Spherical associated homogeneous distributions on R^n

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Abstract

A structure theorem for spherically symmetric associated homogeneous distributions (SAHDs) based on R^n is given. It is shown that any SAHD is the pullback, along the function $|\mathbf{x}|^\lambda$, $\lambda \in \mathbf{C}$, of an associated homogeneous distribution (AHD) on R. The pullback operator is found not to be injective and its kernel is derived (for $\lambda = 1$). Special attention is given to the basis SAHDs, $D_z^m |\mathbf{x}|^z$, which become singular when their degree of homogeneity $z = -n - 2p$, $\forall p \in \mathbb{N}$. It is shown that $\left(D_z^m |\mathbf{x}|^z\right)_{z=-n-2p}$ are partial distributions which can be non-uniquely extended to distributions $\left(\left(D_z^m |\mathbf{x}|^z\right)_e\right)_{z=-n-2p}$ and explicit expressions for their evaluation are derived. These results serve to rigorously justify distributional potential theory in R^n.

1 Introduction

We present a construction of spherical (i.e., $O(n)$-invariant) associated homogeneous distributions (SAHDs) based on R^n, as pullbacks of associated homogeneous distributions (AHDs) based on R. It is shown that any SAHD on R^n can be obtained as the pullback, along the function $|\mathbf{x}|^\lambda$, $\lambda \in \mathbb{C}$, of an AHD on R.

Homogeneous distributions (HDs) on R generalize the concept of homogeneous functions, such as $|x|^z : R\backslash\{0\} \to \mathbb{C}$, which is homogeneous of complex degree z. Associated to homogeneous functions are power-log functions, which arise when taking the derivative with respect to the degree of homogeneity z.

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The set of associated homogeneous distributions with support in (or based on) R, denoted by $\mathcal{H}'(R)$, generalizes these power-log functions, [7], [2], [11]. The set $\mathcal{H}'(R)$ is a subset of the tempered distributions, [14], [15], and is of practical importance because $\mathcal{H}'(R)$ contains the majority of the (1-dimensional) distributions one encounters in physical applications, such as the delta distribution δ, the step distributions $1_{\pm}$, several so called pseudo-functions generated by taking Hadamard's finite part of certain divergent integrals (among which is Cauchy's principal value x^{-1}), Riesz kernels, Heisenberg distributions and many familiar others, [12].

We denote the set of AHDs based on R^n by $\mathcal{H}'(R^n)$. An important subset of $\mathcal{H}'(R^n)$ are the $O(n)$-invariant AHDs on R^n, called SAHDs and of which r^z, $z \in \mathbb{C}$, is a well-known example, having degree of homogeneity z and order of association 0, see e.g., [11, p. 71, p. 98, p. 192]. AHDs based on R^n are important mathematical tools, used in physics and engineering for solving distributional potential (i.e., static field) problems in n-dimensions. SAHDs based on R^n arise in spherically symmetric problems, such as the construction of a fundamental solution (i.e., a Green's distribution) for Poisson's equation and its complex degree generalizations (i.e., involving complex powers of the Laplacian in R^n). We denote the set of SAHDs on R^n by $\mathcal{SH}'(R^n)$. We have the inclusions $\mathcal{SH}'(R^n) \subset \mathcal{H}'(R^n) \subset \mathcal{S}'(R^n) \subset \mathcal{D}'(R^n)$.

Consider the scalar function $T^\lambda : X = R^n \backslash \{\mathbf{0}\} \to Y = R_+$ such that $\mathbf{x} \mapsto y = |\mathbf{x}|^\lambda$ with $\lambda \in \mathbb{C}$. The aim of this paper is to show that any SAHD on R^n can be obtained as the pullback $(T^\lambda)^*$ along T^λ of an AHD on R. This is an interesting result, as it opens a route to extend the properties of the simple and well-understood 1-dimensional AHDs to their $O(n)$-invariant generalizations on R^n. In particular, recent work done by the author showed that the set of AHDs on R can be given the structure of both a convolution algebra and a multiplication algebra over $\mathbb{C}$, see [3], [4], [5] ([8]), [6] ([9]). These algebraic properties of AHDs on R can be extended, under the $O(n)$-invariant function T^λ above, to SAHDs on R^n and the key to this higher dimensional extension of the aforementioned algebras is the here considered pullback relation.

The concept of the pullback of a distribution generalizes the classical concept of a change of variables for a function. Any map $f : Y \to Z$ can be pulled back to a space X by precomposition with a map $T : X \to Y$ as $f \circ T : X \to Z$. Any smooth T represents a homomorphism T^* between the set $C^\infty(Y)$ of smooth functions defined on Y and the set $C^\infty(X)$ of smooth functions defined on X, such that $f \mapsto T^* f = f \circ T$ (for functions this is usually written as $T^* f = f(T(x))$). The homomorphism T^* is called the pullback along the function T. The concept of pullback is more general than that of a change of variables. The latter can not be applied to distributions since they are not functions of the base space, but functionals on a space of (test) functions defined on the base space, here $\mathcal{D}(Y)$. However, it is possible to define the pullback $T^* f \in \mathcal{D}'(X)$ of any distribution $f \in \mathcal{D}'(Y)$ (under certain restrictions on T) in terms of an operation on $\mathcal{D}(Y)$. This results in an indirect definition, such as the one recalled in section 2, to perform a "change of variables" for distributions. One uses the fact that $C^\infty(Y)$ is dense in $\mathcal{D}'(Y)$ (since $\mathcal{D}(Y) \subset C^\infty(Y)$ is) to show that the pullback $T^* f$ exists if precomposition with T maps sequences of smooth functions converging in $\mathcal{D}'(Y)$ to sequences of smooth functions converging in $\mathcal{D}'(X)$. A necessary and sufficient condition for the pullback T^*f to be unique, is that T^* is a sequentially continuous operator, [10, Chapter 7]. Although the pullback of a distribution can be defined along general submersions, see e.g., [10, Theorem 7.2.2], we will only need here the pullback along scalar functions.

We show that the pullback T^*, along the particular scalar function $T \triangleq T^1$, of any AHD on R generates a distribution on R^n that is a linear combination of distributions of the form $D_z^m |\mathbf{x}|^z$, called basis SAHDs. We properly define the distributions $D_z^m |\mathbf{x}|^z$, which are only briefly considered in [11, p. 99], and investigate their properties. Careful attention is given to the cases when the degree of homogeneity z is such that $z + n = -2p \in \mathbb{Z}_{e,-]}$ (even non-positive integers), since the functionals $D_z^m |\mathbf{x}|^z$ possess $(m+1)$-th order poles at $z = -n - 2p$, $\forall p \in \mathbb{N}$.

The here presented study of the distributions $D_z^m |\mathbf{x}|^z$ is placed in the more modern context of pullbacks and extensions, compared to the more classical approach which defines singular distributions as regularizations of certain divergent integrals, e.g., as in [11]. We especially draw attention to the fact that any $\left(D_z^m |\mathbf{x}|^z\right)_{z=-n-2p}$ is a (unique) partial distribution. A partial distribution is a fruitful concept, introduced earlier by the author in [7, Section 3.3], to designate generalized functions that are only defined on a proper subset $\mathcal{D}_r \subset \mathcal{D}$. By definition, a distribution is defined on the whole of $\mathcal{D}$, [15, p. 6]. Our approach to singular distributions is basically a functional extension process that extends a partial distribution to a distribution. Since $\mathcal{D}$ is locally convex, [13, p. 152], [1, pp. 427–431], the (continuous extension version of the) Hahn-Banach theorem applies to $\mathcal{D}$, [13, p. 56]. This theorem guarantees that an extension of a partial distribution defined on any $\mathcal{D}_r \subset \mathcal{D}$ exists as a continuous linear functional on $\mathcal{D}$, hence as a distribution, and that both coincide on $\mathcal{D}_r$, [13, p. 61]. It is natural to use such an extension, denoted $\left(\left(D_z^m |\mathbf{x}|^z\right)_e\right)_{z=-n-2p}$, to define $D_z^m |\mathbf{x}|^z$ at the degree of homogeneity $-n-2p$. We call $\left(\left(D_z^m |\mathbf{x}|^z\right)_e\right)_{z=-n-2p}$ an extension of the partial distribution $\left(D_z^m |\mathbf{x}|^z\right)_{z=-n-2p}$ from $\mathcal{D}_r$ to $\mathcal{D}$.

The Hahn-Banach theorem does not tell how such an extension is to be constructed. We apply a straightforward method to produce a distribution $\left(\left(D_z^m |\mathbf{x}|^z\right)_e\right)_{z=-n-2p}$ on $\mathcal{D}(R^n)$ that is a SAHD and coincides with the partial distribution $\left(D_z^m |\mathbf{x}|^z\right)_{z=-n-2p}$ on $\mathcal{D}_r(R^n)$. This method, first introduced in [7, Section 3.3, eq. (33)] and here applied to SAHDs on R^n, leads to more general results than those found in the classical literature, since the obtained extensions are in general uncountably multi-valued. Any classical regularization is recovered as the unique extension corresponding to a particular branch of this multi-valued spectrum. For (complex) AHDs, the spectrum of multi-valuedness is parametrized by $\mathbb{C}$, hence each value of an extension $\left(\left(D_z^m |\mathbf{x}|^z\right)_e\right)_{z=-n-2p}$ corresponds to a constant $c \in \mathbb{C}$.

We derive explicit expressions for the evaluation of the so constructed multi-valued distributions $\left(\left(D_z^m |\mathbf{x}|^z\right)_e\right)_{z=-n-2p}$. It is found that $\left(\left(D_z^m |\mathbf{x}|^z\right)_e\right)_{z=-n-2p}$ are homogeneous distributions of degree $-n-2p$ and order of association $m+1$. In [11, p. 99] it is incorrectly stated that the particular extension, corresponding to Hadamard's finite part $\left(\left(D_z^m |\mathbf{x}|^z\right)_0\right)_{z=-n-2p}$ (and corresponding to $c = 0$), is associated of order m. That this can not be true is also seen from the result [11, p.195] and by invoking the fact that the Fourier transformation preserves the order of association, [7].

This work extends and generalizes the treatment of SAHDs on R^n in [11]. New results presented here are (i) the concepts of partial distribution and functional extension for defining the occurring singular distributions, (ii) the representation of SAHDs on R^n as pullbacks of AHDs on R, (iii) the kernel of the pullback operator T^*, $\ker T^* \subset \mathcal{H}'(R)$ and (iv) a structure theorem for $\mathcal{SH}'(R^n)$.

The outline of the paper is as follows. We recall the pullback T^* of a distribution along a scalar function $T : X \to Y$ in section 2. We apply this in section 3 to AHDs based on R. In section 4 we investigate the pullback of any distribution along the function T defined above. In section 5, the results from sections 3 and 4 are combined to generate SAHDs on R^n. There, the basis distributions $D_z^m |\mathbf{x}|^z$ are discussed, the general form of an SAHDs on R^n is given and the $\ker T^*$ is derived. In the last section 6, the structure theorem of SAHDs on R^n is proved.

We use the notations introduced in [7]. For convenience, some practical but non-standard notations are repeated here. Define $1_p \triangleq 1$ if p is true, else $1_p \triangleq 0$. Further, $e_m \triangleq 1_{m \in \mathbb{Z}_e}$, hence $e_m = 1$ if m is even, else $e_m = 0$ and similarly $o_m \triangleq 1_{m \in \mathbb{Z}_o}$, hence $o_m = 1$ if m is odd, else $o_m = 0$.

2 Pullback of a distribution on R along a scalar function

Definition 1. *Let $n \in \mathbb{N} : 2 \leq n$, $X \subseteq R^n$, $Y = R$ and $\delta_y \in \mathcal{D}'(Y)$ with $\langle \delta_y, \psi \rangle \triangleq \psi(y)$, $\forall \psi \in \mathcal{D}(Y)$. Let $f \in \mathcal{D}'(Y)$ and $T : X \to Y$ such that $\mathbf{x} \mapsto y = T(\mathbf{x})$ be a C^∞ function with $(dT)(\mathbf{x}) \neq 0$, $\forall \mathbf{x} \in \Sigma_y \triangleq \{\mathbf{x} \in X : T(\mathbf{x}) = y\}$ and $\forall y \in \operatorname{supp} f$. The pullback $T^* f$ of f along T is defined $\forall \varphi \in \mathcal{D}(X)$ as*

$$\langle T^* f, \varphi \rangle \triangleq \langle f, \Sigma_T \varphi \rangle , \tag{1}$$

with

$$(\Sigma_T \varphi)(y) = \langle T^* \delta_y, \varphi \rangle , \tag{2}$$

$$\triangleq \int_{\Sigma_y} \varphi \omega_T . \tag{3}$$

In (3) is ω_T the Leray form of Σ_y, such that $\omega_X = dT \wedge \omega_T$, with ω_X the volume form on X.

The condition on dT is necessary and sufficient for the Leray form to exist on Σ_y. Moreover, although $\omega_X = dT \wedge \omega_T$ does not specify ω_T uniquely in a neighborhood of Σ_y, ω_T is unique on Σ_y, [11, pp. 220-221].

The distribution $\delta_{\Sigma_y} \triangleq T^* \delta_y \in \mathcal{D}'(X)$ represents a delta distribution having as support the level set surface Σ_y of T with level parameter y. We can not speak of *the* delta distribution with support Σ_y since the pullback $T^* \delta_y$, as defined by Definition 1, depends on the equation used to represent the surface Σ_y, through the Leray form, [11, p. 222], [1, p. 439]. It is clear that the delta distribution δ_{Σ_y}, as defined by (2) and (3), is fundamental to define the pullback of any distribution along T.

It is shown in e.g., [10, p. 82, Theorem 7.2.1] that, under the conditions given in Definition 1, $\Sigma_T\varphi \in \mathcal{D}(Y)$, $T^*f \in \mathcal{D}'(X)$ and T^* is a sequentially continuous linear operator.

Theorem 2. *Let $f^z \in \mathcal{D}'(Y)$, depending on a complex parameter z and being complex analytic in a domain $\Omega \subseteq \mathbb{C}$. Let T^* be the pullback from Y to X along a C^∞ function $T : X \subseteq R^n \to Y = R$. Then T^*f^z is complex analytic and moreover*

$$T^*\left(D_z^m f^z\right) = D_z^m\left(T^*f^z\right), \tag{4}$$

$\forall m \in \mathbb{Z}_+$ and $\forall z \in \Omega$.

Proof. (i) Let $m = 1$. Since it is given that f^z is complex analytic in Ω, this means by definition that $d_z\langle f^z, \psi\rangle$ exists. This is a necessary and sufficient condition for the existence of a distribution $D_z f^z$ given by $\langle D_z f^z, \psi\rangle = d_z\langle f^z, \psi\rangle$, $\forall\psi \in \mathcal{D}(Y)$ and $\forall z \in \Omega$, [11, pp. 147-151]. On the other hand, applying (1) to the left-hand side of (4) gives, $\forall\varphi \in \mathcal{D}(X)$,

$$\langle T^*D_z f^z, \varphi\rangle = \langle D_z f^z, \Sigma_T\varphi\rangle .$$

Combining both results yields

$$\langle T^*D_z f^z, \varphi\rangle = d_z\langle f^z, \Sigma_T\varphi\rangle .$$

Applying (1) to the right-hand side of this equation gives

$$\langle T^*D_z f^z, \varphi\rangle = d_z\langle T^*f^z, \varphi\rangle .$$

Hence $d_z\langle T^*f^z, \varphi\rangle$ exists, which implies by definition that T^*f^z is complex analytic in Ω. This is a necessary and sufficient condition for the existence of a distribution $D_z(T^*f^z)$ given by $\langle D_z(T^*f^z), \varphi\rangle = d_z\langle T^*f^z, \varphi\rangle$, so that

$$\langle T^*(D_z f^z), \varphi\rangle = \langle D_z(T^*f^z), \varphi\rangle ,$$

which implies (4) for $m = 1$.

(ii) Since f^z is complex analytic in Ω, $D_z^m f^z$ is also complex analytic in Ω, $\forall m \in \mathbb{Z}_+$. Combining this with (i) and using induction, (4) follows $\forall m \in \mathbb{Z}_+$. ∎

This theorem enables to generate the Taylor series of a pullback distribution $T^*f^z \in \mathcal{D}(R^n)$ directly from the Taylor series of the distribution $f^z \in \mathcal{D}(R)$. In particular, (4) simplifies the calculation of pullbacks of AHDs.

3 Pullback of an AHD on R along a scalar function

Let $\mathbf{X}\cdot\mathbf{D}$ denote the generalized Euler operator and $X_z \triangleq \mathbf{X}\cdot\mathbf{D} - z\,\mathrm{Id}$ the generalized homogeneity operator of degree $z \in \mathbb{C}$ defined on $\mathcal{D}'(R^n)$ (with Id the identity operator), and Y_z the generalized homogeneity operator of degree z defined on $\mathcal{D}'(R)$.

Theorem 3. *Let T^* be the pullback from Y to X along a C^∞ function $T : X \subseteq R^n \to Y = R$ such that $\mathbf{x} \mapsto y = T(\mathbf{x})$, with $(dT)(\mathbf{x}) \neq 0$, $\forall \mathbf{x} \in X$. Let f_0^z be a homogeneous distribution based on Y with degree of homogeneity z. Then holds, $\forall m \in \mathbb{Z}_+$ and $\forall \lambda \in \mathbb{C}$,*

$$X_{\lambda z}^m (T^* f_0^z) = \sum_{l=1}^m p_l^m (x_0, x_\lambda T) \left(T^* \left(D^l f_0^z \right) \right), \tag{5}$$

with $x_\lambda \triangleq \mathbf{x} \cdot \mathbf{d} - \lambda \operatorname{Id}$ the ordinary homogeneity operator of degree λ and p_l^m bivariate polynomials of degree m, satisfying the recursion relations

$$p_1^1 (x_0, h) = h, \tag{6}$$

$$p_k^{m+1} (x_0, h) = x_0 p_k^m (x_0, h) + h p_{k-1}^m (x_0, h). \tag{7}$$

Proof. (i) Under the given conditions, the generalized chain rule is valid so we have for the i-th generalized partial derivative, $\forall f \in \mathcal{D}'(Y)$, $\forall \varphi \in \mathcal{D}(X)$ and $\forall i \in \mathbb{Z}_{[1,n]}$,

$$\langle D_i (T^* f), \varphi \rangle = \langle T^* (Df), (d_i T) \varphi \rangle .$$

Applying this to $x^i \varphi \in \mathcal{D}(X)$, we obtain

$$\left\langle D_i (T^* f), x^i \varphi \right\rangle = \left\langle T^* (Df), (d_i T) x^i \varphi \right\rangle .$$

Using the definition of the multiplication of a distribution with a smooth function, writing the result in terms of the multiplication operator $X^i \triangleq x^i.$ and summing over all values of i gives

$$\langle (\mathbf{X} \cdot \mathbf{D}) (T^* f), \varphi \rangle = \langle T^* (Df), ((\mathbf{x} \cdot \mathbf{d}) T) \varphi \rangle .$$

This is equivalent to, $\forall \lambda \in \mathbb{C}$,

$$\langle (\mathbf{X} \cdot \mathbf{D}) (T^* f), \varphi \rangle - \lambda \langle T^* (Df), T\varphi \rangle = \langle T^* (Df), (x_\lambda T) \varphi \rangle . \tag{8}$$

Applying the definition of the pullback T^*, the fact that T is a scalar function mapping $\mathbf{x} \mapsto y$ and also introducing the multiplication operator $Y \triangleq y.$, we have

$$\begin{aligned} \langle T^* (Df), T\varphi \rangle &= \langle Df, \Sigma_T (T\varphi) \rangle, \\ &= \langle Df, y \Sigma_T \varphi \rangle, \\ &= \langle YDf, \Sigma_T \varphi \rangle, \\ &= \langle T^* (YDf), \varphi \rangle . \end{aligned} \tag{9}$$

In (8) choose $f = f_0^z$, use $YDf_0^z = z f_0^z$ in (9), substitute (9) in (8) and use the operator $X_{\lambda z}$ in the left-hand side of (8). Since $X_\lambda T$ is a smooth function, we obtain (5) for $m = 1$.

(ii) The result for $m > 1$ follows by induction. ∎

Corollary 4. *Let $f_m^z \in \mathcal{H}'(Y)$. If T is not homogeneous, then $T^* f_m^z \notin \mathcal{H}'(X)$.*

Proof. Let f_0^z be a HD on Y. If T is not homogeneous, then $x_\lambda T \neq 0$, $\forall \lambda \in \mathbb{C}$. From Theorem 3 follows that then all $p_k^m \neq 0$, so $X_{\lambda z}^m (T^* f_0^z) \neq 0$, $\forall m \in \mathbb{N}$. This result, together with Theorem 2 and the structure theorem for AHDs on R [2, Theorem 4] (see also (98)), implies that $T^* f_m^z, \forall f_m^z \in \mathcal{H}'(Y)$, is not an AHD on X. ∎

Corollary 4 will be needed in Theorem 14.

Theorem 5. *Let T^* be the pullback along the function T as defined in Theorem 3 and let in addition T be homogeneous of degree $\lambda \in \mathbb{C}$. Then,*

(i) the homogeneity operators X_z and Y_z are related by

$$X_{\lambda z} T^* = \lambda T^* Y_z; \tag{10}$$

(ii) the pullback $T^* f_m^z$ of an AHD f_m^z, of degree of homogeneity z and order of association m based on Y, is again an AHD of the same order of association m and of degree of homogeneity λz, based on X.

Proof. (i) Recalling (8) and using $x_\lambda T = 0$, we get

$$\langle (\mathbf{X} \cdot \mathbf{D}) (T^* f), \varphi \rangle = \lambda \langle T^* (Df), T\varphi \rangle .$$

Using (9) and introducing the homogeneity operators $X_{\lambda z}$ and Y_z, this is equivalently to

$$\langle X_{\lambda z} (T^* f), \varphi \rangle = \lambda \langle T^* (Y_z f), \varphi \rangle .$$

Since f and φ are arbitrary, this implies (10).

(ii) Let $m \in \mathbb{N}$ and f_m^z be any AHD with degree of homogeneity z and order of association m based on Y. By definition, f_m^z satisfies $Y_z f_m^z = f_{m-1}^z$ for some AHD f_{m-1}^z with degree of homogeneity z and order of association $m-1$ based on Y and we define $f_{-1}^z \triangleq 0$. Applying (10) to f_m^z gives

$$X_{\lambda z} (T^* f_m^z) = \lambda T^* f_{m-1}^z. \tag{11}$$

From this follows, by induction over m, that $T^* f_m^z$ is an AHD with degree of homogeneity λz and order of association m based on X. ■

Hence, the pullback T^* of an AHD on R along a homogeneous scalar function T is an order of association preserving homomorphism.

Corollary 6. *If T in Theorem 5 has degree of homogeneity 1, its pullback T^* from Y to X is in addition a homogeneity preserving homomorphism,*

$$X_z T^* = T^* Y_z. \tag{12}$$

Corollary 7. *If T in Theorem 5 has degree of homogeneity 0, $T^* f_m^z$, $\forall f_m^z \in \mathcal{H}'(Y)$, is a homogeneous distribution based on X with degree of homogeneity 0.*

4 Pullback of a distribution on R along the function $|\mathbf{x}|$

Define the function $T : X = R^n \setminus \{\mathbf{0}\} \to Y = R_+$ such that $\mathbf{x} \mapsto r = T(\mathbf{x}) \triangleq |\mathbf{x}|$ with $|\mathbf{x}| \triangleq \left(\left(x^1\right)^2 + ... + \left(x^n\right)^2 \right)^{1/2} > 0$. We have $|dT|(\mathbf{x}) = 1$, $\forall \mathbf{x} \in X$, hence dT is surjective and T is a (scalar) submersion. For $y \in R_+$, $\Sigma_y \triangleq \{\mathbf{x} \in X : |\mathbf{x}| = y\} \subset X$, while for $y \in R_{-]}$, $\Sigma_y = \varnothing$. By (3) holds, $\forall \varphi \in \mathcal{D}(X)$ and $\forall y \in R_+$,

$$(\Sigma_T \varphi)(y) = \int_{\Sigma_y} \varphi \omega_T. \tag{13}$$

We want to extend $\Sigma_T \varphi$ so that it is defined $\forall \varphi \in \mathcal{D}(R^n)$ and $\forall y \in R$. To this end, we change from Cartesian coordinates to spherical coordinates in the integral in (13) (see also Appendix 7.1). We get, $\forall \varphi \in \mathcal{D}(R^n)$ and $\forall y \in R_+$,

$$(\Sigma_T \varphi)(y) = A_{n-1} y^{n-1} (S\varphi)(y), \tag{14}$$

wherein we defined the spherical mean operator S, defined on $\mathcal{D}(R^n)$, by

$$(S\varphi)(y) \triangleq \frac{1}{A_{n-1}} \int_{S^{n-1}} \varphi(y\omega)\, \omega_{S^{n-1}}, \tag{15}$$

with $\omega_{S^{n-1}}$ the volume form on the $(n-1)$-dimensional unit sphere S^{n-1} and A_{n-1} its surface area, given by (120). Clearly, the integral in (15) also exists $\forall y \in R_{-]}$, and it is shown in [11, pp. 72–73] that, $\forall p \in \mathbb{N}$, (i) $\left(d^{2p} S\varphi\right)(0)$ exists and (ii)

$$\left(d^{2p+1} S\varphi\right)(0) = 0, \tag{16}$$

so $S\varphi$ is an even function. Then, eqs. (14)–(15) define $S\varphi$ and $\Sigma_T \varphi$, $\forall y \in R$.

The function $S\varphi$ is of compact support, since φ is. Since $\varphi(y\omega)$ in (15) is obviously jointly continuous in $(y, \omega) \in I \times S^{n-1}$, is $S\varphi$ uniformly continuous in any compact interval I. By induction it follows that $S\varphi$ is smooth in I. Hence the operator S maps from $\mathcal{D}(R^n) \to \mathcal{D}(R)$. Consequently, $\Sigma_T \varphi \in \mathcal{D}(R)$, $\forall \varphi \in \mathcal{D}(R^n)$.

We can now define $T^* f$, in agreement with (1), $\forall f \in \mathcal{D}'(R)$ and $\forall \varphi \in \mathcal{D}(R^n)$, by

$$\langle T^* f, \varphi \rangle \triangleq \left\langle f, y^{n-1} \int_{S^{n-1}} \varphi(y\omega)\, \omega_{S^{n-1}} \right\rangle. \tag{17}$$

We still have to verify that $T^* f$, as defined by (17), is a distribution based on R^n, $\forall f \in \mathcal{D}'(R)$. Theorem 7.2.1 in [10] only guarantees that $T^* f \in \mathcal{D}'(R^n \setminus \{\mathbf{0}\})$ for those distributions $f \in \mathcal{D}'(R)$ such that $\operatorname{supp}(f)$ has a pre-image in R^n under T for which $|dT|(\mathbf{x}) \neq 0$. For any other f, i.e., for which either the pre-image of $\operatorname{supp}(f)$ under T contains the origin (where $(dT)(\mathbf{0})$ does not exist) or either $\operatorname{supp}(f) \subset R_{-]}$ (since then the pre-image of T is not defined) we need to check the linearity and sequential continuity of $T^* f$, $\forall \varphi \in \mathcal{D}(R^n)$.

The linearity of $T^* f$, as defined by (17), is obvious. Further, any sequence $\varphi_\nu \in \mathcal{D}(R^n)$ converging to 0 generates a sequence $(\Sigma_T \varphi)_\nu \in \mathcal{D}(R)$ also converging to 0, due to the uniform continuity of $S\varphi$ in any compact interval. Then, the sequential continuity of f implies the sequential continuity of $T^* f$, showing that T^* is a sequentially continuous operator. Hence, $T^* f \in \mathcal{D}'(R^n)$.

Remarks.

(i) The form (14) for $\Sigma_T \varphi$ and the property (16) of $S\varphi$ imply that the pullback $T^* f$, as defined by (17), is a distribution, even if f itself is only a partial distribution defined on that subset of test functions $\mathcal{D}_{\mathbb{Z}_1}(R)$ having (i) a zero of order $n-1$ at the origin and (ii) which, for n odd, are even (then $\mathbb{Z}_1 = \mathbb{Z}_{[-n,-1]} \cup \mathbb{Z}_{o,-}$) or, for n even, are odd (then $\mathbb{Z}_1 = \mathbb{Z}_{[-n,-1]} \cup \mathbb{Z}_{e,-}$) (for the notation $\mathcal{D}_{\mathbb{Z}_1}(R)$, see [7, Section 2.1, 5]).

(ii) The pullback T^* along the above function T is not injective. Indeed, eq. (17) and the property (16) of $S\varphi$ imply that

$$\left\{ \sum_{l=0}^{n-2} a_l \delta^{(l)} + \sum_{p=0}^{P} b_p \delta^{(n+2p)}, \forall a_l, b_p \in \mathbb{C}, \forall P \in \mathbb{N} \right\} \subset \ker T^*. \tag{18}$$

(iii) The distribution $T^*\delta_y$ in (2) represents a delta distribution having as support the sphere Σ_y with radius y. From (14) follows that

$$\delta_{\Sigma_y} = T^*\delta_y = \delta_y \otimes 1_{(\omega)}, \tag{19}$$

with $1_{(\omega)}$ the one distribution based on S^{n-1}. We can not speak of *the* delta distribution having as support the sphere with radius y, since $\delta_{\Sigma_y} = T^*\delta_y$ depends on the equation used to represent the surface Σ_y, here $|\mathbf{x}| = y$. The equation $|\mathbf{x}|^2 = y^2$ defines the same sphere, but now the function $T_2 : X = R^n \setminus \{\mathbf{0}\} \to Y = R_+$ such that $\mathbf{x} \mapsto r = |\mathbf{x}|^2$ leads to the pullback $\delta_{\Sigma_{y^2}} \triangleq T_2^*\delta_y = \frac{1}{2}\delta_y \otimes 1_{(\omega)} \neq \delta_{\Sigma_y}$.

The pullback T^* along the function T thus performs two actions: (i) possibly an extension from $\mathcal{D}_{\mathbb{Z}_1}(R)$ to $\mathcal{D}(R)$, and (ii) a "change of variables" from $y \mapsto \mathbf{x}$. This can be illustrated more explicitly with the following example.

First, let

$$\Delta \triangleq D_1^2 + D_2^2 + ... + D_n^2 \tag{20}$$

denote the generalized Laplacian defined on $\mathcal{D}'(R^n)$. Define distributions $\Delta^p\delta$, $\forall p \in \mathbb{N}$, based on R^n by

$$\langle \Delta^p \delta, \varphi \rangle \triangleq (\Delta^p \varphi)(\mathbf{0}), \tag{21}$$

where in the right-hand side of (21) Δ denotes the ordinary Laplacian defined on $\mathcal{D}(R^n)$. It is shown in [11, p. 73, eq. (6)] that (Pizetti's formula), $\forall p \in \mathbb{N}$,

$$A_{n-1} \frac{\left(d^{2p} S\varphi\right)(0)}{(2p)!} = \frac{A_{n+2p-1}}{(4\pi)^p} \frac{(\Delta^p \varphi)(\mathbf{0})}{p!}. \tag{22}$$

Now, let $\mathcal{D}_{\mathbb{Z}_{[-k,-1]}}(R)$ stand for the subset of test functions having a zero of order $k-1$ at the origin, $\forall k \in \mathbb{Z}_+$. For any distribution $f \in \mathcal{D}'(R)$ and functions $y^{-k} : R \setminus \{0\} \to R$, the multiplication $y^{-k}.f$ can be defined, $\forall \psi \in \mathcal{D}_{\mathbb{Z}_{[-k,-1]}}(R)$, by

$$\left\langle y^{-k}.f, \psi \right\rangle \triangleq \left\langle f, y^{-k}\psi \right\rangle, \tag{23}$$

since $y^{-k}\psi \in \mathcal{D}(R)$. Hence, $y^{-k}f \triangleq y^{-k}.f$ is a partial distribution defined on $\mathcal{D}_{\mathbb{Z}_{[-k,-1]}}(R)$. For the particular partial distributions $y^{-(n-1)}\delta^{(m)}$, $\forall m \in \mathbb{N}$, (see also Appendix 7.2) (23) gives, $\forall \psi \in \mathcal{D}_{\mathbb{Z}_{[-(n-1),-1]}}(R)$,

$$\left\langle y^{-(n-1)}\delta^{(m)}, \psi \right\rangle = (-1)^m \left(d_y^m \left(y^{-(n-1)}\psi \right) \right)(0). \tag{24}$$

A. Let $m = 2p$, $\forall p \in \mathbb{N}$. On the one hand, using (14), (21), (24) and (22), eq. (17) with $f = y^{-(n-1)}\delta^{(2p)}$ implies that, $\forall p \in \mathbb{N}$,

$$T^* \frac{y^{-(n-1)}\delta^{(2p)}}{(2p)!} = \frac{A_{n+2p-1}}{(4\pi)^p} \frac{\Delta^p \delta}{p!}. \tag{25}$$

Eq. (25) shows that the distributions $\Delta^p\delta$ are proportional to the pullback T^* from Y to X of the partial distributions $y^{-(n-1)}\delta^{(2p)}$, defined on $\mathcal{D}_{Z_{[-(n-1),-1]}}(R)$.

On the other hand, taking the $(n-1+2p)$-th derivative with respect to y of (14), gives

$$\frac{\left(d^{n-1+2p}\Sigma_T\varphi\right)(0)}{(n-1+2p)!} = A_{n-1}\frac{d^{2p}\,(S\varphi)\,(0)}{(2p)!}. \tag{26}$$

Substituting in the right-hand side of (26) the expression (22), using the definition of $\delta^{(m)}$ and applying definition (1), we get, $\forall p \in \mathbb{N}$,

$$T^*\frac{(-1)^{n-1+2p}\,\delta^{(n-1+2p)}}{(n-1+2p)!} = \frac{A_{n+2p-1}}{(4\pi)^p}\frac{\Delta^p\delta}{p!}. \tag{27}$$

Eq. (27) shows that the distributions $\Delta^p\delta$ are also proportional to the pullback T^* from Y to X of the distributions $\delta^{(n-1+2p)}$.

Eqs. (25) and (27) can be summarized as, $\forall p \in \mathbb{N}$,

$$T^*\left(y^{-(n-1)}\frac{\delta^{(2p)}}{(2p)!}\right) = \frac{A_{n+2p-1}}{(4\pi)^p}\frac{\Delta^p\delta}{p!} = T^*\left(\frac{(-1)^{n-1}\,\delta^{(n-1+2p)}}{(n-1+2p)!}\right). \tag{28}$$

B. Let $m = 2p+1$, $\forall p \in \mathbb{N}$. In a similar way as under A we find that

$$T^*\left(y^{-(n-1)}\delta^{(2p+1)}\right) = 0 = T^*\delta^{(n+2p)}. \tag{29}$$

Eqs. (28), (29) and (126) illustrate again that T^* is not injective.

Further, due to (14) holds that $\left\langle T^*\delta^{(l)}, \varphi\right\rangle = 0$, $\forall l \in \mathbb{Z}_{[0,n-2]}$. This result, together with the right equations in (28) and (29), can be summarized as

$$T^*\delta^{(l)} = 0, \forall l \in \mathbb{Z}_{[0,n-2]}, \tag{30}$$

$$T^*\frac{\delta^{(n-1+k)}}{(n-1+k)!} = e_k\,(-1)^{n-1}\,\frac{A_{n+k-1}}{(4\pi)^{k/2}}\frac{\Delta^{k/2}\delta}{(k/2)!}, \forall k \in \mathbb{N}. \tag{31}$$

The distributions $\delta^{(p)}$ in the left-hand sides of (30)–(31) are based on R and the distributions $\Delta^p\delta$ in the right-hand side of (31) are based on R^n. The distributions $\delta^{(p)}_{\Sigma_0} \triangleq T^*\delta^{(p)}$ can be interpreted as spherical multiplet (or p-fold) layers, [11, p. 237], concentrated at an $(n-1)$-dimensional sphere of radius $y = 0$.

5 Pullback of an AHD on R along the function $|\mathbf{x}|$

5.1 The distributions $D_z^m |\mathbf{x}|^z$

Let $m \in \mathbb{N}$.

5.1.1 Pullback of $y_+^z \ln^m |y|$

Regular distributions The distributions $y_+^z \ln^m |y|$ are defined in [11, p. 84], [7, Section 5.2.3]. For $-1 < \mathrm{Re}(z)$, $y_+^z \ln^m |y| = D_z^m y_+^z$ is a regular distribution, so we obtain from (1), $\forall \varphi \in \mathcal{D}(R^n)$,

$$\begin{aligned} \langle T^* (y_+^z \ln^m |y|), \varphi \rangle &= \langle y_+^z \ln^m |y|, \Sigma_T \varphi \rangle, \\ &= \int_0^{+\infty} (y^z \ln^m y)\, \Sigma_T \varphi(y)\, dy. \end{aligned} \tag{32}$$

Substituting herein the expression (14) for $\Sigma_T \varphi$ yields

$$\begin{aligned} \langle T^* (y_+^z \ln^m |y|), \varphi \rangle &= A_{n-1} \int_0^{+\infty} \left(y^{z+n-1} \ln^m y \right) (S\varphi)(y)\, dy, \\ &= \left\langle y_+^{z+n-1} \ln^m |y|, A_{n-1} S\varphi \right\rangle. \end{aligned} \tag{33}$$

As was shown in the previous section, $S\varphi \in \mathcal{D}(R)$. Thus, the right-hand side of (33) can be regarded as the functional value of the regular distribution $y_+^{z+n-1} \ln^m |y|$ for the test function $A_{n-1} S\varphi$. Expression (43) below, for the Laurent series of the function $y_+^w \ln^m y$ about $w = -k \in \mathbb{Z}_-$, shows that $y_+^w \ln^m |y|$ has poles of order $m+1$ at $w = -k \in \mathbb{Z}_-$. However, due to property (16) of the test function $S\varphi$ and the expression for the principal part of the Laurent series of the function $y_+^w \ln^m y$ about $w = -k$, the poles of $y_+^w \ln^m y$ at $w = -k \in \mathbb{Z}_{e,-}$ do not occur in (33). Consequently, the distribution $T^* (y_+^z \ln^m |y|)$ has poles of order $m+1$ only at $z \in \mathbb{Z}_p \triangleq \{-n-2p, \forall p \in \mathbb{N}\}$.

Substituting (15) in (33) gives

$$\langle T^* (y_+^z \ln^m |y|), \varphi \rangle = \int_0^{+\infty} \int_{S^{n-1}} (y^z \ln^m y)\, \varphi(y\omega)\, y^{n-1} \omega_{S^{n-1}} dy. \tag{34}$$

Changing back to Cartesian coordinates in the right-hand side double integral in (34), we get

$$\begin{aligned} \langle T^* (y_+^z \ln^m |y|), \varphi \rangle &= \int_{R^n} \left(|\mathbf{x}|^z \ln^m |\mathbf{x}| \right) \varphi \omega_{R^n}, \\ &= \left\langle |\mathbf{x}|^z \ln^m |\mathbf{x}|, \varphi \right\rangle. \end{aligned} \tag{35}$$

Combining (35) with (33) shows that $|\mathbf{x}|^z \ln^m |\mathbf{x}|$ are regular distributions for $-n < \mathrm{Re}(z)$. Since $y_+^z \ln^m |y| = D_z^m y_+^z$ for $-1 < \mathrm{Re}(z)$, is due to (4) $|\mathbf{x}|^z \ln^m |\mathbf{x}| = D_z^m |\mathbf{x}|^z$ for $-n < \mathrm{Re}(z)$.

In particular for $z = 0$, follows from (35) that, $\forall m \in \mathbb{N}$,

$$\ln^m |\mathbf{x}| = T^* (1_+ \ln^m |y|). \tag{36}$$

Analytic continuations The complex analyticity of the distribution $y_+^z \ln^m |y|$ for $-1 < \mathrm{Re}(z)$ together with the principle of analytic continuation makes that (35) continues to hold, $\forall z \in \mathbb{C} \backslash \mathbb{Z}_p$,

$$|\mathbf{x}|^z \ln^m |\mathbf{x}| = T^* \left(y_+^z \ln^m |y| \right). \tag{37}$$

Similarly we get, $\forall z \in \mathbb{C} \backslash \mathbb{Z}_p$ and $\forall \varphi \in \mathcal{D}(R^n)$, from (33),

$$\langle T^* \left(y_+^z \ln^m |y| \right), \varphi \rangle = \left\langle y_+^{z+n-1} \ln^m |y|, A_{n-1} S \varphi \right\rangle, \tag{38}$$

and from (32),

$$\langle T^* \left(y_+^z \ln^m |y| \right), \varphi \rangle = \langle y_+^z \ln^m |y|, \Sigma_T \varphi \rangle. \tag{39}$$

Invoking (4) and using (37) with $m = 0$, it follows that also $\forall z \in \mathbb{C} \backslash \mathbb{Z}_p$,

$$|\mathbf{x}|^z \ln^m |\mathbf{x}| = D_z^m |\mathbf{x}|^z. \tag{40}$$

Using (37) in (38) further yields, $\forall z \in \mathbb{C} \backslash \mathbb{Z}_{e,-]}$,

$$\left\langle |\mathbf{x}|^{z-n} \ln^m |\mathbf{x}|, \varphi \right\rangle = \left\langle y_+^{z-1} \ln^m |y|, \int_{S^{n-1}} \varphi(y\omega) \, \omega_{S^{n-1}} \right\rangle. \tag{41}$$

We will now derive a more explicit expression in order to evaluate the right-hand side of (41) after analytic continuation. To this end, we first need the following n-dimensional projection operator $T_{p,q}^n : \mathcal{D}(R^n) \to \mathcal{D}(R^n)$ such that $\varphi \mapsto T_{p,q}^n \varphi$, defined by

$$\begin{aligned} \left(T_{p,q}^n \varphi \right)(\mathbf{x}) \;\triangleq\; & \varphi(\mathbf{x}) - \sum_{l=0}^{p+q} \left(\sum_{l_1=0}^{l} \cdots \sum_{l_n=0}^{l} 1_{L=l} \left(\left(\frac{\partial^L \varphi}{(\partial x)^L} \right)(\mathbf{0}) \right) \left(\prod_{i=1}^{n} \frac{\left(x^i\right)^{l_i}}{l_i!} \right) \right) \\ & \left(1_{l<p} + 1_{p \le l} 1_{[+}(1 - |\mathbf{x}|^2) \right), \end{aligned} \tag{42}$$

wherein L is a shorthand for $\sum_{i=1}^n l_i$, $(\partial x)^L$ a shorthand for $\left(\partial x^1\right)^{l_1} \ldots \left(\partial x^n\right)^{l_n}$ and the step function $1_{[+}(x) = 1$ iff $x \ge 0$.

In order to evaluate the right-hand side of (41) after analytic continuation, e.g. for $0 < |z - 1 + k| < 1$ and for any $k \in \mathbb{Z}_+$, we recall the Laurent series of $y_\pm^{z-1} \ln^m |x|$ about $z - 1 = -k$, [7, eq. (117)],

$$\begin{aligned} & \left\langle y_\pm^{z-1} \ln^m |x|, \psi \right\rangle \\ & \quad = (-1)^m \frac{\left\langle \frac{(\mp 1)^{k-1}}{(k-1)!} \delta^{(k-1)}, \psi \right\rangle}{(z-1+k)^{m+1}} + 1_{0 \le p \le k-2} (-1)^m \sum_{l=p}^{k-2} \frac{\left\langle \frac{(\mp 1)^l}{l!} \delta^{(l)}, \psi \right\rangle}{(z-1+l)^{m+1}} \\ & \qquad + \int_{-\infty}^{+\infty} \left(|y|^{z-1} 1_\pm(y) \ln^m |y| \right) \left(T_{p,q} \psi \right)(y) dy, \end{aligned} \tag{43}$$

wherein $p, q \in \mathbb{N} : p + q = k - 1$, $\psi = A_{n-1} S \varphi$ and $T_{p,q} \triangleq T_{p,q}^1$. For the particular choice $p = k - 1, q = 0$, (43) reduces to

$$\left\langle y_\pm^{z-1} \ln^m |x|, \psi \right\rangle = (-1)^m \frac{\left\langle \frac{(\mp 1)^{k-1}}{(k-1)!} \delta^{(k-1)}, \psi \right\rangle}{(z-1+k)^{m+1}} + \left\langle y_{\pm,0}^{z-1} \ln^m |x|, \psi \right\rangle, \tag{44}$$

wherein

$$\left\langle y_{\pm,0}^{z-1}\ln^m|x|,\psi\right\rangle=\int_{-\infty}^{+\infty}\left(|y|^{z-1}1_{\pm}(y)\ln^m|y|\right)(T_{k-1,0}\psi)(y)\,dy. \tag{45}$$

Take $k=2p+2$, $\forall p\in\mathbb{N}$, in (44)–(45). Then, for $0<|z+(2p+1)|<1$, and due to (16), (41) becomes

$$\begin{aligned}\left\langle|\mathbf{x}|^{z-n}\ln^m|\mathbf{x}|,\varphi\right\rangle &= \int_0^{+\infty}\left(y^{z-1}\ln^m|y|\right)\left(T_{2p+1,0}\left(A_{n-1}S\varphi\right)\right)(y)\,dy,\\ &= \int_0^{+\infty}\int_{S^{n-1}}\left(y^{z-n}\ln^m y\right)\left(T^n_{2p+1,0}\varphi\right)(y\omega)\,y^{n-1}\,\omega_{S^{n-1}}\,dy,\\ &= \int_{R^n}\left(|\mathbf{x}|^{z-n}\ln^m|\mathbf{x}|\right)\left(T^n_{2p+1,0}\varphi\right)\omega_{R^n}. \end{aligned}\tag{46}$$

In particular at $z=-(2p+1)$, (46) allows to calculate the functional value of $|\mathbf{x}|^{z-n}\ln^m|\mathbf{x}|$ at the ordinary points $z=-(2p+1)$. The right-hand side of (46) shows that the analytic continuation of the regular distribution $|\mathbf{x}|^z\ln^m|\mathbf{x}|$ is no longer a regular distribution.

Example 8. *In particular for $p=0$, (46) gives, $\forall m\in\mathbb{N}$ and $\forall\varphi\in\mathcal{D}(R^n)$,*

$$\left\langle|\mathbf{x}|^{-n-1}\ln^m|\mathbf{x}|,\varphi\right\rangle$$

$$= A_{n-1}\int_0^{+\infty}\frac{1}{y^2}\begin{pmatrix}(S\varphi)(y)-(S\varphi)(0)\\ -1_{[+}(1-y^2)\left((d(S\varphi))(0)\right)y\end{pmatrix}\ln^m|y|\,dy, \tag{47}$$

$$= \int_{R^n}|\mathbf{x}|^{-n-1}\begin{pmatrix}\varphi(\mathbf{x})-\varphi(\mathbf{0})\\ -1_{[+}(1-|\mathbf{x}|^2)\left(\sum_{i=1}^n\left(\left(\frac{\partial\varphi}{\partial x^i}\right)(\mathbf{0})\right)x^i\right)\end{pmatrix}\ln^m|\mathbf{x}|\,\omega_{R^n}. \tag{48}$$

Remarks.

(i) For $-1<\mathrm{Re}(z)$, $|\mathbf{x}|^z\ln^m|\mathbf{x}|$ can be regarded as the multiplication product $|\mathbf{x}|^z.\ln^m|\mathbf{x}|$ of the regular distributions $|\mathbf{x}|^z$ and $\ln^m|\mathbf{x}|$. By analytic continuation this product is uniquely extended to all $z\in\mathbb{C}\backslash\mathbb{Z}_p$. This justifies our use of the notation $|\mathbf{x}|^z\ln^m|\mathbf{x}|$ in the right-hand side of (35).

(ii) It follows from (39) that, $\forall z\in\mathbb{C}\backslash\mathbb{Z}_{e,-]}$, the distribution $|\mathbf{x}|^z\ln^m|\mathbf{x}|$ is the pullback of the partial distribution $y_+^z\ln^m|y|$, defined on that set of test functions $\mathcal{D}_{\mathbb{Z}_1}(R)$ having (i) a zero of order $n-1$ at the origin and (ii) which, for n odd, are even (i.e., $\mathbb{Z}_1=\mathbb{Z}_{[-n,-1]}\cup\mathbb{Z}_{o,-}$) or, for n even, are odd (i.e., $\mathbb{Z}_1=\mathbb{Z}_{[-n,-1]}\cup\mathbb{Z}_{e,-}$).

(iii) The analytically continued distributions $|\mathbf{x}|^z\ln^m|\mathbf{x}|$ are homogeneous of degree z and have order of association m. This follows from the properties of the analytically continued distributions $y_\pm^z\ln^m|x|$, [7, Section 5.2.2], and Theorem 5.

Extensions We now consider the cases $z+n=-2p\in\mathbb{Z}_{e,-]}$ in (38). The Laurent series of $y_\pm^z$ about $z=-k\in\mathbb{Z}_-$ and holding in $0<|z+k|<1$ are given by, [11, p. 87], [7, Section 4.2.3],

$$y_\pm^z=\frac{\frac{(\mp1)^{k-1}}{(k-1)!}\delta^{(k-1)}}{z+k}+\sum_{m=0}^{+\infty}\left(y_{\pm,0}^{-k}\ln^m|y|\right)\frac{(z+k)^m}{m!}, \tag{49}$$

wherein the distributions $y_{\pm,0}^{-k}\ln^m|y|$, given by (45), are particular extensions of $y_{\pm}^{z}\ln^m|y|$ at the pole $z=-k$, in the sense of [7, Section 3.3, eq. (33)]. Using the sequential continuity of T^*, (37) with $m=0$, (27) and letting $k=n+2p$, we obtain the Laurent series of $|\mathbf{x}|^z$ about $z+n=-2p\in\mathbb{Z}_{e,-]}$ as

$$|\mathbf{x}|^z=\frac{\frac{A_{n+2p-1}}{(4\pi)^p p!}\Delta^p\delta}{z+n+2p}+\sum_{m=0}^{+\infty}\left(T^*\left(y_{+,0}^{-(n+2p)}\ln^m|y|\right)\right)\frac{(z+n+2p)^m}{m!}. \tag{50}$$

Due to the uniform continuity of this series, the Laurent series of $D_z^m|\mathbf{x}|^z$ about $z+n=-2p\in\mathbb{Z}_{e,-]}$ is obtained as

$$D_z^m|\mathbf{x}|^z=(-1)^m\frac{\frac{A_{n+2p-1}}{(4\pi)^p p!}\Delta^p\delta}{(z+n+2p)^{m+1}}+\sum_{l=m}^{+\infty}T^*\left(y_{+,0}^{-(n+2p)}\ln^l|y|\right)\frac{(z+n+2p)^{l-m}}{(l-m)!}. \tag{51}$$

We can now give a meaning to $D_z^m|\mathbf{x}|^z$ at $z+n=-2p\in\mathbb{Z}_{e,-]}$. Expression (51) shows that $\left(D_z^m|\mathbf{x}|^z\right)_{z=-n-2p}$ is a partial AHD, i.e., a generalized function only defined for test functions $\psi\in\mathcal{D}_r(R^n)\triangleq\{\varphi\in\mathcal{D}(R^n):(\Delta^p\varphi)(0)=0\}$. The Hahn-Banach theorem ensures the existence of a distribution $\left(\left(D_z^m|\mathbf{x}|^z\right)_\varepsilon\right)_{z=-n-2p}$, defined $\forall\varphi\in\mathcal{D}(R^n)$ and which coincides with $\left(D_z^m|\mathbf{x}|^z\right)_{z=-n-2p}$ on $\mathcal{D}_r(R^n)\subset\mathcal{D}(R^n)$, called an extension of the partial distribution $\left(D_z^m|\mathbf{x}|^z\right)_{z=-n-2p}$ from $\mathcal{D}_r(R^n)$ to $\mathcal{D}(R^n)$. This extension is generally not unique and not necessarily an AHD. Here we are only interested in constructing AHDs based on R^n, so we restrict our attention to extensions $\left(\left(D_z^m|\mathbf{x}|^z\right)_e\right)_{z=-n-2p}$ which are again an AHD (we indicate extensions which are an AHD by the subscript e and use the subscript ε for a general extension). The subset of distributions which maps $\mathcal{D}_r(U)$ to zero is called the annihilator of $\mathcal{D}_r(U)$ and denoted by $\mathcal{D}_r'^{\perp}(U)$. Any two extensions differ by a generalized function $g\in\mathcal{D}_r'^{\perp}(U)$. Applied to our case here, we find that associated homogeneous extensions are of the form

$$\left(\left(D_z^m|\mathbf{x}|^z\right)_e\right)_{z=-n-2p}=\left(\left(D_z^m|\mathbf{x}|^z\right)_0\right)_{z=-n-2p}+c'\Delta^p\delta, \tag{52}$$

with arbitrary $c'\in\mathbb{C}$. This way, we have extended the partial distributions $\left(D_z^m|\mathbf{x}|^z\right)_{z=-n-2p}$, defined on $\mathcal{D}_r(R^n)$, to the non-unique singular distributions $\left(\left(D_z^m|\mathbf{x}|^z\right)_e\right)_{z=-n-2p}$, defined on the whole of $\mathcal{D}(R^n)$.

The finite part

$$\left(\left(D_z^m|\mathbf{x}|^z\right)_0\right)_{z=-n-2p}\triangleq T^*\left(y_{+,0}^{-(n+2p)}\ln^m|y|\right), \tag{53}$$

is given by (41), (15) and [7, eq. (118)] as

$$\left\langle \left(\left(D_z^m |\mathbf{x}|^z \right)_0 \right)_{z=-n-2p}, \varphi \right\rangle$$

$$\begin{aligned}
&= \left\langle y_{+,0}^{-1-2p} \ln^m |y|, A_{n-1} S\varphi \right\rangle, \\
&= \int_0^{+\infty} \left(y^{-1-2p} \ln^m y \right) \left(T_{2p,0} \left(A_{n-1} S\varphi \right) \right)(y)\, dy, \\
&= \int_0^{+\infty} \int_{S^{n-1}} \left(y^{-n-2p} \ln^m y \right) \left(T_{2p,0}^n \varphi \right)(y\omega)\, y^{n-1} \omega_{S^{n-1}}\, dy, \\
&= \int_{R^n} \left(|\mathbf{x}|^{-n-2p} \ln^m |\mathbf{x}| \right) \left(T_{2p,0}^n \varphi \right) \omega_{R^n}.
\end{aligned} \tag{54}$$

Example 9. *In particular for $p = 0$, (54) gives, $\forall m \in \mathbb{N}$ and $\forall \varphi \in \mathcal{D}(R^n)$,*

$$\left\langle \left(\left(D_z^m |\mathbf{x}|^z \right)_0 \right)_{z=-n}, \varphi \right\rangle$$

$$= A_{n-1} \int_0^{+\infty} \frac{1}{y} \left((S\varphi)(y) - 1_{[+}(1-y^2)(S\varphi)(0) \right) \ln^m y\, dy, \tag{55}$$

$$= \int_{R^n} |\mathbf{x}|^{-n} \left(\varphi(\mathbf{x}) - 1_{[+}(1-|\mathbf{x}|^2) \varphi(\mathbf{0}) \right) \ln^m |\mathbf{x}|\, \omega_{R^n}. \tag{56}$$

Remarks.

(i) The extension $\left(\left(D_z^m |\mathbf{x}|^z \right)_e \right)_{z=-n-2p}$ is of degree $-n-2p$ and associated of order $m+1$, for the same reasons as explained in [7, eq. (121)], but now applied to the distribution $y_{+,e}^{-(n+2p)} \ln^m |y|$.

(ii) Due to [6, eq. (20)] ([9, eq. (20)]) is $y_{+,e}^{-(n+2p)} \ln^m |y| = y_{+,0}^{-(n+2p)} \ln^m |y| + c_+ \delta^{(n+2p-1)}$, $c_+ \in \mathbb{C}$ arbitrary. Then, using (52), (53) and (31) we obtain

$$T^* \left(y_{+,e}^{-(n+2p)} \ln^m |y| \right) = \left(\left(D_z^m |\mathbf{x}|^z \right)_0 \right)_{z=-n-2p} + c'_+ \Delta^p \delta, \tag{57}$$

with the branches of both extensions related by

$$c'_+ = c_+ (-1)^{n-1} (n+2p-1)! \frac{A_{n+2p-1}}{(4\pi)^p\, p!}. \tag{58}$$

(iii) We use the notation $\left(\left(D_z^m |\mathbf{x}|^z \right)_e \right)_{z=-n-2p}$ instead of $|\mathbf{x}|_e^{-n-2p} \ln^m |\mathbf{x}|$, because it is not yet clear if $\left(\left(D_z^m |\mathbf{x}|^z \right)_e \right)_{z=-n-2p}$ is equal to the multiplication of $|\mathbf{x}|_e^{-n-2p}$ by $\ln^m |\mathbf{x}|$. This matter can be resolved after the multiplication algebra constructed for AHDs on R in [6] ([9]) is extended to a multiplication algebra for SAHDs on R^n.

Spherical form From (37), (51), (30) and (52) it thus follows that, $\forall z \in \mathbb{C}$,

$$T^* \left(y_+^z \ln^m |y| \right) = D_z^m |\mathbf{x}|^z, \tag{59}$$

with $y_+^z \ln^m |y|$ replaced by $y_{+,e}^z \ln^m |y|$ for $z \in \mathbb{Z}_-$ and $D_z^m |\mathbf{x}|^z$ replaced by $\left(D_z^m |\mathbf{x}|^z \right)_e$ for $z + n \in \mathbb{Z}_{e,-]}$.

From (34) and for $-1 < \mathrm{Re}(z)$, we can read off the pullback $T^*_{S\to C}$ along the diffeomorphism from spherical to Cartesian coordinates $T_{S\to C}$, defined in Appendix 7.1, of $T^*\left(y_+^z \ln^m |y|\right)$ as

$$\langle T^*\left(y_+^z \ln^m |y|\right), \varphi\rangle = \int_0^{+\infty} \int_{S^{n-1}} \left(r^z \ln^m r \otimes 1_{(\omega)}\right) \varphi(r\omega)\, r^{n-1}\, \omega_{S^{n-1}}\, dr. \quad (60)$$

After analytic continuation we get the distributions $T^*\left(y_+^z \ln^m |y|\right)$, $\forall z+n \in \mathbb{C}\backslash\mathbb{Z}_{e,-]}$, in spherical coordinates as

$$T^*\left(y_+^z \ln^m |y|\right) = r^z \ln^m r \otimes 1_{(\omega)}, \quad (61)$$

or equivalently

$$|\mathbf{x}|^z \ln^m |\mathbf{x}| = r^z \ln^m r \otimes 1_{(\omega)}. \quad (62)$$

At the poles $z+n = -2p \in \mathbb{Z}_{e,-]}$, we mean by $\left(r^{-n-2p} \ln^m r\right)_e$ the distribution defined by

$$\left(r^{-n-2p} \ln^m r\right)_e \otimes 1_{(\omega)} \triangleq \left(\left(D_z^m |\mathbf{x}|^z\right)_e\right)_{z=-n-2p}, \quad (63)$$

with the right-hand side of (63) given by (52).

Example 10. *For instance in R^3, the familiar functional r^{-1} (more precisely, $r^{-1}\otimes 1_{(\omega)}$) is thus a regular distribution, whose functional value is read off from (41) for $z=2$ and $m=0$ as*

$$\begin{aligned}\left\langle r^{-1}\otimes 1_{(\omega)}, \varphi\right\rangle &= \left\langle y_+, \int_{S^2} \varphi(y\omega)\, \omega_{S^2}\right\rangle, \\ &= 4\pi \int_0^{+\infty} y\, (S\varphi)(y)\, dy. \end{aligned} \quad (64)$$

Further, r^{-2} (more precisely, $r^{-2}\otimes 1_{(\omega)}$) is also a regular distribution determined by

$$\begin{aligned}\left\langle r^{-2}\otimes 1_{(\omega)}, \varphi\right\rangle &= \left\langle 1_+, \int_{S^2} \varphi(y\omega)\, \omega_{S^2}\right\rangle, \\ &= 4\pi \int_0^{+\infty} (S\varphi)(y)\, dy. \end{aligned} \quad (65)$$

By contrast, $r^{-3}\otimes 1_{(\omega)}$ is a partial distribution only defined on $\mathcal{D}_r(R^3) = \{\varphi \in \mathcal{D}(R^3) : \varphi(\mathbf{0}) = 0\}$, but which can be non-uniquely extended to a first order AHD $r_e^{-3}\otimes 1_{(\omega)} = |\mathbf{x}|_e^{-3}$, now defined on all of $\mathcal{D}(R^3)$, for which $\left\langle r_e^{-3}\otimes 1_{(\omega)}, \psi\right\rangle = \left\langle r^{-3}\otimes 1_{(\omega)}, \psi\right\rangle$, $\forall \psi \in \mathcal{D}_r(R^3)$, and whose functional value is given by, $\forall \varphi \in \mathcal{D}(R^3)$,

$$\begin{aligned}\left\langle r_e^{-3}\otimes 1_{(\omega)}, \varphi\right\rangle &= \left\langle |\mathbf{x}|_e^{-3}, \varphi\right\rangle, \\ &= \left\langle |\mathbf{x}|_0^{-3}, \varphi\right\rangle + c\,\langle \delta, \varphi\rangle. \end{aligned} \quad (66)$$

More explicitly,

$$\begin{aligned}&\left\langle |\mathbf{x}|_e^{-3}, \varphi\right\rangle \\ &= 4\pi \left(\int_0^1 \frac{(S\varphi)(y) - (S\varphi)(0)}{y}\, dy + \int_1^{+\infty} \frac{(S\varphi)(y)}{y}\, dy\right) + c\,(S\varphi)(0), \quad (67)\\ &= \int_{B^3} \frac{\varphi(\mathbf{x}) - \varphi(\mathbf{0})}{|\mathbf{x}|^3}\, \omega_{R^3} + \int_{R^3\backslash B^3} \frac{\varphi(\mathbf{x})}{|\mathbf{x}|^3}\, \omega_{R^3} + c\varphi(\mathbf{0}), \quad (68)\end{aligned}$$

with $B^n \triangleq \{\mathbf{x} \in R^n : |\mathbf{x}| \leq 1\}$ the closed unit n-dimensional ball and $c \in \mathbb{C}$ arbitrary.

Example 11. *The delta distribution on R^n in spherical coordinates. It is not possible to define the delta distribution δ on R^n in spherical coordinates by a straightforward application of the formula for the pullback along the diffeomorphism $T_{S\to C}$ of Appendix 7.1. The reason being that in order to make $T_{S\to C}$ a diffeomorphism, we must (at least) exclude $\mathbf{0} \in R^n$, but then $T_{S\to C}$ is no longer a diffeomorphism of a neighborhood of the* $\operatorname{supp}\delta = \{\mathbf{0}\}$ *and [10, Theorem 7.1.1] does not apply. However, from (27) follows for $p = 0$ and $\forall\varphi \in \mathcal{D}(R^n)$ that*

$$\left\langle T^* \left(\frac{1}{A_{n-1}} y^{-(n-1)} \delta \right), \varphi \right\rangle = \varphi(\mathbf{0}), \tag{69}$$

which by (1) and (14) is equivalent to

$$\left\langle \frac{1}{A_{n-1}} y^{-(n-1)} \delta, \int_{S^{n-1}} \varphi(y\omega)\, y^{n-1} \omega_{S^{n-1}} \right\rangle = \varphi(\mathbf{0}). \tag{70}$$

In spherical coordinates (70) becomes

$$\left\langle \frac{1}{A_{n-1}} r^{-(n-1)} \delta \otimes 1_{(\omega)}, \varphi \right\rangle = \varphi(\mathbf{0}). \tag{71}$$

From (71) we can read off δ on R^n in spherical coordinates. Notice that its radial part $r^{-(n-1)}\delta / A_{n-1}$ is a distribution defined on $\mathcal{D}(R_+)$, while $y^{-(n-1)}\delta / A_{n-1}$, in the equivalent functional (70), is a partial distribution only defined on $\mathcal{D}_{\mathbb{Z}_{[-n,-1]}}(R)$.

5.1.2 Pullback of $y_-^z \ln^m |y|$

For $-1 < \operatorname{Re}(z)$, $T^*(y_-^z \ln^m |y|)$ is a regular distribution, so we have using (1), $\forall\varphi \in \mathcal{D}(R)$,

$$\begin{aligned}
\langle T^*(y_-^z \ln^m |y|), \varphi \rangle &= \langle y_-^z \ln^m |y|, \Sigma_T \varphi \rangle, \\
&= \int_{-\infty}^{+\infty} (y_-^z \ln^m |y|)\, \Sigma_T \varphi(y)\, dy, \\
&= \int_{-\infty}^{+\infty} (y_+^z \ln^m |y|)\, \Sigma_T \varphi(-y)\, dy.
\end{aligned}$$

Since $S\varphi$ is an even function, it follows from (14) that $(\Sigma_T\varphi)(-y) = (-1)^{n-1}$ $(\Sigma_T\varphi)(y)$. Hence,

$$\begin{aligned}
\langle T^*(y_-^z \ln^m |y|), \varphi \rangle &= (-1)^{n-1} \int_{-\infty}^{+\infty} (y_+^z \ln^m |y|)\, \Sigma_T \varphi(y)\, dy, \\
&= (-1)^{n-1} \langle y_+^z \ln^m |y|, \Sigma_T \varphi \rangle, \\
&= (-1)^{n-1} \langle T^*(y_+^z \ln^m |y|), \varphi \rangle,
\end{aligned}$$

or

$$T^*(y_-^z \ln^m |y|) = (-1)^{n-1} T^*(y_+^z \ln^m |y|). \tag{72}$$

After analytic continuation we find that (72) continues to hold so that, $\forall z \in \mathbb{C}\backslash\mathbb{Z}_p$,

$$T^* \left(y_-^z \ln^m |y|\right) = (-1)^{n-1} D_z^m |\mathbf{x}|^z . \tag{73}$$

At $z + n = -2p \in \mathbb{Z}_{e,-]}$, we find that

$$T^* \left(y_{-,0}^{-(n+2p)} \ln^m |y|\right) = (-1)^{n-1} \left(\left(D_z^m |\mathbf{x}|^z\right)_0\right)_{z=-n-2p}, \tag{74}$$

so that, with $y_{-,e}^{-(n+2p)} \ln^m |y| = y_{-,0}^{-(n+2p)} \ln^m |y| + c_- \delta^{(n+2p-1)}$,

$$T^* \left(y_{-,e}^{-(n+2p)} \ln^m |y|\right) = (-1)^{n-1} \left(\left(D_z^m |\mathbf{x}|^z\right)_0\right)_{z=-n-2p} + c'_- \Delta^p \delta, \tag{75}$$

with the branches of both extensions related by

$$c'_- = c_- (-1)^{n-1} (n+2p-1)! \frac{A_{n+2p-1}}{(4\pi)^p \, p!}. \tag{76}$$

In the process of analytic continuation and the extension process we used the fact that the operator $T_{p,q}^n$, given by (42), preserves the parity of test functions.

Example 12. *The pullback along T of the distributions $(y \pm i0)^z \in \mathcal{D}'(R)$, defined in [11, p. 59], [7] as*

$$(y \pm i0)^z \triangleq y_+^z + e^{\pm i\pi z} y_-^z, \tag{77}$$

are obtained as, $\forall z \in \mathbb{C}\backslash\mathbb{Z}_p$,

$$T^*(y \pm i0)^z = \left(1 - (-1)^n e^{\pm i\pi z}\right) \left(r^z \otimes 1_{(\omega)}\right). \tag{78}$$

Recall the generalized Sokhotskii-Plemelj equations, [12, p. 28 and p. 84], [7, eq. (217)], $\forall k \in \mathbb{Z}_+$,

$$(x \pm i0)^{-k} = \mp i\pi \frac{(-1)^{k-1}}{(k-1)!} \left(\delta^{(k-1)} \pm i\eta^{(k-1)}\right), \tag{79}$$

with the distributions $\eta^{(l)} \triangleq D^l \eta$ and $\eta \triangleq \frac{1}{\pi} x^{-1}$, (see also [7, eq. (176)]). The distributions in (79) are higher degree generalizations of the Heisenberg distributions $\mp\frac{1}{2\pi i}(x \pm i0)^{-1}$. At $z = -k \in \mathbb{Z}_{[-(n-1),-1]}$, we get, using (79) and (30),

$$T^* \eta^{(k-1)} = (-1)^n \frac{2}{\pi} (k-1)! o_{n+k} \left(r^{-k} \otimes 1_{(\omega)}\right). \tag{80}$$

At $z = -n - (2p+1)$, $\forall p \in \mathbb{N}$, we have, now by using (79) and (31),

$$T^* \eta^{(n+2p)} = (-1)^n \frac{2}{\pi} (n+2p)! \left(r^{-n-(2p+1)} \otimes 1_{(\omega)}\right). \tag{81}$$

At $z = -n - 2p$, $\forall p \in \mathbb{N}$, we obtain, using (57), (75), (53), (63) and (31),

$$T^* \frac{(-1)^{n-1} \eta^{(n+2p-1)}}{(n+2p-1)!} = \frac{1}{\pi} \left(c'_+ + (-1)^n c'_- \pm i\pi \frac{A_{n+2p-1}}{(4\pi)^p \, p!}\right) \Delta^p \delta, \tag{82}$$

with the primed constants given by (58) and (76). Eq. (82) can be restated as

$$T^* \eta^{(n+2p-1)} = c \, \Delta^p \delta, \tag{83}$$

with $c \in \mathbb{C}$ arbitrary.

5.2 The normalized distribution Ψ^z

It is convenient to define the normalized distribution, [12, p. 93], [11, p. 74],

$$\Psi^z \triangleq \frac{2}{A_{n-1}} \frac{|\mathbf{x}|^{-n+z}}{\Gamma(z/2)}, \tag{84}$$

which is entire in z by construction. From (59) follows that Ψ^z is related to the normalized distribution $\Phi_+^z \triangleq x_+^{-1+z}/\Gamma(z)$ as

$$\Psi^z = \frac{2}{A_{n-1}} \frac{\Gamma(z-n+1)}{\Gamma(z/2)} T^* \Phi_+^{z-n+1}. \tag{85}$$

The normalized distribution Ψ^z reduces to the following special values at integer values of z.

(i.1) At $z = -2p$, $\forall p \in \mathbb{N}$,

$$\Psi^{-2p} = \frac{1}{A_{n-1}} \frac{(-1)^p A_{n+2p-1}}{(4\pi)^p} \Delta^p \delta. \tag{86}$$

(i.2) At $z = -(2p+1)$, $\forall p \in \mathbb{N}$,

$$\Psi^{-(2p+1)} = \frac{1}{A_{n-1}} \frac{(-1)^{p+1}(2p+1)!}{\pi^{1/2} 2^{2p} p!} |\mathbf{x}|^{-n-(2p+1)}. \tag{87}$$

(ii.1) At $z = 2p+1$, $\forall p \in \mathbb{N}$,

$$\Psi^{2p+1} = \frac{1}{A_{n-1}} \frac{2^{2p+1} p!}{\pi^{1/2}(2p)!} |\mathbf{x}|^{-n+2p+1}. \tag{88}$$

(ii.2) At $z = 2p+2$, $\forall p \in \mathbb{N}$,

$$\Psi^{2p+2} = \frac{1}{A_{n-1}} \frac{2}{p!} |\mathbf{x}|^{-n+2p+2}. \tag{89}$$

The functional Ψ^{-2p}, given by (86), is trivially evaluated using (21). The functionals, given by eqs. (88) and (89), can be directly evaluated using (35) and (33). To evaluate the functionals $\Psi^{-(2p+1)}$, we use the analytic continuation given by (46).

5.3 Kernel of the pullback

Combining (59) with (73) we find that, $\forall p, m \in \mathbb{N}$ and $\forall z \in \mathbb{C}$,

$$T^*\left(|y|^{z-n} \ln^m |y|\right) = 1_{z=-2p} c \Delta^p \delta, n \in \mathbb{Z}_{e,+}, \tag{90}$$

$$T^*\left(|y|^{z-n} \operatorname{sgn} \ln^m |y|\right) = 1_{z=-2p} c \Delta^p \delta, n \in \mathbb{Z}_{o,+}. \tag{91}$$

with $c \in \mathbb{C}$ arbitrary and

$$T^*\left(|y|^{z-n}\ln^m|y|\right) = 2D_z^m|\mathbf{x}|^{z-n}, n \in \mathbb{Z}_{o,+}, \quad (92)$$

$$T^*\left(|y|^{z-n}\operatorname{sgn}\ln^m|y|\right) = 2D_z^m|\mathbf{x}|^{z-n}, n \in \mathbb{Z}_{e,+}, \quad (93)$$

wherein for $z \in \mathbb{Z}_{e,-]}$ it is understood that the distributions are extensions.

Define

$$\mathcal{E}'_{0,L}(R) \triangleq \left\{\sum_{l=0}^{L} a_l\,\delta^{(l)}, \forall a_l \in \mathbb{C}\right\} \subset \mathcal{E}'_0(R) \quad (94)$$

and

$$\mathcal{H}'_e(R) \triangleq \left\{\sum_{l=0}^{m} p_{l,e}(z)\left(|y|^{z-n}\ln^l|y|\right), \forall m \in \mathbb{N}, \forall z \in \mathbb{C}\backslash\mathbb{Z}_{e,-]}\right\}, \quad (95)$$

$$\mathcal{H}'_o(R) \triangleq \left\{\sum_{l=0}^{m} p_{l,o}(z)\left(|y|^{z-n}\operatorname{sgn}\ln^l|y|\right), \forall m \in \mathbb{N}, \forall z \in \mathbb{C}\backslash\mathbb{Z}_{e,-]}\right\}. \quad (96)$$

From (30) and (90)–(93) follows that the pullback T^* along the function $T : X = R^n\backslash\{\mathbf{0}\} \to Y = R$ such that $\mathbf{x} \mapsto y = |\mathbf{x}|$, restricted to $\mathcal{H}'(R)$, has as kernel

$$\ker T^* = \mathcal{E}'_{0,n-2}(R) \cup \left\{\begin{array}{lll} \mathcal{H}'_o(R) & \text{iff} & n \in \mathbb{Z}_{o,+} \\ \mathcal{H}'_e(R) & \text{iff} & n \in \mathbb{Z}_{e,+} \end{array}\right. . \quad (97)$$

6 SAHDs on R^n

6.1 General form

Let $m \in \mathbb{N}$ and $k \in \mathbb{Z}_+$. Let $\Omega \subseteq \mathbb{C}$ be a neighborhood of $z = -k$ and $p_{l,e}, p_{l,o} \in \mathcal{A}(\Omega, \mathbb{C})$, $\forall l \in \mathbb{Z}_{[0,m]}$, complex analytic coefficient functions, independent of y. Denote by f_m^z a general AHD based on R, complex analytic in its degree z in Ω and of order m. From [2, Theorem 4] follows that any f_m^z can be represented in Ω as

$$f_m^z = \sum_{l=0}^{m}\left(p_{l,e}(z)\left(|y|^z\ln^l|y|\right) + p_{l,o}(z)\left(|y|^z\operatorname{sgn}\ln^l|y|\right)\right), \quad (98)$$

with the coefficient functions satisfying, $\forall j \in \mathbb{Z}_{[0,m]}$,

$$\sum_{q=j}^{m}(-1)^q\tbinom{q}{j}\left(d^{q-j}p_{q,e}\right)(l) = 0, \forall l \in \left(\mathbb{Z}_{o,-} \cap \Omega\right), \quad (99)$$

$$\sum_{q=j}^{m}(-1)^q\tbinom{q}{j}\left(d^{q-j}p_{q,o}\right)(l) = 0, \forall l \in \left(\mathbb{Z}_{e,-} \cap \Omega\right). \quad (100)$$

At $z=-k\in\mathbb{Z}_-$, the distribution f_m^{-k} takes the form

$$f_m^{-k}=\left(\sum_{l=0}^{m}\frac{(-1)^l}{l+1}\left(o_k\left(d^{l+1}p_{l,e}\right)(-k)-e_k\left(d^{l+1}p_{l,o}\right)(-k)\right)\right)2\frac{\delta^{(k-1)}}{(k-1)!}$$
$$+\sum_{l=0}^{m}\left(p_{l,e}(-k)\left(|y|_0^{-k}\ln^l|y|\right)+p_{l,o}(-k)\left(\left(|y|^{-k}\operatorname{sgn}\right)_0\ln^l|y|\right)\right). \quad (101)$$

For $T^\lambda : X=R^n\backslash\{\mathbf{0}\}\to Y=R$ such that $\mathbf{x}\mapsto y=|\mathbf{x}|^\lambda$, $\lambda\in\mathbb{C}$, we obtain from Theorem 5, linearity, (98), (101), (62), (63), (52) and (90)–(93) that:

(i) $\forall z+n\in\mathbb{C}\backslash\mathbb{Z}_{e,-]}$,

$$T^*f_m^z=2\sum_{l=0}^{m}\left(o_n p_{l,e}(\lambda z)+e_n p_{l,o}(\lambda z)\right)\left(r^{\lambda z}\ln^l r\otimes 1_{(\omega)}\right), \quad (102)$$

(ii) if $\lambda z+n=-2p\in\mathbb{Z}_{e,-]}$,

$$T^*f_m^z=2\sum_{l=0}^{m}\left(o_n p_{l,e}(-n-2p)+e_n p_{l,o}(-n-2p)\right)\left(\left(r^{-n-2p}\ln^l r\right)_e\otimes 1_{(\omega)}\right). \quad (103)$$

This shows that the radial part of the pullback along T^λ of any AHD f_m^z of degree z, $\forall z+n\in\mathbb{C}\backslash\mathbb{Z}_{e,-]}$, and order of association m based on R, is the multiplication of the distribution $r^{\lambda z}$ with a polynomial of degree m in the regular distribution $\ln r$.

6.2 Structure theorem

Let $R : R^n\to R^n$ such that $\mathbf{x}\mapsto O\mathbf{x}$ with $O\in O(n)$, the orthogonal group of degree n over R. Then, any $f\in\mathcal{D}'(R^n)$ has a pullback R^*f along the diffeomorphism R given by, [10, Chapter 7],

$$\langle R^*f,\varphi\rangle\triangleq\left\langle f,\left|\det\left(R^{-1}\right)'\right|\left(R^{-1}\right)^*\varphi\right\rangle, \quad (104)$$

with $\det\left(R^{-1}\right)'=\pm1$.

A distribution f is called spherically symmetric iff $R^*f=f$. Hence, for any spherically symmetric distribution f holds that

$$\langle f,\varphi\rangle=\left\langle f,\left(R^{-1}\right)^*\varphi\right\rangle. \quad (105)$$

Theorem 13. *For a distribution f to be a spherically symmetric distribution it is necessary and sufficient that f is of the form*

$$f=f_r\otimes 1_{(\omega)}, \quad (106)$$

with $f_r\in\mathcal{D}'(R_+)$ and $1_{(\omega)}$ the one distribution based on S^{n-1}, satisfying $R^*1_{(\omega)}=1_{(\omega)}$.

Proof. (i) Sufficiency. Assume (106) and calculate, $\forall O \in O(n)$ and $\forall \varphi \in \mathcal{D}(R^n)$,

$$\begin{aligned}\left\langle R^*\left(f_r \otimes 1_{(\omega)}\right), \varphi\right\rangle &= \left\langle f_r \otimes 1_{(\omega)}, \left|\det\left(R^{-1}\right)'\right|\left(R^{-1}\right)^* \varphi\right\rangle, \\ &= \left\langle f_r, \left\langle 1_{(\omega)}, \left|\det\left(R^{-1}\right)'\right|\left(R^{-1}\right)^* \varphi\right\rangle\right\rangle, \\ &= \left\langle f_r, \left\langle R^* 1_{(\omega)}, \varphi\right\rangle\right\rangle, \\ &= \left\langle f_r, \left\langle 1_{(\omega)}, \varphi\right\rangle\right\rangle, \\ &= \left\langle f_r \otimes 1_{(\omega)}, \varphi\right\rangle,\end{aligned}$$

hence, $R^* f = f$.

(ii) Necessity. Assume (105). Then, $\forall O \in O(n)$ and $\forall \varphi \in \mathcal{D}(R^n)$,

$$\begin{aligned}\left\langle f_{(r,\theta)}, \varphi(r, \grave{}\,)\right\rangle &= \left\langle f_{(r,\theta)}, \left(R^{-1}\right)^* \varphi(r,\theta)\right\rangle, \\ &= \left\langle f_{(r,\theta)}, \varphi\left(r,\theta'\right)\right\rangle.\end{aligned}$$

This shows that $\left\langle f_{(r,\theta)}, \varphi\left(r,\theta'\right)\right\rangle$ must be independent of the angular dependence of φ, which requires that (106) holds. ∎

Theorem 14. *Structure theorem. Let $T^\lambda : X = R^n \setminus \{\mathbf{0}\} \to Y = R$ such that $\mathbf{x} \mapsto y = |\mathbf{x}|^\lambda$, $\lambda \in \mathbb{C}$. A distribution based on R^n is a spherical associated homogeneous distribution iff it is the pullback along the function T^λ of an associated homogeneous distribution based on R.*

Proof. (i) SAHD on $R^n \Rightarrow \left(T^\lambda\right)^*$AHD on R. Let f be a SAHD on R^n. Being spherically symmetric, f must be of the form (106), due to Theorem 13. Being an AHD on R^n, its radial part f_r in (106) must be an AHD based on R_+, due to the expression (119) of the Euler operator in R^n. This distribution f_r must be of the form given by the right-hand side of (102), due to the structure theorem for one-dimensional AHDs [2, Theorem 4]. Eq. (102) together with Corollary 4, which requires T to be homogeneous, then shows that this form is the pullback along the function T^λ of an AHD based on R.

(ii) $\left(T^\lambda\right)^*$AHD on $R \Rightarrow$ SAHD on R^n. Let f be an AHD on R. The pullback $\left(T^\lambda\right)^* f$ of f along the function T^λ has a form as given by the right-hand side of eq. (102). By Theorem 13 such a distribution is spherically symmetric. Due to expression (119) for the Euler operator in R^n, $\left(T^\lambda\right)^* f$ is an AHD based on R^n. ∎

7 Appendix

7.1 Spherical coordinates

We define a diffeomorphism $T_{S\to C}$, mapping spherical coordinates to Cartesian coordinates, for a domain $\Omega \subset R^n$ with $2 \le n$, such that the range $T_{S\to C} = \Omega$.

Let $\theta \triangleq \left(\theta^p, \forall p \in \mathbb{Z}_{[2,n]}\right)$, $\mathbf{x} \triangleq \left(x^i, \forall i \in \mathbb{Z}_{[1,n]}\right)$ and

$$T_{S\to C} : \Xi \triangleq R_+ \times]0,\pi[^{n-2} \times [0,2\pi[\subset R^n \to X = R^n, \tag{107}$$

such that $\xi = \left(\xi^i, \forall i \in \mathbb{Z}_{[1,n]}\right) = (r,\theta) \mapsto \mathbf{x} = T_{S\to C}(\xi) = \left(r\omega^i(\theta), \forall i \in \mathbb{Z}_{[1,n]}\right) \triangleq r\,\omega$, with $r \in R_+$, $\omega \in S^{n-1}$, $\theta^p \in]0,\pi[$, $\forall p \in \mathbb{Z}_{[2,n-1]}$, and $\theta^n \in [0,2\pi[$. Herein are, $\forall i \in \mathbb{Z}_{[1,n]}$ and $\forall p \in \mathbb{Z}_{[2,n]}$,

$$\omega^i(\theta) \triangleq \left(1_{i=1} + 1_{1<i}\prod_{p=2}^{i}\sin(\theta^p)\right)\left(1_{i=n} + 1_{i<n}\cos\left(\theta^{i+1}\right)\right) \tag{108}$$

and

$$\omega\cdot\omega = \sum_{i=1}^{n}\left(\omega^i\right)^2 = 1. \tag{109}$$

The induced metric on the $(n-1)$-dimensional unit sphere S^{n-1} is given by (implicit summation over i and j), $\forall a,b \in \mathbb{Z}_{[2,n]}$,

$$g_{ab} = \left.\left(\delta_{ij}\frac{\partial x^i}{\partial\xi^a}\frac{\partial x^j}{\partial\xi^b}\right)\right|_{r=1} = 1_{a=b}\left(1_{a=2} + 1_{3\le a}\prod_{p=2}^{a-1}\sin^2(\theta^p)\right). \tag{110}$$

Then, with $g(\theta) \triangleq \det[g_{ab}]$,

$$\sqrt{g(\theta)} = 1_{n=2} + 1_{2<n}\prod_{p=2}^{n-1}\sin^{n-p}(\theta^p) > 0. \tag{111}$$

Hence,

$$|\det dT_{S\to C}(\xi)| = r^{n-1}\sqrt{g(\theta)} > 0, \tag{112}$$

$\forall\xi \in \Xi$, so $T_{S\to C}$ is a diffeomorphism from $\Xi \to R^n$.

Define for $3 \le n$ the set of open half lines

$$L \triangleq \left\{\mathbf{x} = r\,\omega(\theta) \in R^n : \theta^p \in \{0,\pi\}, \forall p \in \mathbb{Z}_{[2,n-1]}, \forall r \in R_+\right\} \tag{113}$$

and the set $\Lambda \triangleq \{\mathbf{0}\} \cup 1_{3\le n} L$. In order for $T_{S\to C}$ to be a diffeomorphism we had to exclude from R^n the set Λ so that $\Omega = R^n \backslash \Lambda$.

Any integral over R^n, stated in Cartesian coordinates and to be converted into spherical coordinates, first has to be restricted to Ω. Under the pullback $T^*_{S\to C}$ this restricted integral transforms into an integral over Ξ. It is usually tacitly understood that Λ is a set of Lebesgue measure zero (which is true by Sard's theorem), so that the final integral is equivalent to the original integral over R^n.

The volume form ω_{R^n} on R^n becomes in spherical coordinates

$$\omega_{R^n} = r^{n-1}\left(dr \wedge \omega_{S^{n-1}}\right), \tag{114}$$

$$\omega_{S^{n-1}} \triangleq \sqrt{g(\theta)}\left(d\theta^2 \wedge d\theta^3 \wedge \ldots \wedge d\theta^n\right), \tag{115}$$

with $\omega_{S^{n-1}}$ the nowhere vanishing volume form on $\Omega \cap S^{n-1}$. Notice that, since $\omega_{S^{n-1}}$ vanishes on $\Lambda \cap S^{n-1}$, $\omega_{S^{n-1}}$ is not a proper volume form on S^{n-1}.

With respect to a coordinate basis $\left\{dx^i, \forall i \in \mathbb{Z}_{[1,n]}\right\}$ for R^n, the operator $\mathbf{d} \triangleq \left(\partial_i, \forall i \in \mathbb{Z}_{[1,n]}\right) : C^\infty(R^n) \to C^\infty(R^n)$ becomes in spherical coordinates

$$\mathbf{d} = \omega\partial_r + \frac{1}{r}\partial_!, \tag{116}$$

with

$$\partial_\omega \triangleq \sum_{p=2}^{n} \frac{\frac{\partial\omega}{\partial\theta^p}}{\left|\frac{\partial!}{\partial\theta^p}\right|^2}\partial_{\theta^p}, \tag{117}$$

$$\left|\frac{\partial\omega}{\partial\theta^p}\right|^2 = 1_{p=2} + 1_{3\le p}\prod_{q=2}^{p-1}\sin^2\left(\theta^q\right). \tag{118}$$

The Euler operator $\mathbf{x}\cdot\mathbf{d} = x^i\partial_i$ (implicit summation over i) then becomes in spherical coordinates

$$\mathbf{x}\cdot\mathbf{d} = r\partial_r. \tag{119}$$

The operator $\omega \cdot \partial_\omega$ is identically zero due to (117) and (109), while $(\partial_\omega \cdot \omega) = n-1$. The operator $\partial_{\omega_s} \cdot \partial_{\omega_s}$ is the Laplace-Beltrami operator (acting on scalar functions) on S^{n-1}.

The surface area of the unit sphere S^{n-1} is given by, $\forall n \in \mathbb{Z}_+$,

$$A_{n-1} \triangleq \int_{S^{n-1}} \omega_{S^{n-1}} = 2\frac{\pi^{n/2}}{\Gamma(n/2)} \tag{120}$$

and the volume of the unit n-dimensional ball it bounds is

$$V_n = \frac{A_{n-1}}{n} = \frac{\pi^{n/2}}{\Gamma(n/2+1)}. \tag{121}$$

7.2 The partial distributions $y^{-k}\delta^{(l)}$

Let $l \in \mathbb{N}$ and $k \in \mathbb{Z}_+$. Define functions $y^{-k} : R\backslash\{0\} \to R$ such that $y \mapsto y^{-k}$ and products $y^{-k}\delta^{(l)} \triangleq y^{-k}.\delta^{(l)}$ by, $\forall\psi \in \mathcal{D}_{\mathbb{Z}_{[-k,-1]}}(R)$,

$$\left\langle y^{-k}.\delta^{(l)}, \psi\right\rangle \triangleq \left\langle \delta^{(l)}, y^{-k}\psi\right\rangle. \tag{122}$$

This definition is legitimate since $y^{-k}\psi \in \mathcal{D}(R)$. However, (122) only defines $y^{-k}\delta^{(l)}$ on $\mathcal{D}_{\mathbb{Z}_{[-k,-1]}}(R) \subset \mathcal{D}(R)$, so $y^{-k}\delta^{(l)}$ is a partial distribution.

Define a new quantity $\left(y^{-k}\delta^{(l)}\right)_0$, $\forall \varphi \in \mathcal{D}(R)$, by

$$\left\langle \left(y^{-k}\delta^{(l)}\right)_0, \varphi \right\rangle \triangleq \left\langle \delta^{(l)}, y^{-k}\left(\varphi(y) - \sum_{j=0}^{k-1} \varphi^{(j)}(0)\frac{y^j}{j!}\right)\right\rangle. \tag{123}$$

Since (123) defines $\left(y^{-k}\delta^{(l)}\right)_0$ on the whole of $\mathcal{D}(R)$, and because it is a linear and sequential continuous functional, it is a distribution. Using the definition for the generalized derivative and the sifting property of δ, (123) can be converted to

$$\left\langle \left(y^{-k}\delta^{(l)}\right)_0, \varphi \right\rangle = \left\langle (-1)^k \frac{l!}{(k+l)!}\delta^{(k+l)}, \varphi \right\rangle, \tag{124}$$

so

$$\left(y^{-k}\delta^{(l)}\right)_0 = (-1)^k \frac{l!}{(k+l)!}\delta^{(k+l)}. \tag{125}$$

It is easily verified that $\left\langle \left(y^{-k}\delta^{(l)}\right)_0, \psi \right\rangle = \left\langle \delta^{(l)}, y^{-k}\psi \right\rangle$, $\forall \psi \in \mathcal{D}_{\mathbb{Z}_{[-k,-1]}}(R)$, so the distribution $\left(y^{-k}\delta^{(l)}\right)_0$ is an extension of the partial distribution $y^{-k}\delta^{(l)}$ from $\mathcal{D}_{\mathbb{Z}_{[-k,-1]}}(R)$ to $\mathcal{D}(R)$. Such an extension is not unique. Any two extensions differ by a distribution which maps $\mathcal{D}_{\mathbb{Z}_{[-k,-1]}}(R)$ to zero. Hence, the general extension is

$$\left(y^{-k}\delta^{(l)}\right)_\varepsilon = (-1)^k \frac{l!}{(k+l)!}\delta^{(k+l)} + \sum_{j=0}^{k-1} c_j\delta^{(j)}, \tag{126}$$

with arbitrary constants $c_j \in \mathbb{C}$, $\forall j \in \mathbb{Z}_{[0,k-1]}$. However, if we are only interested in extensions $\left(y^{-k}\delta^{(l)}\right)_e$ which are homogeneous, we get the unique homogeneous extension

$$\left(y^{-k}\delta^{(l)}\right)_e = \left(y^{-k}\delta^{(l)}\right)_0 = (-1)^k \frac{l!}{(k+l)!}\delta^{(k+l)}. \tag{127}$$

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