$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

Definition 3. (Subordinating Factor Sequence) A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is said to be a ***Subordinating Factor Sequence*** if for the function $f(z)$ of the form (1.1) is analytic, univalent and convex in $\mathbb{U}$, we have the subordination given by

$$\sum_{k=1}^{\infty} a_k b_k z^k \prec f(z) \qquad (z \in \mathbb{U}; a_1 = 1). \tag{1.5}$$

Lemma. The sequence $\{b_k\}_{k=1}^{\infty}$ is Subordinating factor sequence iff

$$Re\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \quad (z \in \mathbb{U}). \tag{1.6}$$

The above lemma is due to Wilf [5].

2 Main Theorem

Theorem . Let $m = 1 - \frac{1}{M}$ $(M > \frac{1}{2})$. Also, let $b \neq 0$; complex with $Re(b) > \frac{-m}{2(1+m)}$ when $m > 0$ and $Re(b) < \frac{-m}{2(1+m)}$ when $m < 0$. If $f(z) \in K_n(b, M)$ then

$$\text{(2.1)} \quad \frac{(1+|b(1+m)+m|)2^{n-1}}{[(1+|b(1+m)+m|)2^n + |b(1+m)|]}(f * g)(z) \prec g(z)$$

$$(z \in \mathbb{U};\ n \in \mathbb{N}_0;\ g(z) \in \mathcal{C})$$

and

$$\text{(2.2)} \quad Re f(z) > -1 - \frac{(1+m)|b|}{(1+|b(1+m)+m|)2^n}.$$

The constant $\frac{(1+|b(1+m)+m|)2^{n-1}}{[(1+|b(1+m)+m|)2^n+|b(1+m)|]}$ is the best estimate.

Proof . Let $f(z) \in K_n(b, M)$ and $g(z) = z + \sum_{k=2}^{\infty} c_k z^k \in \mathcal{C}$. Then

$$\frac{(1+|b(1+m)+m|)2^{n-1}}{[(1+|b(1+m)+m|)2^n + |b(1+m)|]}(f * g)(z)$$

$$\text{(2.3)} \quad = \frac{(1+|b(1+m)+m|)2^{n-1}}{[(1+|b(1+m)+m|)2^n + |b(1+m)|]}\left(z + \sum_{k=2}^{\infty} a_k c_k z^k\right).$$

Thus, by definition 3, (2.1) will hold true if

$$\text{(2.4)} \quad \left\{\frac{(1+|b(1+m)+m|)2^{n-1}}{[(1+|b(1+m)+m|)2^n + |b(1+m)|]}a_k\right\}_{k=1}^{\infty}$$

is a subordinating factor sequence with $a_1 = 1$. In view of Lemma, this is equivalent to

$$\text{(2.5)} \quad Re\left\{1 + \sum_{k=1}^{\infty} \frac{(1+|b(1+m)+m|)2^n}{[(1+|b(1+m)+m|)2^n + |b(1+m)|]}a_k z^k\right\} > 0.$$

Now because $\{k - 1 + |b(1+m) + m(k-1)|\}\, k^n$ $\quad (n \in \mathbb{N}_0; k \geq 2)$ is increasing function of k, we have

$$Re\left\{1+\sum_{k=1}^{\infty}\frac{(1+|b(1+m)+m|)2^n}{[(1+|b(1+m)+m|)2^n+|b(1+m)|]}a_k z^k\right\}$$
$$= Re\{1+\frac{(1+|b(1+m)+m|)2^n}{[(1+|b(1+m)+m|)2^n+|b(1+m)|]}z+$$
$$\frac{1}{[(1+|b(1+m)+m|)2^n+|b(1+m)|]}\sum_{k=2}^{\infty}(1+|b(1+m)+m|)2^n a_k z^k\}$$
$$\geq 1-\frac{(1+|b(1+m)+m|)2^n}{[(1+|b(1+m)+m|)2^n+|b(1+m)|]}r-$$
$$\frac{1}{[(1+|b(1+m)+m|)2^n+|b(1+m)|]}\sum_{k=2}^{\infty}(k-1+|b(1+m)+m(k-1)|)k^n|a_k|r^k\}$$
$$> 1-\frac{(1+|b(1+m)+m|2^n}{[(1+|b(1+m)+m|)2^n+|b(1+m)|]}r-$$
$$\frac{|b(1+m)|}{[(1+|b(1+m)+m|)2^n+|b(1+m)|]}r \qquad > 0. \quad (|z|=r<1)$$

Hence, (2.5) holds true in $\mathbb{U}$ and also the subordination result (2.1) asserted by Theorem 1. The inequality (2.2) follows by taking $g(z)=\frac{z}{1-z}=\sum_{k=1}^{\infty}z^k\in\mathcal{C}$ in (2.1).

Now, consider the function

$$t(z)=z-\frac{|b(1+m)|}{[(1+|b(1+m)+m|)2^n+|b(1+m)|]}z^2 \quad (z\in\mathbb{U})$$

which is a member of the class $K_n(b,M)$. Then by using (2.1), we have

$$\frac{(1+|b(1+m)+m|)2^{n-1}}{[(1+|b(1+m)+m|)2^n+|b(1+m)|]}t(z)\prec\frac{z}{1-z} \qquad (z\in\mathbb{U}).$$

It is easily verified that

$$minRe\left\{\frac{(1+|b(1+m)+m|)2^{n-1}}{[(1+|b(1+m)+m|)2^n+|b(1+m)|]}t(z)\right\}=-\frac{1}{2} \qquad (z\in\mathbb{U}).$$

Then the constant $\frac{(1+|b(1+m)+m|)2^{n-1}}{[(1+|b(1+m)+m|)2^n+|b(1+m)|]}$ cannot be replaced by a larger one, which completes the proof of Theorem.

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