The unicity of best approximation in a space of compact operators

Joanna Kowynia*

Abstract

Chebyshev subspaces of $\mathcal{K}(c_0, c_0)$ are studied. A k-dimensional non-interpolating Chebyshev subspace is constructed. The unicity of best approximation in non-Chebyshev subspaces is considered.

1 Introduction

Let $\mathbb{K}$ be the field of real or complex numbers and let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K}$. Let $\text{ext}S_{X^\star}$ denote the set of all extreme points of $S_{X^\star}$, where $S_{X^\star}$ is the unit sphere in $X^\star$.
For every $x \in X$ we put

$$E(x) = \{f \in \text{ext}S_{X^\star} : f(x) = \|x\|\}. \tag{1}$$

By the Hahn - Banach and the Krein - Milman Theorems, $E(x) \neq \emptyset$.
Let for $Y \subset X$,

$$P_Y(x) = \{y \in Y : \|x - y\| = \text{dist}(x, Y)\}.$$

A linear subspace $Y \subset X$ is called **a Chebyshev subspace** if for every $x \in X$ the set $P_Y(x)$ contains one and only one element.

---

*Research supported by local grant No. 10.420.03
Received by the editors July 2009.
Communicated by F. Bastin.
Key words and phrases : strongly unique best approximation, interpolating subspace, Chebyshev subspace.

Theorem 1 (see [3]) *Assume X is a normed space, $Y \subset X$ is a linear subspace, and let $x \in X \setminus Y$. Then $y_0 \in P_Y(x)$ if and only if for every $y \in Y$ there exists $f \in E(x - y_0)$ with $\Re f(y) \leq 0$.*

Definition (see e. g. [8]) *An element $y_0 \in Y$ is called a strongly unique best approximation for $x \in X$ if there exists $r > 0$ such that for every $y \in Y$,*

$$\|x - y\| \geq \|x - y_0\| + r\|y - y_0\|.$$

The biggest constant r satisfying the above inequality is called **a strong unicity constant**. There exist two main applications of a strong unicity constant: the error estimate of the Remez algorithm (see e. g. [13]), the Lipschitz continuity of the best approximation mapping at x_0 (assuming that there exists a strongly unique best approximation to x_0) (see e. g. [5, 9, 11]).

Theorem 2 (see [17]) *Let $x \in X \setminus Y$ and let Y be a linear subspace of X. Then $y_0 \in Y$ is a strongly unique best approximation for x with a constant $r > 0$ if and only if for every $y \in Y$ there exists $f \in E(x - y_0)$ with $\Re f(y) \leq -r\|y\|$.*

Recall that a k - dimensional subspace $\mathcal{V}$ of a normed space X is called **an interpolating subspace** if for any linearly independent $f_1, f_2, ..., f_k \in \text{ext}S_{X^\star}$ and for every $v \in \mathcal{V}$ the following holds:

$$\text{if} \quad f_i(v) = 0, \quad i = 1, 2, ..., k \quad \text{then} \quad v = 0.$$

Every interpolating subspace is a finite dimensional Chebyshev subspace. If $\mathcal{V} \subset X$ is an interpolating subspace then every $x \in X$ has a strongly unique best approximation in $\mathcal{V}$ (see [2]).

In this paper we consider $X = \mathcal{K}(c_0, c_0)$ (the space of all compact operators from c_0 to c_0 equipped with the operator norm). Here c_0 denotes the space of all real sequences convergent to zero. For any $x = (x_k) \in c_0$ we put

$$\|x\|_\infty = \sup_k \mid x_k \mid .$$

In [8, Theorem 3.1] it has been proved that if $\mathcal{V} \subset \mathcal{K}(c_0, c_0)$ is a finite-dimensional Chebyshev subspace then every $A \in \mathcal{K}(c_0, c_0)$ has a strongly unique best approximation in $\mathcal{V}$.
However, in [8] no example of a non-interpolating Chebyshev subspace has been proposed. If it were true that any finite-dimensional Chebyshev subspace of $\mathcal{K}(c_0, c_0)$ is an interpolating subspace we would have obtained the proof of Theorem 3.1, [8] immediately (see [2] for more details).

The aim of this paper is to show that for every $k < \infty$ there exists a k-dimensional non-interpolating Chebyshev subspace of $\mathcal{K}(c_0, c_0)$.
This result is quite different from the result obtained in [7]. In the space $\mathcal{L}(l_1^n, c_0)$ any finite-dimensional Chebyshev subspace is an interpolating subspace.

Additionally, we discuss the strong unicity of best approximation in some (not necessarily Chebyshev) subspaces of $\mathcal{K}(c_0, c_0)$.

2 k-dimensional Chebyshev subspaces of $\mathcal{K}(c_0, c_0)$

Let $A \in \mathcal{K}(c_0, c_0)$ be represented by a matrix $[a_{ij}]_{i,j\in\mathbb{N}}$. Note that

$$(a_{ij})_{i=1}^{\infty} \in c_0 \text{ for every } j \in \mathbb{N}.$$

Since each row of a matrix $[a_{ij}]_{i,j\in\mathbb{N}}$ corresponds to a linear functional on c_0,

$$(a_{ij})_{j=1}^{\infty} \in l^1 \quad \text{for every } \ i \in \mathbb{N}.$$

Moreover, by the Schur Theorem (see [6])

$$\lim_{i\to\infty}\Big(\sum_{j=1}^{\infty} \mid a_{ij} \mid\Big) = 0.$$

Recall (see[4]) that $\text{ext}S_{\mathcal{K}^*(c_0,c_0)}$ consists of functionals of the form $e_i \otimes x$, where $x \in \text{ext}S_{l^\infty}$ and

$$(e_i \otimes x)(A) = \sum_{j=1}^{\infty} x_j a_{ij}. \tag{2}$$

It is easy to see that

$$\|A\| = \sup_{i\geq 1} \sum_{j=1}^{\infty} \mid a_{ij} \mid .$$

Remark 1 *Let X be a Banach space and let $\mathcal{V}$ be a finite-dimensional subspace with $V_1, V_2, ..., V_k$ as a basis.*
$\mathcal{V}$ is an interpolating subspace if and only if for any linearly independent $f_1, f_2, ..., f_k \in \text{ext}S_{X^\star}$ the determinant of $[f_i(V_j)]_{i,j=1,2,...,k}$ is not equal to zero.

Proof. We apply the definition of a k - dimensional interpolating subspace and the theory of linear equations. This completes the proof. ■

In the sequel, we denote by $\text{lin}\{V_1, V_2, ..., V_k\}$ the k-dimensional subspace of $\mathcal{K}(c_0, c_0)$ with $V_1, V_2, ..., V_k$ as a basis.

Example 1 Let $V = [v_{ij}]_{i,j\in\mathbb{N}}$, where $v_{i1} = \frac{1}{2^i}$, $v_{ij} = 0$, $i, j \in \mathbb{N}$, $j \geq 2$. It is obvious that $\mathcal{V} = \text{lin}\{V\}$ is a one-dimensional interpolating subspace of $\mathcal{K}(c_0, c_0)$.

Theorem 3 *Let $\mathcal{V} = \text{lin}\{V_1, V_2, ..., V_n\}$. Let $V_m = [(v_m)_{ij}]_{i,j\in\mathbb{N}}$, $m = 1, 2, ..., n$. If $\mathcal{V}$ is a Chebyshev subspace then*

$$\begin{vmatrix} f_1(V_1) & . & . & . & f_1(V_n) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ f_n(V_1) & . & . & . & f_n(V_n) \end{vmatrix} \neq 0 \tag{3}$$

for any $f_1, ..., f_n \in \text{ext}S_{\mathcal{K}^*(c_0,c_0)}$ *such that* $f_m = e_{i_m} \otimes x^{i_m}, \quad m = 1, 2, ..., n,$ *where* $i_m \neq i_k$ *for* $m \neq k$.

Proof. Assume (3) does not hold. Therefore there exist $f_1, ..., f_n \in \text{ext}S_{\mathcal{K}^*(c_0,c_0)}$, $f_m = e_{i_m} \otimes x^{i_m}, \quad m = 1, 2, ..., n,$ where $i_m \neq i_k$ for $m \neq k$ such that $\det D = 0$, where

$$D = \begin{bmatrix} f_1(V_1) & . & . & . & f_1(V_n) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ f_n(V_1) & . & . & . & f_n(V_n) \end{bmatrix}.$$

Since $\det D = \det D^T$, there exists $y = (y_1, y_2, ..., y_n) \neq 0$ such that $D^T y = 0$. Consequently,

$$\sum_{m=1}^{n} y_m f_m|_{\mathcal{V}} = 0. \tag{4}$$

Since $y \neq 0$, replacing f_m by $-f_m$ if necessary, we may assume $y_m \geq 0$ for $m = 1, 2, ..., n$ and

$$\sum_{m=1}^{n} y_m = 1.$$

Set $\mathcal{C} = \{ l \in \{1, 2, ..., n\} \ : \ y_l > 0 \}$.
Fix $(d_j)_{j \in \mathbb{N}}$ with the following properties:

$$d_j > 0, \quad j \in \mathbb{N} \quad \text{and} \quad \sum_{j=1}^{\infty} d_j = 1.$$

Define $A = [a_{i_p j}]_{i_p, j \in \mathbb{N}} \in \mathcal{K}(c_0, c_0)$ by

$$a_{i_p j} = 0 \quad \text{for} \quad p \notin \mathcal{C}, \quad j \in \mathbb{N},$$
$$a_{i_p j} = d_j \cdot \text{sgn}\, x^{i_p}(j) \quad \text{for} \quad p \in \mathcal{C}, \quad j \in \mathbb{N}.$$

Note that $\|A\| = 1$ and

$$E(A) = \{f_p \ : \ p \in \mathcal{C}\}.$$

By (4) and Theorem 1, $0 \in \mathcal{P}_{\mathcal{V}}(A)$.
Since $\det D = 0$, there exists $x = (x_1, x_2, ..., x_n) \neq 0$ such that $Dx = 0$.
Put

$$V = \sum_{m=1}^{n} x_m V_m.$$

Note that $V \neq 0$ and $f_m(V) = 0, \quad m = 1, 2, ..., n$.
By Theorem 2, 0 is not a strongly unique best approximation for A in $\mathcal{V}$. By [8,Theorem 3.1], $\mathcal{V}$ is not a Chebyshev subspace and the proof is complete. ∎

Theorem 4 *Let* $\mathcal{V} = \mathrm{lin}\{V\}, \quad V \in \mathcal{K}(c_0, c_0), \quad V \neq 0.$
$\mathcal{V}$ *is a Chebyshev subspace if and only if* $\mathcal{V}$ *is an interpolating subspace.*

Proof. The classical work here is [12]. In l^1, the one-dimensional subspace $\mathrm{lin}\{v\}$ is Chebyshev iff for every $x \in \mathrm{ext} S_{l^\infty}$ the following holds

$$\sum_{j=1}^{\infty} x(j)v(j) \neq 0.$$

Note that for any $x \in c_0$ we obtain $V(x) = [f_1(x), f_2(x), ...]$, where the functionals f_i correspond to elements of l^1.
It is obvious that if for any j, $\mathrm{lin}\{f_j\}$ is not a Chebyshev subspace of l^1, then $\mathrm{lin}\{V\}$ is not a Chebyshev subspace of $\mathcal{K}(c_0, c_0)$.
This proves the theorem. ∎

Note that by a result of Malbrock (see [10] , Theorem 3.3) each one-dimensional subspace $\mathcal{V} = \mathrm{lin}\{V\} \subset \mathcal{L}(c_0, c_0)$ is a Chebyshev subspace iff there exists $\delta > 0$ such that

$$\left| \sum_{j=1}^{\infty} x(j)v_{ij} \right| \geq \delta,$$

where $| x(j) | = 1, \ j \in \mathbb{N}$.

Corollary *Let* $\mathcal{V} \subset \mathcal{K}(c_0, c_0)$ *be a one-dimensional Chebyshev subspace. Every operator* $A \in \mathcal{K}(c_0, c_0)$ *has a strongly unique best approximation in* $\mathcal{V}$.

Proof. Obvious. For more details we refer the reader to [2]. ∎

It is clear that (3) is satisfied for any n-dimensional interpolating subspace. However, (3) is not sufficient for an n-dimensional ($n \geq 2$) subspace to be Chebyshev.

Example 2 Let $\mathcal{V} = \mathrm{lin}\{V_1, V_2\}$, where

$$V_1 = \begin{bmatrix} 1 & 0 & . & . & . \\ \frac{1}{2} & 0 & . & . & . \\ \frac{1}{4} & 0 & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}, V_2 = \begin{bmatrix} 1 & 0 & . & . & . \\ \frac{1}{3} & 0 & . & . & . \\ \frac{1}{9} & 0 & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}.$$

Note that $\mathcal{V}$ satisfies (3). We claim that $\mathcal{V}$ is a non-Chebyshev subspace.
Indeed, define $A = [a_{ij}]_{i,j \in \mathbb{N}}$ by

$$a_{12} = 100, \quad a_{ij} = 0 \quad \text{for each} \quad (i,j) \neq (1,2), \quad i,j \in \mathbb{N}.$$

It follows that

$$A - (\alpha_1 V_1 + \alpha_2 V_2) = \begin{bmatrix} -\alpha_1 - \alpha_2 & 100 & 0 & . & . \\ -\frac{1}{2}\alpha_1 - \frac{1}{3}\alpha_2 & 0 & . & . & . \\ -\frac{1}{4}\alpha_1 - \frac{1}{9}\alpha_2 & 0 & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}, \quad \alpha_1, \alpha_2 \in \mathbb{R}.$$

Hence

$$\|A\| = \|A - (600V_1 - 600V_2)\| = 100 = \inf_{\alpha_1,\alpha_2\in\mathbb{R}} \|A - (\alpha_1 V_1 + \alpha_2 V_2)\|.$$

Theorem 5 *Let $V_1, V_2, ..., V_n$ be given by*

$$V_j = \begin{bmatrix} 0 & 0 & . & . & v_{1j} & 0 & . & . \\ 0 & 0 & . & . & v_{2j} & 0 & . & . \\ 0 & 0 & . & . & v_{3j} & 0 & . & . \\ . & . & . & . & . & & & \\ . & . & . & . & . & & & \\ . & . & . & . & . & & & \\ . & . & . & . & . & & & \end{bmatrix},$$

where $v_{ij} \neq 0$ for each $i \in \mathbb{N}$, $j \in \{1,2,...,n\}$ and

$$\lim_{i\to\infty} v_{ij} = 0 \quad \textit{for each} \quad j \in \{1,2,...,n\}.$$

The following statements are equivalent:

(i) For every choice of distinct $j_1,...,j_k$ from $\{1,2,...,n\}$, $\mathcal{V}(j_1,...,j_k) := \operatorname{lin}\{V_{j_1},...,V_{j_k}\}$ is a Chebyshev subspace of $\mathcal{K}(c_0,c_0)$,

(ii)
$$\forall\, 1 \le k \le n, \quad \forall\, 1 \le j_1 < j_2 < ... < j_k \le n,$$
$$\forall\, 1 \le i_1 < i_2 < ... < i_k,$$
$$\forall\, x_{ml} \in \mathbb{R} : |\, x_{ml} \,| = 1,\ m,\ l = 1,2,...,k$$
$$det[x_{ml} v_{i_m j_l}]_{m=1,2,...,k,\ l=1,2,...,k} \neq 0.$$

Proof. First, we assume that *(ii)* holds.
If $k = 1$ then $\mathcal{V}(j_1)$ is an interpolating subspace for every $j_1 \in \{1,2,...,n\}$.
Let $1 < k < n$ and assume that for any $j_1,...,j_k \in \{1,2,...,n\}, j_p \neq j_q, p \neq q$, $\mathcal{V}_k := \mathcal{V}(j_1,...,j_k)$ is a Chebyshev subspace.
Suppose that there exist $1 \le j_1 < j_2 < ... < j_k < j_{k+1} \le n$ such that

$$\mathcal{V}_{k+1} := \mathcal{V}(j_1,...,j_k,j_{k+1})$$

is a non-Chebyshev subspace. Without loss of generality we can assume that for any $k+1 \in \{1,2,...,n\}$, $j_m = m$, $m = 1,2,...,k+1$. This means precisely that $V_{j_m} = [(V_{j_m})_{ij}]_{i,j\in\mathbb{N}}$, where

$$(V_{j_m})_{ij} = \begin{cases} v_{ij_m}, & j = m \\ 0, & j \neq m \end{cases}$$

for $i \in \mathbb{N}$, $m \in \{1,2,...,k,k+1\}$.
Since $\mathcal{V}_{k+1}$ is a non-Chebyshev subspace, there exists $A = [a_{ij}]_{i,j\in\mathbb{N}} \in \mathcal{K}(c_0,c_0)$ such that $\sharp\, \mathcal{P}_{\mathcal{V}_{k+1}}(A) > 1$. We can assume that $0,\ W \in \mathcal{P}_{\mathcal{V}_{k+1}}(A)$, where $W \neq 0$.

Let $\mathcal{U} = \{i : \ \|e_i \circ A\| = \|A\|\}$. Since $A \in \mathcal{K}(c_0, c_0)$, $\sharp\,\mathcal{U} < \infty$.
For every $i \in \mathcal{U}$ we put

$$E_i = \{x \in \mathrm{ext} S_{l^\infty} \ : \ (e_i \otimes x)(A) = \|A\|\}.$$

Since $0,\ W \in \mathcal{P}_{\mathcal{V}_{k+1}}(A)$, we conclude that for all $i \in \mathcal{U}$ and $x \in E_i$

$$(e_i \otimes x)(W) \geq 0. \tag{5}$$

Let

$$\mathcal{U}_1 = \{i \in \mathcal{U} \ : \exists\, x \in E_i \ : \ (e_i \otimes x)(W) = 0\}.$$

Since $0 \in \mathcal{P}_{\mathcal{V}_{k+1}}(A)$, $\mathcal{U}_1 \neq \emptyset$.
We will prove that for any $i \in \mathcal{U}_1$ and $x, y \in E_i$ such that

$$\begin{aligned} (e_i \otimes x)(W) = (e_i \otimes y)(W) = 0, \\ x(l) = y(l), \quad l = 1, 2, ..., k+1. \end{aligned} \tag{6}$$

On the contrary, suppose that (6) does not hold. Let $x,\ y \in E_i$ be such that

$$(e_i \otimes x)(W) = 0, \qquad (e_i \otimes y)(W) = 0,$$

and

$$x(l) \neq y(l) \quad \text{ for some } \quad l \in \{1, 2, ..., k+1\}.$$

Without loss of generality we can assume

$$x(j) = y(j) \quad \text{for} \quad j = 1, 2, ..., p, \quad p < k+1$$

and

$$x(j) = -y(j) \quad \text{for} \quad j = p+1,\ p+2, ..., k+1.$$

Hence

$$\sum_{j=1}^{p} x(j) w_{ij} = 0, \qquad \sum_{j=p+1}^{k+1} x(j) w_{ij} = 0. \tag{7}$$

As

$$x(j) = -y(j) \quad \text{for} \quad j = p+1,\ p+2, ..., k+1$$

we obtain

$$a_{ij} = 0 \quad \text{for} \quad j = p+1,\ p+2, ..., k+1.$$

By (5),

$$\sum_{j=p+1}^{k} x(j) w_{ij} - x(k+1) w_{i,k+1} \geq 0$$

$$\sum_{j=p+1}^{k} -x(j) w_{ij} + x(k+1) w_{i,k+1} \geq 0.$$

Therefore

$$\sum_{j=p+1}^{k} x(j)w_{ij} = x(k+1)w_{i,k+1}.$$

By (7), $x(k+1)w_{i,k+1} = 0$. Consequently, $w_{i,k+1} = 0$. Hence $W \in \mathcal{V}_k$.
Since $0 \in \mathcal{V}_k$ and $\mathcal{V}_k$ is a Chebyshev subspace, (6) is proved.
We will show that there exists $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0]$,

$$E(A - \alpha W) = \{e_i \otimes x \ : \ i \in \mathcal{U}_1, \ (e_i \otimes x)(W) = 0, \ (e_i \otimes x)(A) = \|A\|\}. \quad (8)$$

We first prove that

$$\sup\{f(A) \ : \ f = e_i \otimes x, \ i \in \mathcal{U} \ : \ f(W) < 0\} \leq$$
$$\|A\| - 2\min\{| \ a_{ij} \ | : i \in \mathcal{U}, \ j \in \{1,2,...,n\}, \ a_{ij} \neq 0\}, \quad (9)$$

where $A = [a_{ij}]_{i,j \in \mathbb{N}}$.
Let $i \in \mathcal{U}$, $f = e_i \otimes x$, $f(W) < 0$. Hence there exists $j_0 \in \{1,2,...,n\}$ satisfying

$$x(j_0) \neq \operatorname{sgn}(a_{ij_0}) \quad \text{for} \quad a_{ij_0} \neq 0.$$

Now, we will show

$$f(A) = \sum_{j=1}^{\infty} x(j)a_{ij} \leq \|A\| - 2 \mid a_{ij_0} \mid \leq$$
$$\|A\| - 2\min\{| \ a_{ij} \ | : \ i \in \mathcal{U}, \ j = 1,2,...,n, \ | \ a_{ij} \ | \neq 0\},$$

and (9) is proved.
We conclude from (9) that there exist $\alpha_0 > 0$, $b > 0$ such that for every $\alpha \in (0, \alpha_0]$,

$$f(A - \alpha W) < b < \|A\|,$$

where $f \in \operatorname{ext} S_{\mathcal{K}^*(c_0,c_0)}$, $f(W) < 0$.
Assume α_0 is so small that

$$\sup_{i \in \mathbb{N} \setminus \mathcal{U}} \|e_i \circ (A - \alpha_0 W)\| < \|A\|.$$

Consequently, if $f \in E(A - \alpha_0 W)$ then $f = e_i \otimes x$, where $i \in \mathcal{U}_1$ and $f(W) = 0$.
Since

$$\|A - \alpha_0 W\| = \|A\| = \operatorname{dist}(A, \mathcal{V}_{k+1}),$$

(8) is proved.
Since $\alpha_0 W \in \mathcal{P}_{\mathcal{V}_{k+1}}(A)$, we conclude (see [16]) that

$$\exists \, 1 \leq q \leq k+2, \qquad \exists \, \lambda_1, ..., \lambda_q > 0, \qquad \sum_{m=1}^{q} \lambda_m = 1$$

such that

$$\sum_{m=1}^{q} \lambda_m (e_{i_m} \otimes x^{i_m})|_{\mathcal{V}_{k+1}} = 0, \quad (10)$$

where $(e_{i_m} \otimes x^{i_m})(A - \alpha_0 W) = \|A - \alpha_0 W\|$.
Let q be the smallest number having property (10). By (6), $i_j \neq i_l$ for $j \neq l, j,\ l \in \{1, 2, ..., q\}$. If $q = k + 2$ then (see [18]) $\alpha_0 W$ is the strongly unique best approximation for A in $\mathcal{V}_{k+1}$, a contradiction.
Suppose that $1 \leq q \leq k + 1$. This contradicts *(ii)*.

Let us assume that $\mathcal{V}_k$ is a Chebyshev subspace of $\mathcal{K}(c_0, c_0)$ for every $1 \leq k \leq n$. Suppose that *(ii)* is false.
Consequently, there exist

$$1 \leq k \leq n, \qquad 1 \leq j_1 < j_2 < ... < j_k \leq n,$$
$$1 \leq i_1 < i_2 < ... < i_k,$$
$$x_{ml} \in \mathbb{R} \ : \mid x_{ml} \mid = 1,\ m,\ l = 1, 2, ..., k$$

satisfying

$$det[x_{ml} v_{i_m j_l}]_{m=1,2,...,k,\ l=1,2,...,k} = 0.$$

It follows that there exist

$$\lambda_1, ..., \lambda_k \in \mathbb{R}, \qquad \sum_{m=1}^{k} \mid \lambda_m \mid > 0$$

such that

$$\sum_{m=1}^{k} \lambda_m (e_{i_m} \otimes x^{i_m})|_{\mathcal{V}_k} = 0, \tag{11}$$

where $x^{i_m} = (x^{i_m}(1), x^{i_m}(2), ...)$, $\quad x^{i_m}(l) = x_{ml}$.
Without loss of generality we can assume

$$\lambda_m > 0, \quad m = 1, 2, ..., k, \quad \sum_{m=1}^{k} \lambda_m = 1.$$

We define an operator $B = [b_{ij}]_{i,j \in \mathbb{N}}$ by

$$b_{ij} = \frac{\text{sgn} x^i(j)}{2^j}, \quad i \in \{i_1, i_2, ..., i_k\},$$
$$b_{ij} = 0, \quad i \notin \{i_1, i_2, ..., i_k\}, \quad j \in \mathbb{N}.$$

Hence $(e_{i_m} \otimes x^{i_m})(B) = \|B\|$, $m = 1, 2, ..., k$. By (11), $0 \in \mathcal{P}_{\mathcal{V}_k}(B)$ and

$$\dim \text{span}\{e_{i_m} \otimes x^{i_m} | \mathcal{V}_k\} < k,$$

where $dim \mathcal{V}_k = k$. Therefore there exists $V \in \mathcal{V}_k \setminus \{0\}$ such that

$$(e_{i_m} \otimes x^{i_m})(V) = 0, \quad m = 1, 2, ..., k.$$

Note that (see the proof of the formula (9))

$$\sup\{f(B) \ : \ f = e_{i_m} \otimes x,\ m = 1, 2, ..., k,\ f(V) < 0\} < \|B\| - \min\{| \ b_{ij} \ | : \ i = i_1, i_2, ..., i_k,\ j = 1, 2, ..., n\}.$$

Hence there exist $\alpha_0 > 0, \quad b > 0$ such that

$$f(B - \alpha_0 V) \leq b < \|B\|, \quad f \in \text{ext}S_{\mathcal{K}^*(c_0,c_0)}, \quad f(V) \leq 0.$$

Consequently, $\|B - \alpha_0 V\| = \|B\|$, a contradiction.
The proof is complete. ∎

Example 3 We will construct an n - dimensional Chebyshev subspace $\mathcal{V} \subset \mathcal{K}(c_0, c_0)$. Let $0 < t_1 < t_2 < ... < t_{n-1}$ be such that

$$\lim_{i\to\infty} \frac{1}{2^i} t_m^i = 0, \quad m = 1, 2, ..., n-1.$$

Define $V_m = [(v_m)_{ij}]_{i,j\in\mathbb{N}}$ by

$$(v_m)_{im} = \frac{1}{2^i} t_m^i, \quad (v_m)_{ij} = 0, \quad i \in \mathbb{N}, \quad j \neq m.$$

Hence $V_m \in \mathcal{K}(c_0, c_0)$ for every $m = 1, 2, ..., n-1$.
Let $\mathcal{V}_{n-1} := \text{lin}\{V_1, V_2, ..., V_{n-1}\}$ satisfy the formula *(ii)* for every $1 \leq k \leq n-1$.
We will construct an operator $V_n \in \mathcal{K}(c_0, c_0)$ such that
$\mathcal{V}_n := \text{lin}\{V_1, V_2, ..., V_{n-1}, V_n\}$ satisfies the formula *(ii)* for every $1 \leq k \leq n$.
Our goal is to find $x \in \mathbb{R}$ such that

$$\lim_{i\to\infty} \frac{1}{2^i} x^i = 0 \tag{12}$$

and

$$W(x, y^1, ..., y^k, i_1, ..., i_k, m_1, ...m_{k-1}) :=$$

$$\begin{vmatrix} y_1^1 \frac{1}{2^{i_1}} t_{m_1}^{i_1} & \cdot & \cdot & \cdot & y_1^{k-1} \frac{1}{2^{i_1}} t_{m_{k-1}}^{i_1} & y_1^k \frac{1}{2^{i_1}} x^{i_1} \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \cdot & \\ y_k^1 \frac{1}{2^{i_k}} t_{m_1}^{i_k} & \cdot & \cdot & \cdot & y_k^{k-1} \frac{1}{2^{i_k}} t_{m_{k-1}}^{i_k} & y_k^k \frac{1}{2^{i_k}} x^{i_k} \end{vmatrix} \neq 0, \tag{13}$$

where $k \in \{1, 2, ..., n\}, \quad i_1, i_2, ..., i_k \in \mathbb{N}, \quad y^1, ..., y^k \in \{-1, 1\}^k, m_1, m_2, ..., m_{k-1} \in \{1, 2, ..., n-1\}$. Since $W(x, y^1, ..., y^k, i_1, ..., i_k, m_1, ...m_{k-1})$ is not totally equal to zero, we conclude that the set of roots of $W(x, y^1, ..., y^k, i_1, ..., i_k, m_1, ...m_{k-1})$ is finite for arbitrary but fixed $y^1, ..., y^k, \quad i_1, ..., i_k, \quad m_1, ...m_{k-1}$.
Therefore for all $y^1, ..., y^k, \quad i_1, ..., i_k, \quad m_1, ...m_{k-1}$ as above, the set of roots of $W(x, y^1, ..., y^k, i_1, ..., i_k, m_1, ...m_{k-1})$ is countable. Since $\mathbb{R}$ is not countable we see that there exists $x \in \mathbb{R}$ satisfying (12) and (13).

Remark 2 *An n-dimensional Chebyshev subspace proposed in Example 3 is a non-interpolating subspace of* $\mathcal{K}(c_0, c_0)$.

Proof. Let us assume that $\mathcal{V}_n = \text{lin}\{V_1, V_2, ..., V_n\}$ is an n-dimensional Chebyshev subspace, where $V_m, \ m = 1, 2, ..., n$ are defined in Example 3.

Put $V = \frac{1}{t_1}V_1 - \frac{1}{t_2}V_2$. Note that $V \neq 0$ and $v_{ij} = 0, \quad j \geq 3, \quad i \in \mathbb{N}$, where $V = [v_{ij}]_{i,j \in \mathbb{N}}$.
It is obvious that there exist $x^1, x^2, ..., x^n \in \text{ext} S_{l^\infty}$ such that $x^m(1) = x^m(2) = 1$, $m = 1, 2, ..., n$ and $f_m := e_1 \otimes x^m$, $m = 1, 2, ..., n$ are linearly independent.
Note that

$$f_m(V) = 0, \quad m = 1, 2, ..., n.$$

This completes the proof. ∎

Lemma *Let X be a normed space and let $\mathcal{V}$ be a finite-dimensional subspace of X. Let $T \in X$. If $0 \in \mathcal{P}_{\mathcal{V}}(T)$ and 0 is not a strongly unique best approximation for T in $\mathcal{V}$ then*

$$\exists\, V \in \mathcal{V}, \quad V \neq 0 \quad : \quad \forall\, f \in E(T) \quad f(V) \geq 0.$$

Proof. Let us assume that

$$\forall\, V \in \mathcal{V}, \quad V \neq 0, \quad \exists\, f \in E(T) \quad : \quad f(V) < 0.$$

Set for any $V \in \mathcal{V}, \quad \|V\| = 1$,

$$-r_V = \inf\{f(V) \ : \ f \in E(T)\},$$
$$-r = \sup\{-r_V \ : \ V \in \mathcal{V}, \ \|V\| = 1\}.$$

We show that $r > 0$.
If not, there exists $(V_n) \subset S_{\mathcal{V}}$ such that $-r_{V_n} \geq -\frac{1}{n}$. Since $\mathcal{V}$ is a finite-dimensional subspace, we may assume that $V_n \to V \in S_{\mathcal{V}}$. Take $f \in E(T), \quad f(V) < 0$. Hence for $n \geq n_0$ there exists $d > 0$ such that

$$-\frac{1}{n} \leq -r_{V_n} \leq f(V_n) < f(V) + d < 0,$$

a contradiction. Therefore

$$\forall\, V \in \mathcal{V}, \quad V \neq 0, \quad \exists\, f \in E(T) \quad : \quad f\left(\frac{V}{\|V\|}\right) < -r.$$

By the above,

$$\forall\, V \in \mathcal{V}, \quad V \neq 0, \quad \exists\, f \in E(T) \quad : \quad f(V) \leq -r\|V\|.$$

Hence 0 is a strongly unique best approximation for T, a contradiction. This proves the lemma. ∎

Theorem 6 *Let $\mathcal{V} \subset \mathcal{K}(c_0, c_0)$ be an n - dimensional subspace such that*

$$\forall\, V \in \mathcal{V}, \quad \forall\, i \in \mathbb{N} \quad \sharp\{j \in \mathbb{N} \ : \ v_{ij} \neq 0\} < \infty,$$

where $V = [v_{ij}]_{i,j \in \mathbb{N}}$ and let $T \in \mathcal{K}(c_0, c_0)$.
T has a unique best approximation in $\mathcal{V}$ if and only if T has a strongly unique best approximation in $\mathcal{V}$.

Proof. Let us assume that 0 is the unique best approximation for T in $\mathcal{V}$. Suppose that 0 is not a strongly unique best approximation.
Hence (see Lemma)

$$\exists\, V \in \mathcal{V}, \quad V \neq 0 \quad : \quad \forall\, f \in E(T) \quad f(V) \geq 0,$$

where $f = e_i \otimes x^i$ for some $x^i \in \text{ext}S_{l^\infty}$.
Put

$$\mathcal{N} = \{i \in \mathbb{N} \,:\, \exists\, x^i \in \text{ext}S_{l^\infty} \,:\, e_i \otimes x^i \in E(T)\}.$$

Since T is compact, we conclude that $\sharp\,\mathcal{N} < \infty$.
For every $i \in \mathcal{N}$ we set

$$E_i = \{x^i \in \text{ext}S_{l^\infty} \,:\, (e_i \otimes x^i)(T) = \|T\|\}.$$

Let $i \in \mathbb{N} \setminus \mathcal{N}$. Hence there exists $b > 0$ such that

$$(e_i \otimes x)(T) < b < \|T\|, \quad x \in \text{ext}S_{l^\infty}.$$

Consequently, there exists $\alpha_0 > 0$ such that for every $\alpha \in (0, \alpha_0]$,

$$\mid (e_i \otimes x)(T - \alpha V) \mid < b.$$

Therefore

$$\sup_{i \in \mathbb{N} \setminus \mathcal{N}} \mid (e_i \otimes x)(T - \alpha V) \mid \leq b < \|T\|.$$

Let $i \in \mathcal{N}$ and let $x^i \notin E_i$. From this we conclude that there exists $j_0 \in \mathbb{N}$ such that

$$\text{sgn}x^i(j_0) \neq \text{sgn}(t_{ij_0}), \quad t_{ij_0} \neq 0,$$

where $T = [t_{ij}]_{i,j \in \mathbb{N}}$.
Set $J = \{j \in \mathbb{N} \,:\, v_{ij} \neq 0\}$.
If $\text{sgn}x^i(j) = \text{sgn}(t_{ij})$ for any $j \in J$, then there exists $y^i \in E_i$ such that

$$(e_i \otimes y^i)(T) = \|T\|, \qquad (e_i \otimes y^i)(V) = (e_i \otimes x^i)(V).$$

By the above,

$$(e_i \otimes x^i)(T - \alpha V) \leq \|T\| - (e_i \otimes y^i)(\alpha V) \leq \|T\|.$$

Let $\text{sgn}x^i(j_0) \neq \text{sgn}(t_{ij_0})$ for some $j_0 \in J$, where $t_{ij_0} \neq 0$. Since J is finite, there exists $\alpha_0 > 0$ such that

$$\|\alpha_0 V\| < \min\{\mid t_{ij} \mid : \, j \in J, \; t_{ij} \neq 0\}.$$

Let $\alpha \in (0, \alpha_0]$. Hence

$$(e_i \otimes x^i)(T - \alpha V) = \sum_{j \in J} x^i(j)(t_{ij} - \alpha v_{ij}) + \sum_{j \notin J} x^i(j)(t_{ij} - \alpha v_{ij}) \leq$$
$$\sum_{j \in J} \mid t_{ij} \mid - 2 \mid t_{ij_0} \mid + \sum_{j \notin J} \mid t_{ij} \mid + \alpha \|V\| =$$
$$\|T\| + \alpha \|V\| - 2 \mid t_{ij_0} \mid < \; \|T\|.$$

Finally,

$$\|T - \alpha V\| = f(T - \alpha V),$$

where $f = e_i \otimes x^i, \quad i \in \mathcal{N}, \quad x^i \in E_i$.
Hence

$$\|T - \alpha V\| = f(T - \alpha V) \leq \|T\|.$$

The proof is complete. ∎

Acknowledgments The author wishes to thank Professor Lewicki for his remarks and suggestions concerning this article.

References

[1] A. ALEKSIEWICZ, Analiza funkcjonalna, PWN, Warszawa, 1969

[2] D. A. AULT, F. R. DEUTSCH, P. D. MORRIS, J. E. OLSON, Interpolating Subspaces in Approximation Theory, *J. Approx. Theory* **3** (1970), 164-182

[3] B. BROSOWSKI, R. WEGMANN, Charakterisierung bester Approximationen in normirierten Vektorraumen, *J. Approx. Theory* **3** (1970), 369-397

[4] H. S. COLLINS, W. RUESS, Weak compactness in spaces of compact operators and vector-valued functions, *Pacific J. Math.* **106**, (1983), 45-71

[5] F. DEUTSCH, W. LI, Strong uniqueness, Lipschitz continuity and continuous selections for metric projections in L_1, *J. Approx. Theory* **66** (1991), No. 2, 198-224

[6] R. E. EDWARDS, Functional Analysis Theory and Applications, Dover Publications, Inc. New York, 1994

[7] J. KOWYNIA, Best approximation in finite dimensional Chebyshev subspaces of $L(l_1^n, c)$, *East J. Approx.* **12** (2006), No. 4, 407-416

[8] G. LEWICKI, Best Approximation in Finite Dimensional subspaces of L(W, V), *J. Approx. Theory* **81**, No. 2, (1995), 151-165

[9] W. LI, Strong uniqueness and Lipschitz continuity, of metric projections: a generalization of classical Haar theory *J. Approx. Theory* **56**, No. 2, (1989), 164-184

[10] J. MALBROCK, Chebyshev subspaces in the space of bounded linear operators from c_0 to c_0, *J. Approx. Theory* **9**, No. 2, (1973), 149-164

[11] G. NÜRNBERGER, Strong unicity constants in Chebyshev approximation *Numerical methods of approx. theory* **8**, (Oberwolfach 1986), 144-154

[12] R. R. PHELPS, Chebyshev subspaces of finite dimension in L^1, *Proc. Amer. Math. Soc.* **17**, No. 3, (1966), 646-652

[13] E. REMEZ, General computation Methods of Chebyshev approximation, Atomic Energy, Voroneck, 1957

[14] W. RUDIN, Functional Analysis, McGraw-Hill Book Company, New York, 1973

[15] W. M. RUESS, C. STEGALL, Extreme points in duals of operator spaces, *Math. Ann.* **261** (1982), 535-546

[16] I. SINGER, Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces, Springer, Berlin, Heidelberg, New York, 1970

[17] A. WÓJCIK, Characterization of strong unicity by tangent cones, *Approximation and Function Spaces, Proceedings of the International Conference, Gdańsk*, (1979) (Z. Ciesielski, Ed), 854-866, PWN, Warszawa/North-Holland, Amsterdam, 1981

[18] A. WÓJCIK, J. SUDOLSKI, Some remarks on strong uniqueness of best approximation, *J. Approx. Theory and its Appl.* **6**, No. 2, (1990), 44-78

Department of Mathematics, AGH University of Science and Technology,
Al. Mickiewicza 30, 30 - 059 Cracow, Poland
e-mail: kowynia@wms.mat.agh.edu.pl