Criteria of existence of bounded approximate identities in topological algebras

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Abstract

Some results and criteria of existence concerning bounded approximate identities in Banach algebras are extended to the topological algebras setting. We thereby prove that the bidual of a commutative locally C*-algebra with either of the two Arens products is a unital commutative algebra, and that a quasinormable Fréchet m-convex algebra has a left (resp. right) bounded approximate identity if and only if it can be represented as an inverse limit of Banach algebras each of which has a left (resp. right) bounded approximate identity.

1 Introduction

The notion of a bounded approximate identity first appeared in harmonic analysis and soon gained ground in Banach algebra theory due to its connection with factorization problems and Johnson's amenability (see [6, 2.9] for a succinct account of results and references). In topological homology, which studies the homological properties of topological algebras, it is closely related both to amenability and the notion of flatness. Specifically, a Banach algebra A is amenable if and only if it has a bounded approximate identity and it is flat as Banach A-bimodule [12, VII.2.20]. Our attempt to extend this characterization of amenability to the context of locally convex non-normed algebras, resulted in [25, §5], gave rise to this paper.

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The fundamental concepts and techniques on which the aforementioned extension is based had been already developed by the author since 2005 in an earlier extended form of [23]. However, while [23] was being reduced in size during the refereeing process, the arXiv version of [22] was revised and completed, without acknowledgment, using some ideas and results of the original version of [23], which we now include in the present paper. More precisely, Propositions 3.1 and 3.2, Theorems 3.9(1; i⇔ii) and 4.2(2), and Corollary 4.4 of this paper have also appeared in [22]. However, Proposition 3.2 was mentioned without proof in [22], and we prove Theorem 4.2(2) and Corollary 4.4 in a more direct way. Moreover, by introducing the dual module of a locally convex module and extending the notions of approximate units and of Arens regularity to the topological algebra setting, the proofs of all the other results are very streamlined and rather direct. Finally, the present manuscript is largely based on the doctoral thesis of the author, completed in 2006. The manuscript [25] also contains some results of that thesis, using auxiliary results of the present paper.

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2 Preliminaries

The topological vector spaces (tvs) considered throughout are complex and Hausdorff. We denote by $\mathcal{N}_0(X)$ the filter basis of all balanced 0-neighborhoods in a tvs X and by $\mathcal{B}(X)$ the family of all non-empty bounded subsets of X.

A tvs which is an (associative) algebra with separately continuous multiplication is called *topological algebra*. A topological algebra with locally convex underlying tvs is called *locally convex* (lca). In particular, an *m-convex algebra* is a lca whose topology is defined by a family of m-(i.e., submultiplicative) seminorms. According to the Arens-Michael representation theorem, such an algebra is topological algebraically embedded into an inverse limit of Banach algebras. More precisely, if $(s_\mu)_{\mu \in M}$ is a defining family of m-seminorms for an m-convex algebra A, then each $A/\mathrm{Ker}(s_\mu)$ is a normed algebra with the norm $\|a + \mathrm{Ker}(s_\mu)\|_\mu := s_\mu(a)$ and $A \underset{\to}{\subseteq} \varprojlim (A/\mathrm{Ker}(s_\mu))^{\tilde{}}$ holds up to a topological algebra embedding, which turns isomorphism if A is complete (cf. [3], [18]). In case A is complete involutive and the s_μ are C*-seminorms (i.e., $s_\mu(a^*a) = s_\mu(a)^2$ $\forall\, a \in A$), the previous normed algebras are C*-algebras and A is called *locally C*-algebra* [13]. For properties and examples of locally C*-algebras, we refer to [9] and [21].

A net $(e_\lambda)_{\lambda \in \Lambda}$ in a topological algebra A is called *left* (resp. *right*) *approximate identity* (ai) if, for every $a \in A$, $\lim_\lambda e_\lambda a = a$ (resp. $\lim_\lambda a e_\lambda = a$) holds and *approximate identity* if it is both a left and a right ai. If the previous limit/-s is/are taken for the weak topology of A, then the ai is called *weak*. An ai whose elements form a bounded set is called *bounded* (bai).

A left module over a topological algebra A which is a tvs with separately continuous outer multiplication is called *topological left A-module*. A *topological right* or *bi-module over A* is similarly defined. Naturally, a *locally convex module* (lcm) is a topological module with locally convex underlying tvs.

If the outer multiplication of a topological left A-module X is $\mathcal{B}(X)$- (resp. $\mathcal{B}(A)$-) hypocontinuous, that is, for each $V \in \mathcal{N}_0(X)$ and $B \in \mathcal{B}(X)$, there is a $U \in \mathcal{N}_0(A)$ such that $U \cdot B \subset V$ (resp. for each $V \in \mathcal{N}_0(X)$ and $C \in \mathcal{B}(A)$, there is a $U \in \mathcal{N}_0(X)$ such that $C \cdot U \subset V$), then we call X *left* (resp. *right*) *hypotopological*. We say that X is *hypotopological* if it is left and right hypotopological. These notions are accordingly modified for a topological right A-module. Obviously, every topological module with continuous outer multiplication is hypotopological. Moreover, every lc left (resp. right) module over a barreled lca is left (resp. right) hypotopological (cf. [15, 40.2(3)]). In a left or right hypotopological left A-module X, $C \cdot B \in \mathcal{B}(X)$ holds, for any $C \in \mathcal{B}(A)$, $B \in \mathcal{B}(X)$. Indeed, if X is e.g. right hypotopological, take $U \in \mathcal{N}_0(X)$. By the $\mathcal{B}(A)$-hypocontinuity of X's outer multiplication, there is $V \in \mathcal{N}_0(X)$ such that $C \cdot V \subset U$. Since B is bounded, there is $\rho > 0$ with $B \subset \rho V$. Hence $C \cdot B \subset \rho U$ and so $C \cdot B$ is bounded. A topological algebra A is termed *left*, *right* or just *hypotopological* if it is respectively left, right or just hypotopological as a topological left (or right) A-module.

Let X be a left A-lcm. Then its dual space X' is a right A-module with respect to (w.r.t.) the map $X' \times A \to X' \colon (f,a) \mapsto f \cdot a \colon X \to \mathbb{C} \colon x \mapsto f(a \cdot x)$. It is easy to see that this map is separately continuous for the weak* topology on X'. For the strong topology, it is continuous w.r.t. the first variable and continuous w.r.t. the second if and only if $A \times X \to X_\sigma$ is $\mathcal{B}(X)$-hypocontinuous (which occurs when A is barreled lc; ibid.). If we consider a right A-lcm X, then X' is a left A-module w.r.t. $A \times X' \to X' \colon (a,f) \mapsto a \cdot f \colon X \to \mathbb{C} \colon x \mapsto f(x \cdot a)$, which has analogous continuity properties. Given two topological A-bimodules X and Y, the vector space ${}_A h(X,Y)$ of all continuous left A-module morphisms $X \to Y$ is an A-bimodule w.r.t. $(a,T) \mapsto a \cdot T \colon x \mapsto T(x \cdot a)$, $(T,a) \mapsto T \cdot a \colon x \mapsto T(x) \cdot a$ and the vector space $h_A(X,Y)$ of all continuous right A-module morphisms $X \to Y$ is an A-bimodule w.r.t. $(a,T) \mapsto a \cdot T \colon x \mapsto a \cdot T(x)$, $(T,a) \mapsto T \cdot a \colon x \mapsto T(a \cdot x)$. In particular, the last two products render the vector space $h(X,Y)$ $(= {}_0h(X,Y))$ of continuous (linear) operators $X \to Y$ an A-bimodule [12, §0.4.2].

3 Approximate identities and approximate units

Proposition 3.1. *Let A be a topological algebra. Then A has a left ai if and only if, for each $F \subset A$ finite and $U \in \mathcal{N}_0(A)$, there exists $u \in A$ with $a - ua \in U \; \forall\, a \in F$. In particular, A has a left bai if and only if there is a $B \in \mathcal{B}(A)$ such that for each $F \subset A$ finite and $U \in \mathcal{N}_0(A)$, there exists $u \in B$ with $a - ua \in U \; \forall\, a \in F$.*

Proof. Considering both cases, (⇒) follows from repeated application of the definitions. (⇐) If $\mathcal{F}$ is the family of all non-empty finite sets in A, then $\mathcal{F} \times \mathcal{N}_0(A)$ is a directed set w.r.t. the order $(F_1,U_1) \le (F_2,U_2) \overset{\text{def}}{\Longleftrightarrow} F_1 \subset F_2$ and $U_1 \supset U_2$. By the hypothesis, $\forall\,(F,U) \in \mathcal{F} \times \mathcal{N}_0(A) \;\exists\, e_{F,U} \in A \colon\; a - e_{F,U}\, a \in U \;\forall\, a \in F$ (resp. $\exists\, B \in \mathcal{B}(A) \;\forall\,(F,U) \in \mathcal{F} \times \mathcal{N}_0(A) \;\exists\, e_{F,U} \in B \colon\; a - e_{F,U}\, a \in U \;\forall\, a \in F$). This yields that $(e_{F,U})_{(F,U) \in \mathcal{F} \times \mathcal{N}_0(A)}$ is a left ai (resp. bai) for A. ∎

Proposition 3.2. *If a hypotopological algebra A has a left and a right bai, then it has a bai.*

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a left and $(f_\mu)_{\mu\in M}$ a right bai for A. We will show that $(f_\mu \circ e_\lambda)_{(\lambda,\mu)\in\Lambda\times M}$, where $f_\mu \circ e_\lambda := f_\mu + e_\lambda - f_\mu e_\lambda$, is a bai. Take $a \in A$ and $U \in \mathcal{N}_0(A)$. Choose $V \in \mathcal{N}_0(A)$ such that $V + V \subset U$. By the right hypocontinuity of A's multiplication, there is $W \in \mathcal{N}_0(A)$ with $f_\mu W \subset V\, \forall\, \mu \in M$. For some $\lambda_0 \in \Lambda$, we have $e_\lambda a - a \in V \cap W\, \forall\, \lambda \geq \lambda_0$. Therefore, for every $\lambda \geq \lambda_0$ and $\mu \in M$, we get $(f_\mu \circ e_\lambda)\, a - a = f_\mu(a - e_\lambda a) + (e_\lambda a - a) \in f_\mu W + V \subset V + V \subset U$, which proves that $(f_\mu \circ e_\lambda)$ is a left ai for A. Similarly, $(f_\mu \circ e_\lambda)$ is a right ai. To show that it is bounded, let $U \in \mathcal{N}_0(A)$ and choose $V \in \mathcal{N}_0(A)$ with $V + V + V \subset U$. Since (e_λ), (f_μ) and $(f_\mu e_\lambda)$ are bounded, there exist $\rho_1, \rho_2, \rho_3 > 0$ with $e_\lambda \in \rho_1 V$, $f_\mu \in \rho_2 V$ and $f_\mu e_\lambda \in \rho_3 V\ \forall (\lambda, \mu) \in \Lambda \times M$. Setting $\rho = \max\{\rho_1, \rho_2, \rho_3\}$, we get $f_\mu \circ e_\lambda \in \rho(V + V + V) \subset \rho\, U\ \forall\, (\lambda, \mu) \in \Lambda \times M$. ∎

The A-bimodule $h(A, X)_b$ in the next lemma carries the topology of uniform convergence on the bounded sets of A: a 0-neighborhood basis of this tvs is formed by the sets $\mathcal{U}(B, U) := \{T \in h(A, X) \colon T(B) \subset U\}$, $B \in \mathcal{B}(A)$, $U \in \mathcal{N}_0(X)$ [14, 2.10.A]. The map $j_X \colon X \to h_A(A, X)_b \colon\ x \mapsto j_X(x) \colon A \to X \colon a \mapsto x \cdot a$ considered thereto is an A-bimodule morphism.

Lemma 3.3. *Let A be a right hypotopological algebra and X a topological A-bimodule whose left outer multiplication is $\mathcal{B}(X)$-hypocontinuous. Then $h(A, X)_b$ is a topological A-bimodule. If the right outer multiplication of X is $\mathcal{B}(A)$-hypocontinuous, then $j_X \in {}_A h_A(X, h_A(A, X)_b)$.*

Proof. Say $L \colon (a, T) \mapsto a \cdot T \colon A \to X \colon b \mapsto a \cdot T(b)$ and $R \colon (T, a) \mapsto T \cdot a \colon A \to X \colon b \mapsto T(ab)$ are the outer multiplications of $h(A, X)_b$. Let $a \in A$ and $\mathcal{U}(B, U)$ a 0-neighborhood in $h(A, X)_b$ where $B \in \mathcal{B}(A)$, $U \in \mathcal{N}_0(X)$. Choose $V \in \mathcal{N}_0(X)$ such that $a \cdot V \subset U$. Then $a \cdot T \in \mathcal{U}(B, U)\ \forall\, T \in \mathcal{U}(B, V)$ and $T \cdot a \in \mathcal{U}(B, U)$ $\forall\, T \in \mathcal{U}(aB, U)$. Namely, L is continuous w.r.t. the second variable and R is continuous w.r.t. the first variable. Now, let $T \in h(A, X)$ and $\mathcal{U}(B, U)$ as before. By the $\mathcal{B}(X)$-hypocontinuity of the left outer multiplication of X, there is $W_1 \in \mathcal{N}_0(A)$ such that $W_1 \cdot T(B) \subset U$. Thus, $a \cdot T \in \mathcal{U}(B, U)\ \forall\, a \in W_1$ and so L is continuous w.r.t. the first variable. In addition, if $W_2 \in \mathcal{N}_0(A)$ is such that $T(W_2) \subset U$, by the right hypocontinuity of A's multiplication, there is a $W_3 \in \mathcal{N}_0(A)$ with $W_3\, B \subset W_2$. Hence, $T \cdot a \in \mathcal{U}(B, U)\ \forall\, a \in W_3$ and so R is continuous w.r.t. the second variable. Finally, it is easily seen that the continuity of j_X is equivalent to the $\mathcal{B}(A)$-hypocontinuity of the right outer multiplication of X. ∎

Corollary 3.4. *Let A be a barreled lca and X a lc A-bimodule. Then the next hold:*
(1) $h(A, X)_b$ is a lc A-bimodule and, if X is barreled, then $j_X \in {}_A h_A(X, h_A(A, X)_b)$.
(2) $h(A, X'_b)_b$ is a lc A-bimodule and $j_{X'} \in {}_A h_A(X'_b, h_A(A, X'_b)_b)$.

Proof. (1) stems from 3.3. (2) Since, for every $B \in \mathcal{B}(X)$ and $C \in \mathcal{B}(A)$, one has $C \cdot B \in \mathcal{B}(X)$ and $(C \cdot B)^{\mathrm{o}} \cdot C \subset B^{\mathrm{o}}$, the right outer multiplication of X'_b is $\mathcal{B}(A)$-hypocontinuous. The left outer multiplication of X'_b being $\mathcal{B}(X'_b)$-hypocontinuous as A is barreled, we apply 3.3 again. ∎

A topological left A-module X that coincides with the closed linear span of $A \cdot X := \{a \cdot x\colon\ a \in A,\ x \in X\}$ is called *essential*. We note that a right hypotopological left A-module X is essential if and only if a left bai $(e_\lambda)_{\lambda\in\Lambda}$ for A is a left bai for X (i.e., $x = \lim_\lambda e_\lambda\, x\ \forall\, x \in X$) [23, 3.2].

Proposition 3.5. *Let A be a barreled lca with a left bai and X an essential quasi-barreled lc A-bimodule. Then $h_A(A, X'_b)_b = X'_b$ holds up to a topological isomorphism of lc A-bimodules.*

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a left bai for A. We shall show that $j_{X'}$ in 3.4 is a topological isomorphism. Since X is essential, $j_{X'}$ is injective. To show that $j_{X'}$ is surjective, let $T \in h_A(A, X'_b)_b$. Since $(T(e_\lambda))_\lambda$ is bounded and X is quasi-barreled (hence any bounded set in X'_b is equicontinuous [15, 39.3(3)]), by the Alaoglu-Bourbaki theorem there is $f_T \in X'$ with $T(e_\lambda) \to f_T$ in X'_{σ^*}. So, for all $a \in A$, $x \in X$, $(j_{X'}(f_T)(a))(x) = \lim_\lambda T(e_\lambda)(a \cdot x) = \lim_\lambda T(e_\lambda a)(x) = T(a)(x)$. Now, set $C \equiv \{e_\lambda\colon \lambda \in \Lambda\}$ and let $B \in \mathcal{B}(X)$. For each $T \in h_A(A, X'_b)$ with $|T(e_\lambda)(x)| \leq 1$ $(x \in B,\ \lambda \in \Lambda)$, we have $|f_T(x)| \leq 1$, i.e., $\mathcal{U}(C, B^\circ) \subset j_{X'}(B^\circ)$. Hence $j_{X'}^{-1}$ is continuous. ∎

A continuous operator $D\colon A \to X$ from a topological algebra A into a topological A-bimodule X is called *derivation* if it satisfies the "Leibnitz identity": $D(ab) = a \cdot D(b) + D(a) \cdot b$, for every $a, b \in A$. A derivation of the form $D_x\colon A \to X\colon a \mapsto a \cdot x - x \cdot a$, where $x \in X$, is called *inner*, generated by x.

Proposition 3.6. *Let A be a barreled lca with a right bai and X a quasi-barreled lc A-bimodule such that $A \cdot X = 0$. Then each derivation from A into X'_b is inner.*

Proof. Let $(e_\lambda)_\lambda$ be a right bai for A and $D\colon A \to X'_b$ a derivation. Arguing as in 3.5, we get an $f_D \in X'$ such that $D(e_\lambda) \to f_D$ in X'_{σ^*}. Then, for any $a \in A$ and $x \in X$, we have $D(a)(x) = \lim_\lambda D(ae_\lambda)(x) = \lim_\lambda D(e_\lambda)(x \cdot a) = f_D(x \cdot a - a \cdot x) = (a \cdot f_D - f_D \cdot a)(x)$. ∎

A fruitful weakening of the notion of an ai is given now (see [8, I.9] for normed algebras).

Definition 3.7. A topological algebra A is said to have *left* (resp. *right*) *approximate units* if, for each $a \in A$ and $U \in \mathcal{N}_0(A)$, there exists $u \in A$ with $a - ua \in U$ (resp. $a - au \in U$) and *approximate units* if, for each $a \in A$ and $U \in \mathcal{N}_0(A)$, there exists $u \in A$ with $a - ua \in U$ and $a - au \in U$. If there is a $B \in \mathcal{B}(A)$ such that, for each $a \in A$ and $U \in \mathcal{N}_0(A)$, there exists $u \in B$ with $a - ua \in U$ (resp. $a - au \in U$, both), then A is said to have *left* (resp. *right*, just) *bounded approximate units* (baus, for short).

Lemma 3.8. *A right hypotopological algebra A has left baus if and only if there is $B \in \mathcal{B}(A)$ such that, for each $F \subset A$ finite and $U \in \mathcal{N}_0(A)$, there exists $u \in B$ with $a - ua \in U\ \forall\, a \in F$.*

Proof. Consider $B \in \mathcal{B}(A)$ as in 3.7. Take $F = \{a_1, \ldots, a_n\} \subset A$ and $U \in \mathcal{N}_0(A)$. Choose $V \in \mathcal{N}_0(A)$ with $V + V + V \subset U$ and $W \in \mathcal{N}_0(A)$ with $W \subset V$, $BW \subset V$ and $WF \subset V$. Then, pick $W^+ \in \mathcal{N}_0(A_+)$ such that $W^+ \cap A = W$

and set $B_i^+ = (e - B)^{n-i} \subset A_+ \; \forall\, i = 1, \ldots, n$ (where e is the identity of A_+). For each $i \in \{1, \ldots, n\}$, there is $W_i^+ \in \mathcal{N}_0(A_+)$ such that $B_i^+ W_i^+ \subset W^+$. Clearly, $W_i \equiv W_i^+ \cap A \in \mathcal{N}_0(A)$ and $B_i^+ W_i \subset W$. Now choose $u_1, \ldots, u_n \in B$ successively such that $(e - u_i) \cdots (e - u_1)\, a_i \in W_i \; \forall\, i = 1, \ldots, n$. And define $v \in A$ by $e - v = (e - u_n) \cdots (e - u_1)$. Then, for every $i \in \{1, \ldots, n\}$, we have $a_i - va_i = (e - u_n) \cdots (e - u_{i+1})(e - u_i) \cdots (e - u_1)\, a_i \in B_i^+ W_i \subset W$. By picking $u \in B$ such that $v - uv \in W$, we finally get, for each $i \in \{1, \ldots, n\}$, $a_i - ua_i = (a_i - va_i) + (v - uv)\, a_i + u(va_i - a_i) \in W + WF + BW \subset V + V + V \subset U$. ∎

In view of the following theorem, recall that a topological algebra A is called *polynomially generated* by a set S if the smallest subalgebra generated by S (i.e., $\mathbb{C}_0[S] = \{p(a_1, \ldots, a_n) \colon \; p \in \mathbb{C}[X_1, \ldots, X_n],\; p(0) = 0,\; a_1, \ldots, a_n \in S,\; n \in \mathbb{N}\}$) is dense in A.

Theorem 3.9. *Let A be a right hypotopological algebra. Then:*
(1) The following are equivalent:
(i) A has a left bai. (ii) A has left baus. (iii) A has a weak left bai.
(2) If A has a dense subset D, then it has a left bai if and only if there is a $B \in \mathcal{B}(A)$ such that, for each $a \in D$ and $U \in \mathcal{N}_0(A)$, there exists $u \in B$ with $a - ua \in U$.
(3) If A is polynomially generated by a set S, then $(e_\lambda)_{\lambda \in \Lambda}$ is a left bai for A if and only if $e_\lambda b \to b$, for every $b \in S$.

Proof. (1; i⇔ii) stems from 3.1 and 3.8. In (1; i⇔iii), (2) and (3), we need only prove the "if" part.

(1; i⇐iii) Let $(e_\lambda)_{\lambda \in \Lambda}$ be a weak left bai of A and set $B \equiv \operatorname{co}\{e_\lambda \colon \lambda \in \Lambda\}$. Take $a \in A$ and $U \in \mathcal{N}_0(A)$. Then $a \overset{\sigma}{=} \lim_\lambda e_\lambda a \in \overline{Ba}^{\sigma} = \overline{Ba}$, where the second equality holds due to the fact that the closure of a convex subset of A is the same for all (A, A')-compatible topologies [26, IV.3.1]. So, there exists $u \in B$ such that $a - ua \in U$ and the claim follows from (1; ii⇒i).

(2) Take $a \in A$ and $U \in \mathcal{N}_0(A)$. Choose $V \in \mathcal{N}_0(A)$ with $V + V + V \subset U$ and $W \in \mathcal{N}_0(A)$ with $W \subset V$ and $BW \subset V$. Pick $d \in D$ with $a - d \in W$ and $u \in B$ with $d - ud \in V$. Then $a - ua = (a - d) + (d - ud) + u(d - a) \in W + V + BW \subset V + V + V \subset U$.

(3) Clearly, $e_\lambda b \to b$, for any $b \in \mathbb{C}_0[S]$. So, the conclusion follows by arguing as in (2). ∎

Proposition 3.10. *Let A be a metrizable right hypotopological algebra with a left bai and $S \subset A$ a countable set. Then there exists a closed separable subalgebra D of A that contains S and has a sequential left bai.*

Proof. Let $S = \{a_n \colon n \in \mathbb{N}\}$, $(U_n)_{n \in \mathbb{N}}$ a 0-neighborhood basis in A and $B \in \mathcal{B}(A)$ as in 3.7 (cf. 3.9). Define inductively a sequence $(e_n)_{n \in \mathbb{N}}$ in B as follows: Choose $e_1 \in B$ with $a_1 - e_1 a_1 \in U_1$. Assuming that $e_1, \ldots, e_n$ are specified, set $Y_n = \{a_1, \ldots, a_{n+1}, e_1, \ldots, e_n\}$ and choose, on the basis of 3.8, $e_{n+1} \in B$ such that $b - e_{n+1} b \in U_n \; \forall\, b \in Y_n$. Put $Y = \bigcup_{n \in \mathbb{N}} Y_n = \{a_n, e_n \colon n \in \mathbb{N}\}$ and $D = \overline{\mathbb{C}_0[Y]}$. Since $b = \lim_n e_n b \; \forall\, b \in Y$, 3.9(3) implies that $(e_n)_{n \in \mathbb{N}}$ is a left bai for D. ∎

Corollary 3.11. *If a metrizable, separable right hypotopological algebra has a left bai, then it also has a sequential left bai.*

Proposition 3.12. *Let I and J be closed right respectively left ideals in a right hypotopological algebra A. If I has left baus, then $I+J$ is a closed subspace of A.*

Proof. In view of [6, A.3.11], the claim will hold if the canonical map $(I+J)/J \to I/(I\cap J)$ is continuous. Let $\pi_1\colon I+J \to (I+J)/J$, $\pi_2\colon I \to I/(I\cap J)$ be the quotient maps and assume that the left baus of I are contained in some $B \in \mathcal{B}(I)$. Let $U \in \mathcal{N}_0(I)$. As the topology of A is defined by all the continuous F-seminorms on A [14, 2.7.3], there is a 0-neighborhood of the form $U_0 = \{a \in I\colon q(a) < \varepsilon\}$, with q F-seminorm and $\varepsilon > 0$, inside of U. Choose $V, W \in \mathcal{N}_0(I)$ with $V + V \subset U_0$ and $BW \subset V$ and a 0-neighborhood $W_0 := \{a \in I\colon p(a) < \delta\} \subset W$ with p F-seminorm and $\delta > 0$. Take $a \in I$ with $\pi_1(a) \in \pi_1(W_0)$, i.e., $\dot{p}(a+J) := \inf\{p(a+b)\colon b \in J\} < \delta$. Then, there exist $b \in J$ with $p(a-b) < \delta$ and $u \in B$ with $a - ua \in V$. Thus, we have $ub \in I\cap J$ and $a - ub = (a-ua) + u(a-b) \in V + BW \subset V + V \subset U_0$. Hence, $\dot{q}(a + I\cap J) \le q(a-ub) < \varepsilon$, that is, $\pi_2(a) \in \pi_2(U_0) \subset \pi_2(U)$ and the proof is finished. ■

Proposition 3.13. *Let I be a closed two-sided ideal in a right hypotopological algebra A. If I and A/I have both left bais, then A has a left bai too.*

Proof. Suppose that the left baus of I and A/I are contained in B_1 and B_2, respectively. Let $a \in A$ and $U \in \mathcal{N}_0(A)$. Choose $V, W \in \mathcal{N}_0(A)$ such that $V+V+V \subset U$ and $W \subset V$, $B_1W \subset V$, and a 0-neighborhood $W_0 = \{a \in A\colon q(a) < \varepsilon\} \subset W$, with q F-seminorm and $\varepsilon > 0$. Then, if $\pi\colon A \to A/I$ is the quotient map, there exists $\pi(u) \in B_2$ with $\pi(a-ua) \in \pi(W_0)$ and so there is $b \in I$ with $a - ua + b \in W_0$. On the other side, there exists $v \in B_1$ with $b - vb \in V$. Hence, we have $a - (v\circ u)a = (a-ua+b) - v(a-ua+b) + (vb-b) \in W + B_1W + V \subset V+V+V \subset U$ and $v \circ u$ belongs to $B_1 + B_2 - B_1B_2 \in \mathcal{B}(A)$. ■

Proposition 3.14. *If I and J are two-sided respectively left ideals in a right hypotopological algebra that both have left bais, then $I \cap J$ has a left bai too.*

Proof. Suppose that the left baus of I and J are contained in B_1 and B_2, respectively. Take $a \in I\cap J$ and $U \in \mathcal{N}_0(I\cap J)$. Choose $V \in \mathcal{N}_0(I\cap J)$ with $V + V \subset U$ and $V_1 \in \mathcal{N}_0(I)$ with $V_1 \cap (I\cap J) = V$. Since the product $I \times J \to I$ is $\mathcal{B}(I)$-hypocontinuous, there is $W_2 \in \mathcal{N}_0(J)$ with $B_1W_2 \subset V_1$. Set $W = W_2 \cap (I\cap J)$. Then $B_1W \subset V$. Pick $u_1 \in B_1$, $u_2 \in B_2$ with $a - u_1a \in V_1$ and $a - u_2a \in W_2$. Then, we have $a - u_1u_2\,a = (a-u_1a) + u_1(a-u_2a) \in V + B_1W \subset V+V \subset U$ and u_1u_2 belongs to $B_1B_2 \in \mathcal{B}(I\cap J)$. ■

4 Approximate identities and the bidual locally convex algebra

Let A be a left hypotopological lca. Then, as Gulick [11] showed, its bidual A''_b, denoted in future by A'', is a lca (m-convex if A is such) w.r.t. the first, or left, Arens product "$\Box$". This product is defined via the bilinear maps $A' \times A \to A'\colon (f,a) \mapsto fa\colon A \to \mathbb{C}\colon b \mapsto f(ab)$ and $A'' \times A' \to A'\colon (F,f) \mapsto Ff\colon A \to \mathbb{C}\colon a \mapsto F(fa)$ by: $A'' \times A'' \to A''\colon (F,G) \mapsto F \Box G\colon A' \to \mathbb{C}\colon f \mapsto F(Gf)$ (see [1]). It is an extension of A's multiplication such that, for every $G \in A''$, $F \mapsto F \Box G$ is continuous for $\sigma(A'', A')$ and, for every $a \in A$, $G \mapsto \hat{a} \Box G$

is continuous for $\sigma(A'', A')$, where $\hat{a} \equiv J(a)$ and $J\colon A \hookrightarrow A''$ is the canonical embedding [11, 3.4, 3.6].

In the Banach algebra setting, the second, or right, Arens product "$\Diamond$" is another extension of the multiplication of the original algebra A to A''. This is defined via the bilinear maps $A' \times A \to A'\colon (f, a) \mapsto af\colon A \to \mathbb{C}\colon b \mapsto f(ba)$ and $A'' \times A' \to A'\colon (F, f) \mapsto fF\colon A \to \mathbb{C}\colon a \mapsto F(af)$ by: $A'' \times A'' \to A''\colon (F, G) \mapsto F \Diamond G\colon A' \to \mathbb{C}\colon f \mapsto G(fF)$ (see [2]).

Since $(A'', \Diamond) = ((A^{\mathrm{op}})'', \Box)^{\mathrm{op}}$, Gulick's result yields the well-definition of the second Arens product in the general case where A is a right hypotopological lca, hence $(A'', \Diamond)$ is a lca too. It is readily seen that, in this case, $\Diamond$ has the following continuity properties: for each $F \in A''$, $G \mapsto F \Diamond G$ is continuous for $\sigma(A'', A')$ and, for each $a \in A$, $F \mapsto F \Diamond \hat{a}$ is continuous for $\sigma(A'', A')$. It is also easily verified that if $\phi\colon A \to B$ is a continuous morphism between left (resp. right) hypotopological lcas, then $\phi''\colon (A'', \Box) \to (B'', \Box)$ (resp. $(A'', \Diamond) \to (B'', \Diamond)$) is a continuous algebra morphism as well. If A is unital with identity e_A, then $\hat{e}_A$ is the identity of both $(A'', \Box)$ and $(A'', \Diamond)$.

In particular, the notion of Arens regularity introduced in [2] can be extended from the context of Banach algebras to the context of hypotopological lcas.

Definition 4.1. A hypotopological lca A is called *Arens regular* if $\Box$ and $\Diamond$ coincide on A''.

Since in a commutative hypotopological lca A, $F \Box G = G \Diamond F$ holds for every $F, G \in A''$, A is Arens regular if and only if $(A'', \Box)$ is commutative. Gulick [11] called an m-convex algebra *bicommutative* if its bidual with $\Box$ is commutative. He proved that a complete m-convex algebra is bicommutative if it can be represented as an inverse limit of bicommutative Banach algebras [11, 5.2]. This result combined with the fact that every C*-algebra is Arens regular [5, 7.1] implies that *every commutative locally C*-algebra is Arens regular*. Since, moreover, bicommutativity is inherited by subalgebras (cf. [11, 5.3]), *every subalgebra of a commutative locally C*-algebra is Arens regular*. By the continuity properties of the two Arens products and the fact that $J(A)$ is dense in A''_{σ^*}, it is easily verified that a hypotopological algebra A is Arens regular if and only if for each $F \in A''$, $G \mapsto F \Box G$ is continuous for $\sigma(A'', A')$ or, equivalently, for each $G \in A''$, $F \mapsto F \Diamond G$ is continuous for $\sigma(A'', A')$.

As regards bais, we have the next

Theorem 4.2. *(1) A right hypotopological lca A has a left bai if and only if $(A'', \Diamond)$ has a left identity.*
(2) A left hypotopological lca A has a right bai if and only if $(A'', \Box)$ has a right identity.
(3) A hypotopological lca A has a bai if and only if $(A'', \Box)$ has a right identity and $(A'', \Diamond)$ has a left identity.
(4) An Arens regular lca A has a bai if and only if $(A'', \Box)$ is unital.

Proof. (1) ($\Rightarrow$) Let $(e_\lambda)_{\lambda \in \Lambda}$ be a left bai for A. Since $\{\hat{e}_\lambda\colon \lambda \in \Lambda\}$ is equicontinuous in A'', by the Alaoglu-Bourbaki theorem it is relatively weak* compact. Hence, there is $L \in A''$ with $\hat{e}_\lambda \to L$ in A''_{σ^*}. We thus have $(fL)(a) = L(af) = \lim_\lambda f(e_\lambda a) = f(a)\ \forall\, a \in A$, $f \in A'$ and so $fL = f\ \forall\, f \in A'$. As a result, $(L \Diamond F)(f) = F(fL) = F(f)\ \forall\, F \in A''$, $f \in A'$.

($\Leftarrow$) Let L be a left identity of $(A'', \Diamond)$. Viewing $\{L\}$ as an equicontinuous set in A'', there is $B \in \mathcal{B}(A)$ with $L \in B^{oo} = \overline{J(B)}^{\sigma^*}$, where the equality holds due to the bipolar theorem. Hence, there is $(e_\lambda)_\lambda$ in B with $\hat{e}_\lambda \to L$ in A''_{σ^*}. We thus have $\lim_\lambda f(e_\lambda a) = \lim_\lambda (af)(e_\lambda) = \lim_\lambda \hat{e}_\lambda(af) = L(af) = (fL)(a) = (L \Diamond \hat{a})(f) = f(a)\ \forall\, a \in A,\ f \in A'$ and so $(e_\lambda)_\lambda$ is a weak left bai for A. The conclusion now follows from 3.9(1).

(2) is proven similarly. (3) follows from (1), (2) and 3.2, while (4) follows from (3). ■

Since any locally C*-algebra has a bai [13, 2.6], Theorem 4.2 immediately implies the next result concerning the bidual of a locally C*-algebra; in this respect, see also [24, 3.5].

Corollary 4.3. *The bidual of a commutative locally C*-algebra equipped with either Arens product is a unital commutative m-convex algebra.*

The category of Fréchet (resp. Banach) left A-lcms and continuous morphisms appearing in the next corollary is denoted by A-Frmod (resp. A-Banmod); the notation is modified accordingly for the categories of right modules. Recall that a sequence $(\mathcal{C})\ 0 \to X \overset{f}{\to} Y \overset{g}{\to} Z \to 0$ of morphisms between lcms is said to be *split* (resp. *admissible*) if it is topologically isomorphic, as a sequence of lcms (resp. of lcs), to $0 \to X \overset{i}{\to} X \oplus Z \overset{p}{\to} Z \to 0$ (i, p are the canonical embedding respectively projection). $(\mathcal{C})$ splits if and only if $(\mathcal{C})$ is exact, with f topologically injective and g retraction (i.e. having a right inverse morphism) or, equivalently, with f coretraction (i.e. having a left inverse morphism) and g open [12, III.1.8]. A module $J \in A$-Banmod such that the contravariant functor ${}_A h(\cdot, J)\colon A\text{-Banmod} \to \mathsf{VS}$ sends short admissible sequences of Banach left A-modules to exact sequences of vector spaces is called *injective* [12, III.1.14]. Recall also that a lcs E is called *quasinormable* [10] if for every equicontinuous set $C \subset E'$, there is a 0-neighborhood V in E such that, on C, the topology induced by $b(E', E)$ coincides with the topology of uniform convergence on V. The class of quasinormable Fréchet spaces contains all Banach spaces, Fréchet-Schwartz (hence Fréchet nuclear) spaces, quojections and prequojections (see [4]). The reader is referred to [10], [14], [16], [20] for more information on quasinormable spaces and to [7], [17] for quojections and prequojections.

Corollary 4.4. *A quasinormable Fréchet m-convex algebra A has a left bai if and only if it has an Arens-Michael representation $A = \varprojlim A_n$ such that each A_n has a left bai.*

Proof. ($\Rightarrow$) In general, if an m-convex algebra $(A, (s_\mu)_{\mu \in M})$ has a left bai $(e_\lambda)_\lambda$, then we know that each one of the canonical maps of the Arens-Michael analysis $A \underset{\to}{\subseteq} \varprojlim A_\mu$ of A (where $A_\mu \equiv (A/\mathrm{Ker}(s_\mu))\tilde{}$), say $\pi_\mu\colon A \to A_\mu$, takes $(e_\lambda)_\lambda$ to a left bai in A_μ (cf. [9, 11.4]).

($\Leftarrow$) By 4.2, we need to prove that $(A'', \Diamond)$ has a left identity. Consider the admissible sequences $(\mathcal{C})\ 0 \to A \overset{i}{\to} A_+ \overset{p}{\to} \mathbb{C} \to 0$, $(\mathcal{C}_n)\ 0 \to A_n \overset{i_n}{\to} (A_n)_+ \overset{p_n}{\to} \mathbb{C} \to 0$ in Frmod-A, respectively Banmod-A_n ($n \in \mathbb{N}$). Since, for each $n \in \mathbb{N}$, A_n has a left bai, $\mathbb{C}' \cong \mathbb{C}$ is an injective Banach left A_n-module (with the trivial multiplication) [12, VII.1.20]. So, as the dual sequence $(\mathcal{C}'_n)$ is admissible in A_n-Banmod, the sequence

$$({}_{A_n}h(\mathcal{C}'_n, \mathbb{C}))\ 0 \to {}_{A_n}h(A'_n, \mathbb{C}) \xrightarrow{f_n} {}_{A_n}h((A_n)'_+, \mathbb{C}) \xrightarrow{g_n} \mathbb{C} \to 0$$

is exact, where $f_n \equiv {}_{A_n}h(i'_n, id_{\mathbb{C}})$ and $g_n \equiv {}_{A_n}h(p'_n, id_{\mathbb{C}})$ with $g_n(F) = F(p_n)$ (it is $p'_n(1) = p_n$). Note that ${}_{A_n}h(A'_n, \mathbb{C})$ coincides with the left annihilator $A''^{\perp}_n := \{F \in A''_n \colon F \lozenge A''_n = 0\}$ of $(A''_n, \lozenge)$. Since moreover π_n has dense range, each term of $(\mathcal{C}'_n)$ can be rendered an A-module. This allows viewing $({}_{A_n}h(\mathcal{C}'_n, \mathbb{C}))$ as $({}_{A}h(\mathcal{C}'_n, \mathbb{C}))$. We let H_n denote ${}_{A}h(A'_n, \mathbb{C})$. Moreover, on the basis of 4.2, we let E_n denote a left identity for $(A''_n, \lozenge)$.

Let $\pi_{mn} \colon A_m \to A_n$ $(n \leq m)$ be the linking maps of the inverse spectrum formed by A_n's. Then, for any $n \leq m$, $\pi''_{mn}(H_m) \subset H_n$. Indeed, for each $F \in H_m$, we have $\pi''_{mn}(F) \lozenge \pi''_{mn}(A''_m) = \pi''_{mn}(F \lozenge A''_m) = 0$ and so, by the continuity of $\lozenge$ w.r.t. the second variable for $\sigma(A''_n, A'_n)$, $\pi''_{mn}(F) \lozenge A''_n = \pi''_{mn}(F) \lozenge \overline{\pi''_{mn}(A''_m)}^{\sigma*} = 0$. Therefore, $(H_n, \pi''_{mn}|_{H_m})$ is an inverse spectrum of Banach subspaces of (A''_n, π''_{mn}). In fact, each H_n is topologically embedded into A''_n, since $P_n \colon A''_n \to A''_n \colon F \mapsto F - F \lozenge E_n$ is a continuous projection onto H_n and hence the canonical embedding $j_n \colon H_n \hookrightarrow A''_n$ is a coretraction.

Since A is quasinormable, the first derived inverse limit functor $\varprojlim^1(A''_n, \pi''_{mn})$ vanishes [19, 7.5, Pr. 6.2] (for a simpler proof of this, see [24, 3.6]). Equivalently, this means that the map $\Psi \colon \prod A''_n \to \prod A''_n \colon (F_n) \mapsto (F_n - \pi''_{n+1,n}(F_{n+1}))$ is open onto its image [27, 3.2.8, 3.3.1]. But then, $\check{\Psi} \colon \prod H_n \to \prod H_n \colon (F_n) \mapsto (F_n - \pi''_{n+1,n}|_{H_{n+1}}(F_{n+1}))$ is open too, since $\check{\Psi} = \Psi \circ \prod j_n$. Namely, $\varprojlim^1(H_n, \pi''_{mn}|_{H_m}) = 0$. This implies that the "inverse limit sequence" $(\varprojlim {}_{A}h(\mathcal{C}'_n, \mathbb{C}))$ is exact; see [19, (5.5)] and/or [27, 3.1.5]. A being quasinormable yields also that $\varprojlim {}_{A}h(A'_n, \mathbb{C}) = {}_{A}h(\varinjlim A'_n, \mathbb{C}) = {}_{A}h(A', \mathbb{C})$ hold up to isomorphisms of vector spaces [16, 25.13] (in fact, up to topological isomorphisms of Fréchet spaces, when ${}_{A}h(\varinjlim A'_n, \mathbb{C})$ and ${}_{A}h(A', \mathbb{C})$ are considered with the strong topology [26, Ex. 6, p. 116]). Similarly,
$\varprojlim {}_{A}h((A_n)'_+, \mathbb{C}) = {}_{A}h(A'_+, \mathbb{C})$ and $\varprojlim \mathbb{C} = \mathbb{C}$. Consequently, the exact inverse limit sequence takes the form

$$(\varprojlim {}_{A}h(\mathcal{C}'_n, \mathbb{C}))\ 0 \to {}_{A}h(A', \mathbb{C}) \xrightarrow{f} {}_{A}h(A'_+, \mathbb{C}) \xrightarrow{g} \mathbb{C} \to 0,$$

where $f = {}_{A}h(i', id_{\mathbb{C}})$ and $g = {}_{A}h(p', id_{\mathbb{C}})$ with $g(F) = F(p)$.

By the surjectivity of g, there is $F \in {}_{A}h(A'_+, \mathbb{C})$ with $g(F) = 1$, i.e., $F \circ p' = id_{\mathbb{C}}$. Hence, $(\mathcal{C}')$ and thus also $(\mathcal{C}'')$ splits. Let $\rho \colon A''_+ \to A''$ be a left inverse of i'' and set $E = \rho(\hat{e})$. Then, we have $E \lozenge \hat{a} = E \cdot a = \rho(\hat{a}) = \rho(\widehat{i(a)}) = \rho(i''(\hat{a})) = \hat{a}$ $\forall\, a \in A$. But since $\overline{J(A)}^{\sigma*} = A''$ and $\lozenge$ is continuous w.r.t. the second variable for $\sigma(A'', A')$, we finally get $E \lozenge A'' = A''$. ∎

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