Coefficient estimates for close-to-convex functions with argument β

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Abstract

This paper deals with coefficient estimates for close-to-convex functions with argument β ($-\pi/2 < \beta < \pi/2$). By using Herglotz representation formula, sharp bounds of coefficients are obtained. In particular, we solve the problem posed by A. W. Goodman and E. B. Saff in [2]. Finally some complicate computations yield the explicit estimate of the third coefficient.

1 Introduction

Let $\mathcal{A}$ be the family of functions f analytic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{A}_1$ be the subset of $\mathcal{A}$ consisting of functions f which are normalized by $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}_1$ is said to be starlike (denoted by $f \in \mathcal{S}^*$) if f maps $\mathbb{D}$ univalently onto a domain starlike with respect to the origin.

Let

$$\mathcal{P}_\beta = \left\{ p \in \mathcal{A} : \ p(0) = 1, \operatorname{Re} e^{i\beta} p > 0 \right\}.$$

Here and hereafter we always suppose $-\pi/2 < \beta < \pi/2$. It is easy to see that

$$p \in \mathcal{P}_\beta \Leftrightarrow \frac{e^{i\beta} p - i \sin \beta}{\cos \beta} \in \mathcal{P}_0. \qquad (1)$$

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Herglotz representation formula (see [4]) together with (1) yield the following equivalence

$$p \in \mathcal{P}_\beta \Leftrightarrow p(z) = \int_{\partial\mathbb{D}} \frac{1+e^{-2i\beta}xz}{1-xz} d\mu(x) \tag{2}$$

for a Borel probability measure μ on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$. This correspondence is 1-1.

Since $\mathcal{P}_0$ is the well-known Carathéodory class, we call $\mathcal{P}_\beta$ the tilted Carathéodory class by angle β. Some equivalent definitions and basic estimates are known (for a short survey, see [7]).

Definition 1. A function $f \in \mathcal{A}_1$ is said to be close-to-convex (denoted by $f \in \mathcal{CL}$) if there exist a starlike function g and a real number $\beta \in (-\pi/2, \pi/2)$ such that

$$\frac{zf'}{g} \in \mathcal{P}_\beta.$$

This definition involving a real number β is slightly different from the original one due to Kaplan [5]. An equivalent definition of $\mathcal{CL}$ by using Kaplan class and some related sets of univalent functions can be found in [6]. If we specify the real number β in the above definition, the corresponding function is called a close-to-convex function with argument β and we denote the class of all such functions by $\mathcal{CL}(\beta)$ (see [1, II, Definition 11.4]). Note that the union of class $\mathcal{CL}(\beta)$ over $\beta \in (-\pi/2, \pi/2)$ is precisely $\mathcal{CL}$ while the intersection is the class of convex functions. These results were given in [2] without proof. Since the former one is obvious, we will only give an outline of the proof of the latter one. Choose a sequence $\{\beta_n\} \subset (-\pi/2, \pi/2)$ such that $\beta_n \to \pi/2$ as $n \to \infty$. The assertion follows from the facts that the class of starlike functions is compact in the sense of locally uniform convergence and any function sequence $\{p_n\}$ where $p_n \in \mathcal{P}_{\beta_n}$ converges to the constant function 1 locally uniform as $\beta_n \to \pi/2$.

In the literature, when studying the close-to-convex functions, some authors focus only on the case $\beta = 0$. A. W. Goodman and E. B. Saff [2] were the first to point out explicitly that $\mathcal{CL}(\beta)$ and $\mathcal{CL}$ are different when $\beta \neq 0$ and more deeply the class $\mathcal{CL}(\beta)$ has no inclusion relation with respect to β. Therefore it is useful to consider the individual class $\mathcal{CL}(\beta)$. The present paper follows their way in this direction and improves their result concerning the class $\mathcal{CL}(\beta)$;

Theorem A (Goodman-Saff [2]) Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{CL}(\beta)$ for a $\beta \in (-\pi/2, \pi/2)$. Then

$$|a_n| \leq 1 + (n-1)\cos\beta.$$

for $n = 2, 3, \cdots$. If either $n = 2$ or $\beta = 0$, the inequality is sharp.

In the above mentioned paper, they also stated that the problem of finding the maximum for $|a_n|$ in the class $\mathcal{CL}(\beta)$ was difficult for $n \geq 3$. With regard to their problem, in the present paper we shall establish the following theorems:

Theorem 1. *Suppose* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{CL}(\beta)$ *for a* $\beta \in (-\pi/2, \pi/2)$, *then the sharp inequality*

$$|a_n| \leq \frac{2\cos\beta}{n} \max_{|u|=1} \left| \frac{n}{1+e^{-2i\beta}} + \sum_{k=1}^{n-1} k u^{n-k} \right|. \tag{3}$$

holds for $n = 2, 3, \cdots$. *Extremal functions are given by*

$$f'(z) = \frac{1}{(1-yz)^2} \frac{1+e^{-2i\beta} y u_n z}{1 - y u_n z}$$

for $y \in \partial\mathbb{D}$, *where* $u_n \in \partial\mathbb{D}$ *is a point at which the above maximum is attained.*

We mention here that it seems that there are no extremal functions other than the form given above in Theorem 1. Theorem A follows from Theorem 1 immediately by the elementary inequality

$$\left| \frac{n}{1+e^{-2i\beta}} + \sum_{k=1}^{n-1} k u^{n-k} \right| \leq \frac{n}{2\cos\beta} + \frac{n(n-1)}{2}$$

for any $u \in \partial\mathbb{D}$.

The expression in (3) is implicit. When $n = 3$, we can give a more concrete estimate and also show the extremal functions are unique;

Theorem 2. *Suppose* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{CL}(\beta)$, *then the sharp inequality*

$$|a_3| \leq \frac{2\cos\beta}{3} \sqrt{5 + \frac{9}{4\cos^2\beta} + \frac{13}{1-t_0}} \tag{4}$$

holds, where t_0 *is the unique root of the equation*

$$t^3 - \left(\frac{4}{3}\cos^2\beta + 6\right) t^2 + \left(\frac{40}{9}\cos^2\beta + 9\right) t + 4\cos^2\beta - 4 = 0 \tag{5}$$

in $0 \leq t < 1$. *Equality holds in (4) if and only if*

$$f'(z) = \frac{1}{(1-yz)^2} \frac{1+e^{-2i\beta} y u_3 z}{1 - y u_3 z}$$

for some $y \in \partial\mathbb{D}$, *where*

$$u_3 = \begin{cases} 1 - \frac{t_0}{2} - i\sqrt{t_0 - \frac{t_0^2}{4}} \frac{\beta}{|\beta|}, & \text{when } \beta \neq 0; \\ 1, & \text{when } \beta = 0. \end{cases}$$

Remark 1. Comparing Theorem A and Theorem 2, it is not difficult to see that

$$1+2\cos\beta=\frac{2\cos\beta}{3}\sqrt{5+\frac{9}{4\cos^2\beta}+\frac{13}{1-t_0}}$$

if and only if

$$t_0=\frac{9-9\cos\beta}{9+4\cos\beta}.$$

Since this t_0 is a root of (5) in $[0,1)$ only when $\beta=0$, Theorem A is sharp only when $\beta=0$ for $n=3$.

Finally we give an example to show how Theorem 2 works.

Example. Let $\beta=\pi/4$. Applying Mathematica, we may get the root of equation (5) which belongs to $[0,1)$ is $0.201\cdots$, therefore in this case

$$|a_3|\lessapprox 2.394$$

which is less than $1+\sqrt{2}\approx 2.414$ by Theorem A.

2 Proof of Theorems

In order to prove our theorems, we shall need the following lemma

Lemma 1. *(see [3] p. 52) If $f\in\mathcal{S}^*$, then there exists a Borel probability measure ν on $\partial\mathbb{D}$ such that*

$$f(z)=\int_{\partial\mathbb{D}}\frac{z}{(1-yz)^2}d\nu(y).$$

Proof of Theorem 1 :

Equivalence (2) and Lemma 1 imply that if $f\in\mathcal{CL}(\beta)$, then there exist two Borel probability measures μ and ν on $\partial\mathbb{D}$ such that f' can be represented as

$$f'(z)=\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}}\frac{1}{(1-yz)^2}\frac{1+e^{-2i\beta}xz}{1-xz}d\mu(x)d\nu(y).$$

Thus in order to estimate the coefficients of f, it is sufficient to estimate those of functions

$$\frac{1}{(1-yz)^2}\frac{1+e^{-2i\beta}xz}{1-xz}$$

when $|x|=|y|=1$.

Since

$$\frac{1}{(1-yz)^2}\frac{1+e^{-2i\beta}xz}{1-xz}=\sum_{n=0}^{\infty}\left\{(n+1)y^n+\sum_{k=0}^{n-1}(k+1)(1+e^{-2i\beta})y^kx^{n-k}\right\}z^n$$

implies

$$\begin{aligned}|na_n|&\le\max_{|x|=|y|=1}\left|ny^{n-1}+\sum_{k=0}^{n-2}(k+1)(1+e^{-2i\beta})y^kx^{n-1-k}\right|\\&=\max_{|x|=|y|=1}\left|n+\sum_{k=1}^{n-1}k(1+e^{-2i\beta})(x/y)^{n-k}\right|\end{aligned}$$

after letting $u = x/y$, we can easily obtain (3). The extremal functions can be obtained easily by the proof of this theorem. ■

Proof of Theorem 2: By Theorem 1, we have the sharp inequality

$$|a_3| \leq \frac{2\cos\beta}{3} \max_{-\pi<\alpha\leq\pi} \sqrt{h(\alpha)}.$$

where

$$h(\alpha) = \left|1 + 2e^{i\alpha} + \frac{3}{1+e^{-2i\beta}} e^{2i\alpha}\right|^2. \tag{6}$$

Straightforward calculations give

$$\begin{aligned} h(\alpha) &= 5 + \frac{9}{4\cos^2\beta} + 4\cos\alpha + \frac{3\cos(\beta+2\alpha)+6\cos(\beta+\alpha)}{\cos\beta} \\ &= 5 + \frac{9}{4\cos^2\beta} + (10\cos\alpha + 3\cos 2\alpha) - 3\tan\beta(\sin 2\alpha + 2\sin\alpha), \end{aligned} \tag{7}$$

and

$$\begin{aligned} h'(\alpha) &= -4\sin\alpha - \frac{12\sin\frac{2\beta+3\alpha}{2}\cos\frac{\alpha}{2}}{\cos\beta} \\ &= -(10\sin\alpha + 6\sin 2\alpha) - 6\tan\beta(\cos 2\alpha + \cos\alpha), \end{aligned} \tag{8}$$

$$h''(\alpha) = -(10\cos\alpha + 12\cos 2\alpha) + 6\tan\beta(2\sin 2\alpha + \sin\alpha). \tag{9}$$

Since $h'(\pi) = 0$ and $h''(\pi) < 0$, $h(\alpha)$ attains a local maximum $h(\pi) = (9 - 8\cos^2\beta)/(4\cos^2\beta)$ at π. It follows from $h(\pi) < h(0)$ that π is not a global maximum point of $h(\alpha)$. Since $h(\alpha)$ is periodic and continuous, its maximum point exists over $(-\pi, \pi)$, thus we may suppose that $h(\alpha)$ attains its maximum at some point α_0 in $(-\pi, \pi)$, then

$$h'(\alpha_0) = 0 \tag{10}$$

and

$$h''(\alpha_0) \leq 0. \tag{11}$$

Combining (8) and (10), we may represent $\tan\beta$ in term of α_0;

$$\tan\beta = -\frac{5\sin\alpha_0 + 3\sin 2\alpha_0}{3(\cos\alpha_0 + \cos 2\alpha_0)}. \tag{12}$$

Substituting it into (9) shows

$$\begin{aligned} h''(\alpha_0) &= -(10\cos\alpha_0 + 12\cos 2\alpha_0) - 2(2\sin 2\alpha_0 + \sin\alpha_0)\frac{5\sin\alpha_0 + 3\sin 2\alpha_0}{\cos\alpha_0 + \cos 2\alpha_0} \\ &= -\frac{2(11 + 11\cos\alpha_0 + 4\sin^2\alpha_0\cos\alpha_0)}{\cos\alpha_0 + \cos 2\alpha_0}. \end{aligned} \tag{13}$$

Since

$$11 + 11\cos\alpha + 4\sin^2\alpha\cos\alpha > 0$$

whenever $-\pi < \alpha < \pi$, hence from (11) and (13), we deduce that

$$\cos\alpha_0 + \cos 2\alpha_0 > 0$$

which is fulfilled only when $\cos\alpha_0 > 1/2$ i.e. $\alpha_0 \in (-\pi/3, \pi/3)$.

Let $g(\alpha_0)$ denote the quantity given in the right hand side of (12). Since $g'(\alpha) < 0$ over $(-\pi/3, \pi/3)$, there exists one and only one α_0 which satisfies (10) and (11) and $h(\alpha)$ assumes its maximum

$$5 + \frac{9}{4\cos^2\beta} + \frac{13}{1 - 4\sin^2\frac{\alpha_0}{2}}$$

at α_0.

(8) and (10) also imply

$$\cos\frac{\alpha_0}{2}\left(2\sin\frac{\alpha_0}{2} + 3\frac{\sin\frac{3\alpha_0+2\beta}{2}}{\cos\beta}\right) = 0. \qquad (14)$$

Since $\alpha_0 \neq \pi$, after letting $x_0 = \sin(\alpha_0/2)$, (14) implies that x_0 is the unique root of the following equation

$$11x - 12x^3 + 3\tan\beta\sqrt{1-x^2}(1-4x^2) = 0.$$

in $(-1/2, 1/2)$. Writing $t_0 = 4x_0^2$ and $t = 4x^2$, we get t_0 is a root of equation (5) in $[0, 1)$.

Let $v(t)$ be the polynomial in the left hand of (5), it is easy to verify that $v(0) \leq 0$, $v(1) > 0$ and $v'(t) > 0$ in $0 \leq t < 1$ which together assure the uniqueness of root $t_0 \in [0, 1)$ of equation (5).

Therefore Theorem 2 is complete. ∎

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