Uncountably Generated Algebras of Everywhere Surjective Functions

Richard M. Aron* José A. Conejero† Alfredo Peris‡
Juan B. Seoane-Sepúlveda§

Abstract

We show that there exists an uncountably generated algebra every non-zero element of which is an everywhere surjective function on $\mathbb{C}$, that is, a function $f : \mathbb{C} \to \mathbb{C}$ such that, for every non void open set $U \subset \mathbb{C}$, $f(U) = \mathbb{C}$.

1 Preliminaries and Main Result

This note contributes to the search for what are often large algebraic structures (infinite dimensional spaces, infinitely generated algebras, among others) of functions on $\mathbb{R}$ or $\mathbb{C}$ having certain *pathological* properties. The search for large algebraic structures of functions with pathological properties has lately attracted the attention of many authors.

Let us recall that a set M of functions satisfying some special property is called *lineable* if $M \cup \{0\}$ contains an infinite dimensional vector space and *spaceable* if $M \cup \{0\}$ contains a closed infinite dimensional vector space. More specifically,

*Partially supported by MEC and FEDER, Project MTM2005-08210.

†Partially supported by MEC and FEDER, Project MTM2007-64222, GVA, Project GVA/2010/091, and UPV, Project PAID-06-09-2932.

‡Partially supported by MEC and FEDER, Project MTM2007-64222 and by Generalitat Velenciana, Project PROMETEO/2008/101.

§Partially supported by MEC and FEDER, Project MTM2009-07848.

Received by the editors September 2008.

Communicated by F. Bastin.

2000 *Mathematics Subject Classification :* Primary 46E25; Secondary 15A03.

Key words and phrases : Lineability, spaceability, algebrability, everywhere surjective functions.

we will say that M is *μ-lineable* if $M \cup \{0\}$ contains a vector space of dimension μ, where μ is a cardinal number. Similarly, we can also define the notion of *algebrability* [5]. Here we will consider a slightly simplified definition:

Definition 1.1. *Let $\mathcal{L}$ be an algebra. A set $A \subset \mathcal{L}$ is said to be β-algebrable if there exists an algebra $\mathcal{B}$ so that $\mathcal{B} \subset A \cup \{0\}$ and card$(Z) = \beta$, where β is a cardinal number and Z is a minimal system of generators of $\mathcal{B}$. Here, by $Z = \{z_\alpha : \alpha \in \Lambda\}$ is a minimal system of generators of $\mathcal{B}$, we mean that $\mathcal{B} = \mathcal{A}(Z)$ is the algebra generated by Z, and for every $\alpha_0 \in \Lambda$, $z_{\alpha_0} \notin \mathcal{A}(Z \setminus \{z_{\alpha_0}\})$. We also say that A is algebrable if A is β-algebrable for β infinite.*

Remark 1.2. *Observe that, if Z is a minimal infinite system of generators of $\mathcal{B}$, then $\mathcal{A}(Z') \neq \mathcal{B}$ for any $Z' \subset \mathcal{B}$ such that card$(Z') <$ card(Z). The result is not true for finite systems of generators: Take $X = \mathbb{C}^2$ with coordinate-wise multiplication. X is a Banach algebra with unit $(1,1)$. The set $\{(1,0),(0,1)\}$ is a minimal system of generators of X. However, X is also single generated by $u = (1,i)$: Consider $P : X \to X, P(s,t) = (s^2,t^2)$. Note that $P(u) = (1,-1)$ and so we get*

$$\frac{1}{1+i}(u - P(u)) = (0,1) \in X.$$

Similarly, we also have $(1,0) \in X$.

This terminology of *lineability* and *spaceability* was first introduced by Enflo and Gurariy in [8] (see also [3]) while the term *algebrability* did not appear until recently in [5]. Lebesgue [9, 15] was the first to give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that for every non-trivial interval I, $f(I) = \mathbb{R}$. Let $\mathcal{S}$ denote the set of *everywhere surjective functions on* $\mathbb{C}$, that is, functions $f : \mathbb{C} \to \mathbb{C}$ with the property that for every open set $U \subset \mathbb{C}$, $f(U) = \mathbb{C}$. Such functions can be found in a similar way as the example of Lebesgue in $\mathbb{R}$. It was shown in [3] that $\mathcal{S}$ is 2^c-lineable, where c denotes the continuum. Usually, obtaining algebrability is more complex than obtaining lineability. Several results in this direction have been achieved lately. In [10] the authors proved the c-algebrability of the set of C^∞ functions with constant Taylor expansion on $\mathbb{R}$. Several different directions in this topic have also been considered by Bayart and Quarta in [7]. They proved, among other things, that the set of continuous nowhere differentiable functions is algebrable. Besides, in [12] Bandyopadhyay and Godefroy studied the algebraic structure of the set of norm attaining functionals on a Banach space. The interested reader can refer to [1, 2, 4, 5, 6, 11, 13, 14] for further results in this topic. Our present contribution to this area is an improvement of a result appearing in [5], where the authors showed that there exists an infinitely (and countably) generated algebra every non-zero element of which is an everywhere surjective function on $\mathbb{C}$. Here, we take that result to a next step:

Theorem 1.3. *$\mathcal{S}$ contains an uncountably generated algebra $\mathcal{A}$. That is, there is an algebra $\mathcal{A} \subset \mathcal{S} \cup \{0\}$ such that the subalgebra generated by any countable set $A \subset \mathcal{A}$ is strictly contained in $\mathcal{A}$. In other words, $\mathcal{S}$ is c-algebrable.*

Proof. Let $(Q_j)_{j=1}^{\infty}$ be a countable basis of open sets of $\mathbb{C}$, of the form

$$Q_j := \{z = x + iy \ : \ a_j < x < b_j \text{ and } c_j < y < d_j\},$$

for some $a_j, b_j, c_j, d_j \in \mathbb{R}$, for every $j \in \mathbb{N}$. Inductively, we select copies of the Cantor set $C_j \subset]a_j, b_j[$, such that $C_{j+1} \cap (\cup_{k=1}^{j} C_k) = \emptyset$, $j \in \mathbb{N}$. Then, for every $j \in \mathbb{N}$ we can choose $h_j :]c_j, d_j[\to \mathbb{C}$ and $\phi_j : \mathcal{C} \to C_j$ bijections, where $\mathcal{C} \subset [0,1]$ is the ternary Cantor set. For each $\alpha \in \mathcal{C}$, let us define $f_\alpha : \mathbb{C} \to \mathbb{C}$ by

$$f_\alpha(z) := \begin{cases} h_j(\Im(z)) & \text{if } \Re(z) = \phi_j(\alpha) \text{ and } \Im(z) \in]c_j, d_j[\text{ for some } j, \\ 1 & \text{otherwise,} \end{cases}$$

where $\Re(z)$ and $\Im(z)$ denote, respectively, the real part and the imaginary part of z. Clearly, all these functions are everywhere surjective. We fix $\alpha_0 \in \mathbb{C}$ and consider the algebra $\mathcal{A}$ generated by the family $\{f_{\alpha_0} f_\alpha \ : \ \alpha_0 \neq \alpha \in \mathcal{C}\}$. If $f \in \mathcal{A} \setminus \{0\}$, we write $f = p(f_{\alpha_0} f_{\alpha_1}, \dots, f_{\alpha_0} f_{\alpha_n})$ for some $n \in \mathbb{N}$ and $p \in \mathbb{C}[z_1, \dots, z_n]$ with $p(0) = 0$. In order to prove that $f \in \mathcal{S}$, let us define $q(z) := p(z, \dots, z)$. Thus two cases can occur:

Case 1: *$q(z)$ is non-constant.*
In this case, given any $z \in \mathbb{C}$, we find $\tilde{z} \in \mathbb{C}$ so that $q(\tilde{z}) = z$. For any non-empty and open set $U \subset \mathbb{C}$, we select $j \in \mathbb{N}$ with $Q_j \subset U$. If we fix $t \in]c_j, d_j[$ satisfying $h_j(t) = \tilde{z}$, then for $z' := \phi_j(\alpha_0) + it \in U$, we have $f_{\alpha_0}(z') = \tilde{z}$ and $f_\alpha(z') = 1$ if $\alpha \neq \alpha_0$. Therefore

$$f(z') = p(f_{\alpha_0} f_{\alpha_1}, \dots, f_{\alpha_0} f_{\alpha_n})(z') = p(\tilde{z}, \dots, \tilde{z}) = q(\tilde{z}) = z.$$

Case 2: *$q(z)$ is constant.*
This necessarily implies $q = 0$. For each $k = 1, \dots, n$, we can decompose p as $z_k p_k + q_k$, where $p_k \in \mathbb{C}[z_1, \dots, z_n]$, and q_k is a $(n-1)$-variable polynomial depending on z_j, $j \neq k$. If we fix all variables in p and p_k as 1, except the k-th variable, equal to z, we obtain polynomials $r_k(z)$ and $s_k(z)$, respectively. Easily, $r_k(z)$ is constant if and only if $s_k(z) = 0$. If for some k the corresponding $r_k(z)$ is non-constant, we proceed as in case 1, with $r_k(z)$ and α_k, to get that, given arbitrary $z \in \mathbb{C}$ and $U \subset \mathbb{C}$ open, there are $\tilde{z} \in \mathbb{C}$ and $z' \in U$ with $r_k(\tilde{z}) = z$ and $f_{\alpha_k}(z') = \tilde{z}$. Therefore $f(z') = r_k(\tilde{z}) = z$ and $f \in \mathcal{S}$. If this is not the case, then $s_k(z) = 0$, $k = 1, \dots, n$. We will show that this yields a contradiction. Indeed, given any $z \in \mathbb{C}$, we either have $f_{\alpha_k}(z) = 1$, $k = 1, \dots, n$, which implies $f(z) = q(f_{\alpha_0}(z)) = 0$, or there is some j so that $z' := f_{\alpha_j}(z) \neq 1$. Thus $f_{\alpha_k}(z) = 1$ for $k \neq j$ and

$$\begin{aligned} f(z) &= r_j(z') = z' s_j(z') + q_j(1, \dots, 1) = q_j(1, \dots, 1) \\ &= s_j(1) + q_j(1, \dots, 1) = r_j(1) = q(1) = 0. \end{aligned}$$

That is, $f = 0$, which is a contradiction.

Therefore we have shown that $\mathcal{A} \subset \mathcal{S} \cup \{0\}$. To see that $\mathcal{A}$ is uncountably generated, we just have to show that $f_{\alpha_0} f_\alpha \neq p(f_{\alpha_0} f_{\alpha_1}, \ldots, f_{\alpha_0} f_{\alpha_n})$ for any $n \in \mathbb{N}, p \in \mathbb{C}[z_1, \ldots, z_n]$ if $\alpha \neq \alpha_k, k = 0, \ldots, n$. Proceeding by contradiction, let $z \in \mathbb{C}$ be such that $f_\alpha(z) \notin \{1, q(1)\}$. Then $\Re(z) = \phi_j(\alpha)$ for some $j \in \mathbb{N}$. This implies $\Re(z) \neq \phi_j(\alpha_i)$, $i = 0, \ldots, n, j \in \mathbb{N}$, which gives $f_{\alpha_i}(z) = 1, i = 0, \ldots, n$. That is, $f_\alpha(z) \neq p(1, \ldots, 1) = p(f_{\alpha_0} f_{\alpha_1}, \ldots, f_{\alpha_0} f_{\alpha_n})(z)$. ∎

References

[1] R. M. Aron, J. A. Conejero, A. Peris, and J. B. Seoane-Sepúlveda. Powers of hypercyclic functions for some classical hypercyclic operators. *Integr. Equat. Oper. Theory*, 58(4):591–596, 2007.

[2] R. M. Aron, D. García, and M. Maestre. Linearity in non-linear problems. *RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, 95(1):7–12, 2001.

[3] R. M. Aron, V. I. Gurariy, and J. B. Seoane-Sepúlveda. Lineability and spaceability of sets of functions on $\mathbb{R}$. *Proc. Amer. Math. Soc.*, 133(3):795–803 (electronic), 2005.

[4] R. M. Aron, D. Pérez-García, and J. B. Seoane-Sepúlveda. Algebrability of the set of non-convergent Fourier series. *Studia Math.*, 175(1):83–90, 2006.

[5] R. M. Aron and J. B. Seoane-Sepúlveda. Algebrability of the set of everywhere surjective functions on $\mathbb{C}$. *Bull. Belg. Math. Soc. Simon Stevin*, 14:25-31, 2007.

[6] F. Bayart. Linearity of sets of strange functions. *Michigan Math. J.* 53:291–303, 2007.

[7] F. Bayart and L. Quarta. Algebras in sets of queer functions. *Isr. J. Math.* 158:285–296, 2007.

[8] P. Enflo and V. I. Gurariy. On lineability and spaceability of sets in function spaces. Unpublished notes.

[9] B. R. Gelbaum and J. M. H. Olmsted. *Counterexamples in analysis*. Dover Publications Inc., Mineola, NY, 2003. Corrected reprint of the second (1965) edition.

[10] F. J. García-Pacheco, N. Palmberg, and J. B. Seoane-Sepúlveda. Lineability and algebrability of pathological phenomena in analysis. *J. Math. Anal. Appl.*, 326(2):929–939, 2007.

[11] F. J. García-Pacheco, M. Martín, and J. B. Seoane-Sepúlveda. Lineability, spaceability, and algebrability of certain subsets of function spaces. *Taiwanese J. Math.*, 13 (2009), no 4, 1257–1369.

[12] P. Bandyopadhyay and G. Godefroy. Linear structures in the set of norm-attaining functionals on a Banach space. *J. Convex Anal.*, 13(3-4): 489–497, 2006.

[13] V. I. Gurariy and L. Quarta. On lineability of sets of continuous functions. *J. Math. Anal. Appl.*, 294(1):62–72, 2004.

[14] S. Hencl. Isometrical embeddings of separable Banach spaces into the set of nowhere approximatively differentiable and nowhere Hölder functions. *Proc. Amer. Math. Soc.*, 128(12):3505–3511, 2000.

[15] H. Lebesgue. *Leçons sur l'intégration et la recherche de fonctions primitives*, volume 136 of 8°. Gauthier-Villars. VII, Paris, 1904.

Department of Mathematical Sciences,
Kent State University, Kent, Ohio, 44242, USA.
Email: aron@math.kent.edu

IUMPA,
Departamento de matemática Aplicada,
Universidad Politécnica de Valencia,
E-46022 Valencia, Spain.
Emails: aconejero@mat.upv.es and aperis@mat.upv.es

Departamento de Análisis Matemático,
Facultad de Ciencias Matemáticas,
Universidad Complutense de Madrid,
Plaza de las Ciencias 3, E-28040 Madrid, Spain.
Email: jseoane@mat.ucm.es