# Endomorphism Rings of H-comodule Algebras

F.Van Oystaeyen Y.H. Zhang

**Abstract**

In this note we provide a general method to study the endomorphism rings of H-comodule algebras over subalgebras. With this method we may derive the endomorphism rings of crossed products and some duality theorems for graded rings.

## Introduction

In this set-up we study a Hopf Galois extension $A/B$ with Hopf algebra $H$, in paticular we look at a right coideal subalgebra $K$ of $H$ and a Hopf subalgebra $L$ of $H$ and produce in Theorem 2.1, a description of $A \otimes_{A(K)} A$ and $A \otimes_{A(L)} A$. In Section 3 we focus on endomorphism rings of $H$-comodule algebras over some subalgebras. Using the smash products in view of [D] and [Ko], and in the situation of Section 2, we obtain a description of $\mathrm{End}_{A(K)}(A)$ and $\mathrm{End}_{A(L)}(A)$, cf.Theorem 3.1. As a consequence we have obtained a unified treatment for results on graded rings or crossed product algebras e.g, for skewgroup rings cf.[RR], for duality theorem for graded rings cf.[NRV], for crossed products cf.[Ko].

## 1 Preliminaries

$H$ is a Hopf algebra, over a fixed field $k$, with comultiplication $\Delta$, counit $\epsilon$ and antipode $S$. Throughout this note we assume that the antipode $S$ is bijective. A subspace $K$ of $H$ is said to be a left coideal if $\Delta(K) \subseteq H \otimes K$. So a left coideal

Received by the editors October 1993
Communicated by A. Verschoren

subalgebra $K$ of $H$ is both a coideal of $H$ and a subalgebra of $H$. Similarly we may define a right coideal subalgebra. We refer to [Sw] for full detail on Hopf algebras, and in particular we use the sigma notation: $\Delta h = \sum h_{(1)} \otimes h_{(2)}$, for $h \in H$. Now let us review some definitions and notations.

**Definition 1.1** *An algebra $A$ is called a* **right $H$-comodule algebra** *if $A$ is a right $H$-comodule module with comodule structure map $\rho$ being an algebra map. Similarly we may define a left $H$-module algebra. Dually, a coalgebra $C$ is said to be an $H$-module coalgebra if $C$ is a left $H$-module satifying compatibility condition: $\Delta_C(h \cdot c) = \sum h_{(1)} \cdot c_{(1)} \otimes h_{(2)} \cdot c_{(2)}$.*

**Definition 1.2** *A triple $(C, A, H)$ is said to be a* **Smash data** *if $H$ is a Hopf algebra, $C$ is a left $H$-coalgebra and $A$ is a right $H$-comodule algebra. We call $(H, A, D)$ an* **opposite smash date** *if $D$ is a right $H$-module coalgebra.*

We say that $A/B$ is a right $H$-extension if $A$ is an $H$-comodule algebra and $B$ is its invariant subalgebra $A^{coH} = \{b \in A | \rho(b) = b \otimes 1\}$. A right $H$-extension is said to be Galois if the canonical map $\beta$:

$$A \otimes_B A \longrightarrow A \otimes H, \ \beta(a \otimes b) = \sum ab_{(0)} \otimes b_{(1)}$$

is an isomorphism. Since the antipode $S$ is bijective, another canonical map

$$\beta' : A \otimes_B A \longrightarrow A \otimes H, \ \beta'(a \otimes b) = \sum a_{(0)} b \otimes a_{(1)}$$

is then an isomorphism too.

Let $V$ be a right $H$-comodule and $W$ a left $H$-comodule. Recall that a cotensor product $V \square_H W$ is defined by the following equalizer diagram:

$$0 \longrightarrow V \square_H W \longrightarrow V \otimes W \underset{1 \otimes \rho_W}{\overset{\rho_V \otimes 1}{\rightrightarrows}} V \otimes H \otimes W.$$

For an $H$-comodule algebra $A$ and a left coideal subalgebra $K$ of $H$, we write $A(K)$ for the subalgebra $\rho^{-1}(A \otimes K) = A \square_H K$ of $A$. In case $K$ is a Hopf subalgebra $A(K)$ is a $K$-comodule algebra with the same invariant subalgebra as $A$. We write $K^+$ for $ker(\epsilon) \cap K$, the augmentation of $K$.

# 2 Induced isomorphisms

**Theorem 2.1** *Suppose that $A/B$ is right $H$-Galois. Then for any left coideal subalgebra $K$ of $H$ and Hopf subalgebra $L$,*

*1) $\beta_K : A \otimes_{A(K)} A \longrightarrow A \otimes H/K^+H, \ a \otimes b \mapsto ab_{(0)} \otimes \overline{b_{(1)}}$ is a left $A$-linear isomorphism if $A_B$ is flat.*

*2) $\beta'_K : A \otimes_{A(L)} A \longrightarrow A \otimes H/HL^+, \ a \otimes b \mapsto a_{(0)} b \otimes \overline{a_{(1)}}$ is a right $A$-linear isomorphism if ${}_BA$ is flat.*

**proof.** Let $\pi$ denote the projection $H \longrightarrow \overline{H} = H/K^+H$. The following diagram is commutative:

$$\begin{array}{ccccccccc} 0 \longrightarrow & D & \longrightarrow & A \otimes_B A & \xrightarrow{\phi} & A \otimes_{A(K)} A & \longrightarrow 0 \\ & \downarrow \beta|_D & & \downarrow \beta & & \downarrow \beta_K & \\ 0 \longrightarrow & A \otimes K^+H & \longrightarrow & A \otimes H & \xrightarrow{1\otimes\pi} & A \otimes \overline{H} & \longrightarrow 0 \end{array}$$

Here $D$ is the kernel of the canonical map $\phi$. Since $\beta$ is bijective, $\beta|_D$ is injective and $\beta_K$ is surjective. So $\beta|_D$ is surjective if and only if $\beta_K$ is injective. However, to show $\beta|_D$ is surjective it is enough to check $\beta^{-1}(1 \otimes K^+H) \subseteq D$. For any $x \in K, h \in H$, take an element $\sum x_i \otimes y_i$ and $\sum a_j \otimes b_j$ in $A \otimes_B A$ such that

$$\beta(\sum x_i \otimes y_i) = 1 \otimes h, \quad \beta(\sum a_j \otimes b_j) = 1 \otimes x.$$

Then

$$\begin{aligned} \beta(\sum_{i,j} x_i a_j \otimes b_j y_i) &= \sum_{i,j} x_i a_j b_{j(0)} y_{i(0)} \otimes b_{j(1)} y_{i(1)} \\ &= \sum_i x_i y_{i(0)} \otimes x y_{i(1)} \\ &= 1 \otimes xh \end{aligned}$$

On the other hand, since $A_B$ is flat we have isomorphism $(A \otimes_B A)\square_H K \simeq A \otimes_B (A\square_H K) = A \otimes_B A(K)$. Now the following commutative diagram induces the isomorphism $A \otimes_B A(K) \simeq A \otimes K$.

$$\begin{array}{ccc} A \otimes_B A(K) & \xrightarrow{\sim} & (A \otimes_B A)\square_H K \\ \downarrow & & \downarrow \text{can} \otimes 1 \\ A \otimes K & \xrightarrow{\sim} & (A \otimes H)\square_H K \end{array}$$

This allow us to pick the elements $\{b_j\}$ in $A(K)$ such that $1 \otimes x = \beta(\sum a_j \otimes b_j)$. It follows that

$$\begin{aligned} \sum x_i a_j \otimes_{A(K)} b_j y_i &= \sum_i x_i (\sum_j a_j b_j) \otimes y_i \\ &= \epsilon(x) \sum x_i \otimes y_i \end{aligned}$$

Now, if $x \in K^+$, then

$$\beta^{-1}(1 \otimes hx) = \sum x_i a_j \otimes_B b_j y_i \in D.$$

Consquentely, the left $A$-linearlity of $\beta^{-1}$ entails that $\beta|_D$ is surjective.
The bijectivity of $\beta'_L$ may be established in a similar way. □

Let $\overline{\rho}$ indicate the composite map:

$$A \xrightarrow{\rho} A \otimes H \longrightarrow A \otimes \overline{H}.$$

There is a subalgebra $A_K$ of $A$ induced by $\overline{\rho}$ as follows:

$$A_K = \{a \in A | \overline{\rho}(a) = a \otimes \bar{1}\}.$$

It is easy to see that $A(K) \subseteq A_K$ and the canonical map

$$\beta^K : A \otimes_{A_K} A \longrightarrow A \otimes \overline{H}$$

is well-defined. Moreover, We have the following commutative diagram

$$\begin{array}{ccc} A \otimes_{A(K)} A & \xrightarrow{can.} & A \otimes_{A_K} A \\ \beta_K \searrow & & \swarrow \beta^K \\ & A \otimes \overline{H} & \end{array}$$

**Corrollary 2.2** *With assumptions as in Theorem 2.1, the map*

$$\beta^K : A \otimes_{A_K} A \longrightarrow A \otimes \overline{H}$$

*is bijective.*

**proof.** Since *can.* is surjective, the statement follows from the above commutative diagram and the bijectivity of $\beta_K$. □

The above corollary tell us that $A/A_K$ is $\overline{H}$-Galois extension introduced in [Sch]. In [Sch] Schneider proved Corollary 2.2 by assuming that $H$ is $\overline{H}$-faithfully coflat, or $H$ is $K$-projective cf.[Sch;1.4, 1.8]. Now what is the relationship between $A(K)$ and $A_K$? We have the following:

**Corrollary 2.3** *With assumptions as above, we have:*

$$A_K = \{a \in A | a \otimes_{A(K)} 1 = 1 \otimes_{A(K)} a \in A \otimes_{A(K)} A\}.$$

**proof.** The left hand side is contained in the right hand side since $a \otimes 1 - 1 \otimes a \in ker(can.), a \in A_K$. Conversely, suppose $a \otimes_{A(K)} 1 = 1 \otimes_{A(K)} a$ then $(1 \otimes \pi)\rho(a) = \beta_K(1 \otimes a) = \beta_K(a \otimes 1) = a \otimes \bar{1}$. It follows that $a \in A_K$ and the left hand side contains the one on the right. □

# 3 Endomorphism rings

In [NRV;2.18] the following algebra isomorphism has been established:

$$(R\#G) * L \cong \mathrm{M}_{|L|}(R\#G/L),$$

where $G$ is a finite group, $R$ is a $G$-graded ring, $L$ is a subgroup of $G$ and $R\#G$ is nothing but $R\#\mathbb{Z}G^*$ the usual smash product over $\mathbb{Z}$. This is just the duality theorem for graded rings. We derive an anologous result for $H$-Galois extensions generalizing the foregoing duality theorem as well as [Ko].

Let $A$ be a right $H$-comodule algebra, $E$ a left $H$-module algebra. In view of [Ta] we may form a smash product $A\#E$ as follows:

$A\#E = A \otimes E$ as underlying space, and the multiplication is given by $(a\#b)(c\#d) = \sum ac_{(0)}\#(c_{(1)} \cdot b)d,\ a, c \in A, b, d \in E.$

If $C$ is a left $H$-module coalgebra, then $C^*$ is a right $H$-module algebra via $c^* \leftharpoonup h(x) = c^*(h \cdot x)$. Similarly, if $C$ is a right $H$-module coalgebra, then $C^*$ is a left $H$-module algebra. Let $A$ be a right $H$-comodule algebra, $C$ a left $H$-module coalgebra. Recall that a generalized smash product algebra is defined as follows cf.[D]:

1). $\#(C, A) = \mathrm{Hom}(C, A)$ as underlying spaces, the multiplication is

2). $(f \cdot g)(c) = \sum f(g(c_{(2)})_{(1)} \cdot c_{(1)})g(c_{(2)})_{(0)},\ f, g \in \#(C, A), c \in C.$

This is an an associative algebra with unit $\epsilon : c \mapsto \epsilon(c)1$. In case $C$ is finite dimensional, $\mathrm{Hom}(C, A) \simeq C^* \otimes A$. In this situation, the generalized smash product is exactly the usual smash product $A\#C^*$. Similarly, we may define the **opposit smash product** for the opposit smash data $(H, A, D)$ as follows:

1). $\#^0(D, A) = \mathrm{Hom}(D, A)$ as underlying spaces, the multiplication is:

2). $(f \cdot g)(d) = \sum f(d_{(2)})_{(0)}g(d_{(1)} \leftharpoonup f(d_{(2)})_{(1)}),\ f, g \in \mathrm{Hom}(D, A), c \in D.$

Let $K$ be a left coideal subalgebra of $H$. Then $H/K^+H$ is a right $H$-module coalgebra. In the situation of Theorem 2.1, the canonical map

$$\beta :\ A \otimes_{A(K)} A \longrightarrow A \otimes H/K^+H$$

is a left $A$-module isomorphism. Now $\beta$ induces a bijective map $\mathrm{Hom}_{A-}(-, A)$ and the following diagram of linear isomorphisms:

$$\begin{array}{ccc} \mathrm{Hom}_{A-}(A \otimes H/K^+H, A) & \longrightarrow & \mathrm{Hom}_{A-}(A \otimes_{A(K)} A, A) \\ \Updownarrow & & \| \\ \#^0(H/K^+H, A) & \longrightarrow & \mathrm{End}_{A(K)-}(A). \end{array}$$

Where the bottom bijective map is just the canonical map:$f \mapsto (a \mapsto \sum a_{(0)}f(\overline{a_{(1)}}))$ and is an anti-algebra isomorphism. Write $|H/L|$ for the dimension of the space $H/L$. In view of Theorem 2.1 we obtain:

**Theorem 3.1** *Suppose that $A/B$ is an $H$-Galois extension For any left coideal subalgebra $K$ and Hopf subalgebra $L$ of $H$,*

*a) $\#^0(H/K^+H, A) \cong \mathrm{End}_{A(K)-}(A)^{op}$ if $A_B$ is flat.*

*b) $\#(H/HL^+, A) \cong \mathrm{End}_{-A(L)}(A)$ if ${}_BA$ is flat.*

*c) In situation of b), if $|H/L| < \infty$, then* $\mathrm{End}_{-A(L)}(A) \cong A\#(H/HL^+)^*$.

Let $A$ be an $H$-module algebra, $\sigma : H \otimes H \longrightarrow A$ an invertible cocycle. We may form a crossed product $A\#_\sigma H$ (for full detail we refer to [BCM]). It is well-known that $A\#_\sigma H/A$ is a right $H$-Galois extension, and $A\#_\sigma H$ is flat both as a left and a right $A$-module. In view of the forgoing theorems, we obtain the following generalizations of [NRV] and [Ko].

**Corrollary 3.2** *Suppose that $A$ is an $H$-module algebra and $\sigma$ is an invertible cocycle. For any left coideal subalgebra $K$ and Hopf subalgebra $L$ of $H$,*
*a)* $\#^0(H/K^+H, A\#_\sigma H) \cong \mathrm{End}_{A\#_\sigma K-}(A\#_\sigma H)^{op}$.
*b)* $\#(H/HL^+, A\#_\sigma H) \cong \mathrm{End}_{-A\#_\sigma L}(A\#_\sigma H)$.

**Example 3.3** *Let $H = kG$, and let $L$ be a subgroup of $G$ such that $|G/L| < \infty$. Put $\overline{H} = kG/L$, the left coset of $L$ in $G$, which is a left $H$-module coalgebra. Since $H$ is cocommutative $\overline{H} = H/HL^+$, here $L^+ = (kL)^+$. As a consequence of the foregoing theorems we now obtain the following propertiy of skew group ring $A * G$:*

$$(A * G) * \overline{H}^* \cong \mathrm{End}_{-A*L}(A * G).$$

*In particular, if $G$ is a finite group and $L$ is the commutator subgroup $G'$, then $\overline{H}^*$ is nothing but the charactor subgroup $X$ on $G$. Therefore, the above equality is exactly* [RR;5.1], *i.e,*

$$(A * G) * X \cong \mathrm{End}_{-A*G'}(A * G).$$

**Example 3.4** (cf.[NRV;2.18]) *Let $G$ be a finite group and $R$ is a graded algebra of type $G$. $L$ is a subgroup of $G$. Put $K = (kG/L)^*$, and $H = kG^*$. Then $K \subseteq H$, is a right $H$-module coalgebra. In view of the forgoing theporems we have*

$$\begin{aligned}(R\#kG^*) * L &\cong \#(H/HK^+, R\#kG^*)\\ &\cong \mathrm{End}_{R\#K}(R\#kG^*)\\ &\cong \mathrm{M}_{|L|}(R\#K)\end{aligned}$$

**Example 3.5** (cf.[NRV;2.20]) *Notations as above but now $R$ is a crossed product graded algebra, and $L$ has finite index $n = [G : L] < \infty$, $K = k(G/L)^*$. Then*

$$\begin{aligned}R\#G/L &= R\#K \cong \#(K^*, R)\\ &= \#(G/L, R) = \#(H/HL^+, R)\\ &\cong \mathrm{End}_{R(L)}(R) = \mathrm{End}_{R^K}(R)\\ &\cong \mathrm{M}_n(R^{(L)})\end{aligned}$$

# References

[BCM] R.Blattner, M.Cohen and S.Montgomery, Crossed products and Inner Actions of Hopf Algebras, Trans.Amer.Math.Soc.298(1986), 671-711.

[D] Y.Doi, Cleft Comodules and Hopf Modules, Comm. Alg. 12(1984), 1155-1169.

[Ko] M.Koppinen, Variations on The Smash Product with Application to Group-Graded Rings, Preprint.

[NRV] C.Năstăsescu, S.Raianu and F.Van Oystaeyen, Modules Graded by G-sets. Math. Z.203,605-627(1990).

[RR] I.Reiten, C.Riedtmann, Skew Group Rings in Representation Theory, J.Alg.92,224-282,1985.

[Sch] H-J. Schneider, Normal Basis and Transitivity of Crossed Products for Hopf Algebras, J.Alg.152, 289-312(1992).

[Sw] M.E.Sweedler, Hopf algebras, Benjamin New York,1969.

[Ta] M. Takeuchi, $\mathrm{Ext}_{ad}(Sp\ R, \mu^A) \simeq \cap Br(A/k)$, J.Alg. 67(1980),436-475.

F.Van Oystaeyen
Dept. of Mathematics
University of Antwerp (UIA)
B-2610,Wilrijk,Belgium

Y.H. Zhang
Dept. of Mathematics
University of Antwerp (UIA)
B-2610,Wilrijk,Belgium