Compact and quasicompact homomorphisms between differentiable Lipschitz algebras

H. Mahyar A. H. Sanatpour

Abstract

In this note we consider homomorphisms between differentiable Lipschitz algebras $Lip^n(X, \alpha)$ $(0 < \alpha \leq 1)$ and $lip^n(X, \alpha)$ $(0 < \alpha < 1)$, where X is a perfect compact plane set. We give sufficient conditions implying the compactness and power compactness of these homomorphisms. Moreover, we investigate under what conditions a quasicompact homomorphism between these algebras is power compact. We also give a necessary condition for a homomorphism between these algebras to be quasicompact and in certain cases to be power compact. Finally, using these results, by giving an example we show that there exists a quasicompact homomorphism between these algebras which is not power compact.

1 Introduction

Let X be a compact plane set and $0 < \alpha \leq 1$. The Lipschitz algebra $Lip(X, \alpha)$, of order α, is the algebra of all complex-valued functions f on X for which

$$p_\alpha(f) = \sup \left\{ \frac{|f(z) - f(w)|}{|z - w|^\alpha} : z, w \in X \text{ and } z \neq w \right\} < \infty.$$

The subalgebra of those functions $f \in Lip(X, \alpha)$ for which $|f(z) - f(w)|/|z - w|^\alpha \to 0$ as $|z - w| \to 0$ is denoted by $lip(X, \alpha)$. These Lipschitz algebras were first studied by Sherbert ([17], [18]). The algebras $Lip(X, \alpha)$ for $0 < \alpha \leq 1$

Received by the editors June 2008 - In revised form in July 2009.
Communicated by F. Bastin.
2000 *Mathematics Subject Classification* : Primary 46J15;Secondary 47B48.
Key words and phrases : Compact; power compact; quasicompact; homomorphisms; differentiable Lipschitz algebras.

and $lip(X,\alpha)$ for $0 < \alpha < 1$ are Banach function algebras on X if equipped with the norm $\|f\|_\alpha = \|f\|_X + p_\alpha(f)$, where $\|f\|_X = \sup\limits_{z\in X}|f(z)|$. Moreover, for every α, β with $0 < \alpha < \beta \leq 1$ we have

$$p_\alpha(f) \leq p_\beta(f)(\mathrm{diam}X)^{\beta-\alpha}, \quad (f \in Lip(X,\beta))$$

and $Lip(X,\beta) \subseteq lip(X,\alpha) \subseteq Lip(X,\alpha)$. These inclusions are proper when X contains infinitely many points (see [2], [11]).

In this paper we consider certain subalgebras of Lipschitz algebras, namely differentiable Lipschitz algebras. A complex-valued function f on a perfect plane set X is called differentiable on X if at each point $z_0 \in X$ the following limit exists

$$f'(z_0) = \lim_{\substack{z\to z_0\\ z\in X}} \frac{f(z)-f(z_0)}{z-z_0}.$$

Let X be a perfect compact plane set, $0 < \alpha \leq 1$, and $n \in \mathbb{N}$. The algebra of all complex-valued functions f on X whose derivatives up to order n exist and $f^{(k)} \in Lip(X,\alpha)$ for each k $(0 \leq k \leq n)$, is denoted by $Lip^n(X,\alpha)$. Similarly the algebras $lip^n(X,\alpha)$ are defined for $0 < \alpha < 1$. These differentiable Lipschitz algebras were first studied in [10] and [14]. The algebras $Lip^n(X,\alpha)$ and $lip^n(X,\alpha)$ with the norm

$$\|f\|_{n,\alpha} = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\alpha}{k!} = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_X + p_\alpha(f^{(k)})}{k!}, \qquad (f \in Lip^n(X,\alpha))$$

are normed function algebras on X. In fact, $lip^n(X,\alpha)$ is a closed subalgebra of $Lip^n(X,\alpha)$. It is also obvious that $Lip^m(X,\alpha) \subseteq Lip^n(X,\alpha)$ and $lip^m(X,\alpha) \subseteq lip^n(X,\alpha)$ for each $m,n \in \mathbb{N}$ with $n \leq m$. These differentiable Lipschitz algebras are not necessarily complete. To investigate the completeness of these algebras we introduce Dales-Davie algebras.

Let X be a perfect compact plane set and $n \in \mathbb{N}$. The algebra of n-times continuously differentiable functions on X is denoted by $D^n(X)$. These algebras were originally studied by Dales and Davie in [7]. The algebra $D^n(X)$ with the norm

$$\|f\|_n = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_X}{k!}, \qquad (f \in D^n(X))$$

is a normed function algebra on X. As it was shown in [14] the completeness of $D^1(X)$ implies that all algebras $D^n(X)$, $Lip^n(X,\alpha)$ and $lip^n(X,\alpha)$ are complete. There are some examples of X such that $D^1(X)$ is incomplete ([4], [5], [8]). As proved in [5] and [10], $D^1(X)$ is complete if and only if for every $z \in X$ there exists a constant c_z such that for every $f \in D^1(X)$ and every $w \in X$,

$$|f(z) - f(w)| \leq c_z|z-w|(\|f\|_X + \|f'\|_X).$$

Moreover, we recall the definition of *regularity* to give a sufficient condition for the completeness of $D^1(X)$.

Definition 1.1. Let X be a compact plane set which is connected by rectifiable arcs. Let $\delta(z,w)$ be the geodesic metric on X, the infimum of the lengths of the arcs joining z and w.

(i) X is *pointwise regular* if for each $z \in X$ there exists a constant $c_z > 0$ such that for every $w \in X$, $\delta(z,w) \leq c_z|z-w|$.

(ii) X is *uniformly regular* if there exists a constant $c > 0$ such that for every $z,w \in X$, $\delta(z,w) \leq c|z-w|$.

Note that every convex compact plane set is obviously uniformly regular. There are also non-convex uniformly regular sets, like the Swiss cheese defined in [13]. Dales and Davie in [7] showed that $D^1(X)$ is complete whenever X is a finite union of uniformly regular sets. The proof given in [7] is also valid when X is a finite union of pointwise regular sets [10]. The following condition which will be used in the sequel, is also a sufficient condition for the completeness of $D^1(X)$ (see [3]).

Definition 1.2. A perfect compact plane set X is said to satisfy the $(*)$-*condition* if there exists a constant $c > 0$ such that for every $z,w \in X$ and $f \in D^1(X)$,

$$|f(z)-f(w)| \leq c|z-w|(\|f\|_X + \|f'\|_X).$$

It is known that every uniformly regular set satisfies the $(*)$-condition (see [7]). Note that if X satisfies the $(*)$-condition, then $D^1(X)$ is a proper subalgebra of $Lip(X,1)$ and the norms of $D^1(X)$ and $Lip(X,1)$ are equivalent on $D^1(X)$. In other words, $D^1(X)$ is a closed subalgebra of $Lip(X,1)$ [14, Theorem 2.2.22]. For any such X we therefore have

$$D^{n+1}(X) \subseteq Lip^n(X,1) \subseteq lip^n(X,\alpha) \subseteq Lip^n(X,\alpha) \subseteq D^n(X).$$

Also, the norms of $D^{n+1}(X)$ and $Lip^n(X,1)$ are equivalent on $D^{n+1}(X)$. For further results about these algebras we refer the reader to [3], [4], [5], [7], [8], [10], [11], [14], and [15].

We recall that a function algebra A on a compact Hausdorff space X is *natural* if every nonzero complex homomorphism on A is an evaluation homomorphism at some point of X [6, 4.1.3]. In the case where A is a Banach function algebra on X, it is natural if its maximal ideal space M_A coincides with X. As proved in [10], the algebras $Lip^n(X,\alpha)$ and $lip^n(X,\alpha)$ are natural, when X is uniformly regular. However, applying the same method used by Jarosz in [12], one can show that the algebras $Lip^n(X,\alpha)$ and $lip^n(X,\alpha)$ are natural for every perfect compact plane set X (see also [8]).

Let A and B be Banach function algebras on compact plane sets X and Y, respectively. Then a homomorphism $T : A \to B$ is said to be induced by a map $\varphi : Y \to X$ if $Tf = f \circ \varphi$ for every $f \in A$. It is known that if A and B are natural, then every unital homomorphism $T : A \to B$ is induced by a continuous map $\varphi : Y \to X$. If A contains the coordinate function Z, then obviously $\varphi \in B$. It is interesting to know under what conditions a map $\varphi : Y \to X$ induces a homomorphism $T : A \to B$. In other words, under what conditions

$f \circ \varphi \in B$ whenever $f \in A$. In this paper, we consider homomorphisms between differentiable Lipschitz algebras. In Section 2, we investigate when a selfmap $\varphi : X \to X$ in $Lip^{\ell}(X, \alpha)$ induces a (compact) homomorphism $T : Lip^{m}(X, \alpha) \to Lip^{\ell}(X, \alpha)$. In Section 3, we study power compact and quasicompact homomorphisms between these algebras.

2 Compact homomorphisms between differentiable Lipschitz algebras

Let X be a perfect compact plane set satisfying the $(*)$-condition and m, ℓ be two positive integers with $m \geq \ell$. Since $Lip^{m}(X, \alpha)$ contains the coordinate function Z, if a selfmap $\varphi : X \to X$ induces a homomorphism $T : Lip^{m}(X, \alpha) \to Lip^{\ell}(X, \alpha)$, then $\varphi \in Lip^{\ell}(X, \alpha)$. Conversely, we show that every selfmap $\varphi : X \to X$ in $Lip^{\ell}(X, \alpha)$ induces a homomorphism $T : Lip^{m}(X, \alpha) \to Lip^{\ell}(X, \alpha)$, i.e., if $\varphi \in Lip^{\ell}(X, \alpha)$ then $f \circ \varphi \in Lip^{\ell}(X, \alpha)$ for every $f \in Lip^{m}(X, \alpha)$. To do this, let $f \in Lip^{m}(X, \alpha)$ and $\varphi \in Lip^{\ell}(X, \alpha)$. Since $m \geq \ell$, using Faà di Bruno's formula [1, page 823], for each k $(0 \leq k \leq \ell)$ we have

$$(f \circ \varphi)^{(k)} = \sum_{j=0}^{k} (f^{(j)} \circ \varphi) \cdot h_{j,k}, \tag{2.1}$$

where

$$h_{j,k} = \sum_{a} \left(\frac{k!}{a_1! a_2! \cdots a_k!} \prod_{i=1}^{k} \left(\frac{\varphi^{(i)}}{i!} \right)^{a_i} \right),$$

the sum $\sum_a$ is taken over all non-negative integers $a_1, a_2, \ldots, a_k$ satisfying $a_1 + a_2 + \cdots + a_k = j$ and $a_1 + 2a_2 + \cdots + ka_k = k$. Note that $h_{j,k} \in Lip(X, \alpha)$ for every $k = 1, 2, ..., \ell$ and $j = 1, 2, ..., k$, since $\varphi \in Lip^{\ell}(X, \alpha)$ and $h_{j,k}$ is a combination of the derivatives of φ up to order k. On the other hand, since X satisfies the $(*)$-condition and φ is continuously differentiable we have

$$p_\alpha(f \circ \varphi) \leq c^\alpha p_\alpha(f) \left(\|\varphi\|_X + \|\varphi'\|_X \right)^\alpha, \qquad (f \in Lip(X, \alpha)).$$

Hence, $f^{(j)} \circ \varphi \in Lip(X, \alpha)$ for each $0 \leq j \leq k$. Therefore, according to (2.1), $(f \circ \varphi)^{(k)} \in Lip(X, \alpha)$ for each $k = 0, 1, ..., \ell$ and hence $f \circ \varphi \in Lip^{\ell}(X, \alpha)$.

We now show that such a homomorphism is compact when $m > \ell$.

Theorem 2.1. *Let X be a perfect compact plane set satisfying the $(*)$-condition and m, ℓ be two positive integers with $m > \ell$. Let $T : Lip^{m}(X, \alpha) \to Lip^{\ell}(X, \alpha)$ be a homomorphism induced by the selfmap $\varphi : X \to X$. Then T is compact.*

Proof. For the compactness of T let (f_n) be a bounded sequence in $Lip^{m}(X, \alpha)$ with $\|f_n\|_{m,\alpha} = \sum_{k=0}^{m} \frac{\|f_n^{(k)}\|_\alpha}{k!} \leq 1$. Since $\|f_n^{(k)}\|_X + p_\alpha(f_n^{(k)}) = \|f_n^{(k)}\|_\alpha \leq k!$, the sequence $(f_n^{(k)})$ is bounded and equicontinuous in $C(X)$ for each $k = 0, 1, ..., m$. So by the Arzela-Ascoli Theorem, (f_n) has a subsequence (f_{n_j}) such that $(f_{n_j}^{(k)})$ is uniformly convergent on X for all k ($0 \le k \le m$). Then by using the $(*)$-condition, one can find an m-times continuously differentiable function f on X such that

$$\|f_{n_j}^{(k)} - f^{(k)}\|_X \to 0 \qquad \text{as } j \to \infty,$$

for each $k = 0, 1, ..., m$. We show that (Tf_{n_j}) is convergent in $Lip^\ell(X, \alpha)$.

In general, when X satisfies the $(*)$-condition for every continuously differentiable function g on X, we have

$$p_\alpha(g) \le c(\operatorname{diam} X)^{1-\alpha}(\|g\|_X + \|g'\|_X),$$

which implies that

$$\begin{aligned}\|f_{n_j} - f\|_{\ell,\alpha} &= \sum_{k=0}^{\ell} \frac{\|f_{n_j}^{(k)} - f^{(k)}\|_\alpha}{k!} \\ &\le \Big(1 + c(\operatorname{diam} X)^{1-\alpha}\Big) \sum_{k=0}^{\ell} \frac{\|f_{n_j}^{(k)} - f^{(k)}\|_X + \|f_{n_j}^{(k+1)} - f^{(k+1)}\|_X}{k!}.\end{aligned}$$

Therefore (f_{n_j}) converges in $Lip^\ell(X, \alpha)$, since $m > \ell$. On the other hand, by the discussion before this theorem, φ induces an endomorphism of $Lip^\ell(X, \alpha)$ which is automatically continuous since $Lip^\ell(X, \alpha)$ is semisimple. So, the convergence of (f_{n_j}) in $Lip^\ell(X, \alpha)$ implies that $(Tf_{n_j}) = (f_{n_j} \circ \varphi)$ is convergent in $Lip^\ell(X, \alpha)$. ■

It is interesting to see that when X is a perfect compact plane set with nonempty interior, every selfmap $\varphi : X \to X$ in $Lip^\ell(X, \alpha)$ with $\varphi(X) \subseteq \operatorname{int} X$ induces a homomorphism $T : Lip^m(X, \alpha) \to Lip^\ell(X, \alpha)$ for every $m, \ell \in \mathbb{N}$. If $Lip^\ell(X, \alpha)$ is complete, this is an immediate consequence of the holomorphic functional calculus. Otherwise, by [3, Lemma 1.5] one can obtain a compact set $Y \subseteq \operatorname{int} X$ containing $\varphi(X)$ and a constant $c_0 > 0$ such that for every analytic function f in $\operatorname{int} X$ and every $z, w \in X$,

$$|f(\varphi(z)) - f(\varphi(w))| \le c_0 |\varphi(z) - \varphi(w)|(\|f\|_Y + \|f'\|_Y).$$

Consequently, for every analytic function f in $\operatorname{int} X$, we have

$$p_\alpha(f \circ \varphi) \le c_0 p_\alpha(\varphi)(\|f\|_Y + \|f'\|_Y). \tag{2.2}$$

Now let $f \in Lip^m(X, \alpha)$. Then f is analytic in $\operatorname{int} X$, so f is infinitely differentiable in $\operatorname{int} X$ and its all derivatives are analytic in $\operatorname{int} X$. Hence (2.2) yields

$$p_\alpha(f^{(k)} \circ \varphi) \le c_0 p_\alpha(\varphi)(\|f^{(k)}\|_Y + \|f^{(k+1)}\|_Y), \qquad (k = 0, 1, 2, ...). \tag{2.3}$$

Therefore,

$$\begin{aligned}\|f^{(k)} \circ \varphi\|_\alpha &= \|f^{(k)} \circ \varphi\|_X + p_\alpha(f^{(k)} \circ \varphi) \\ &\le \|f^{(k)}\|_Y + c_0 p_\alpha(\varphi)(\|f^{(k)}\|_Y + \|f^{(k+1)}\|_Y) \\ &\le c_1(\|f^{(k)}\|_Y + \|f^{(k+1)}\|_Y), \qquad (k = 0, 1, 2, ...)\end{aligned} \tag{2.4}$$

where $c_1 = 1 + c_0 p_\alpha(\varphi)$. Using Faà di Bruno's formula (2.1) and (2.3), for every $z, w \in X$ with $\varphi(z) \neq \varphi(w)$ we have

$$\begin{aligned}
\frac{|(f\circ\varphi)^{(k)}(z) - (f\circ\varphi)^{(k)}(w)|}{|z-w|^\alpha} \leq & \frac{1}{|z-w|^\alpha}\sum_{j=0}^{k} |f^{(j)}(\varphi(z))h_{j,k}(z) - f^{(j)}(\varphi(w))h_{j,k}(w)| \\
\leq & \sum_{j=0}^{k} \frac{|f^{(j)}(\varphi(z)) - f^{(j)}(\varphi(w))|}{|z-w|^\alpha}|h_{j,k}(z)| \\
& + \sum_{j=0}^{k} |f^{(j)}(\varphi(w))| \frac{|h_{j,k}(z) - h_{j,k}(w)|}{|z-w|^\alpha} \\
\leq & c_0 p_\alpha(\varphi) \sum_{j=0}^{k} (\|f^{(j)}\|_Y + \|f^{(j+1)}\|_Y)\|h_{j,k}\|_X \\
& + \sum_{j=0}^{k} \|f^{(j)}\|_Y p_\alpha(h_{j,k}),
\end{aligned}$$

for all $k = 0, 1, 2, \ldots \ell$. Therefore $(f \circ \varphi)^{(k)} \in Lip(X, \alpha)$ for all $k = 0, 1, 2, \ldots \ell$. It follows that φ induces a homomorphism $T : Lip^m(X, \alpha) \to Lip^\ell(X, \alpha)$, for every $m, \ell \in \mathbb{N}$.

We next show that for every two positive integers m, ℓ, a selfmap φ of X with $\varphi(X) \subseteq \mathrm{int}X$ induces a compact homomorphism $T : Lip^m(X, \alpha) \to Lip^\ell(X, \alpha)$ (see Theorem 3.1 and Remark 3.2 in [3]). To prove this, we need the completeness of algebras of the type $Lip^n(X, \alpha)$, and for this we can assume that $D^1(X)$ is complete.

Theorem 2.2. *Let X be a perfect compact plane set with nonempty interior such that $D^1(X)$ is complete. Let m and ℓ be two positive integers. Assume that φ in $Lip^\ell(X, \alpha)$ is a selfmap of X with $\varphi(X) \subseteq intX$ and $T : Lip^m(X, \alpha) \to Lip^\ell(X, \alpha)$ is the homomorphism induced by φ. Then T is compact.*

Proof. Let (f_n) be a bounded sequence in $Lip^m(X, \alpha)$ with $\|f_n\|_{m,\alpha} = \sum_{k=0}^{m} \frac{\|f_n^{(k)}\|_\alpha}{k!} \leq 1$. Then $\|f_n\|_X + p_\alpha(f_n) = \|f_n\|_\alpha \leq 1$. Hence by the Arzela-Ascoli Theorem (f_n) has a uniformly convergent subsequence (f_{n_j}) in $C(X)$. So it is uniformly Cauchy on X, that is, $\|f_{n_i} - f_{n_j}\|_X \to 0$ as $i, j \to \infty$. We show that $(Tf_{n_i}) = (f_{n_i} \circ \varphi)$ is $\|.\|_{\ell,\alpha}$-Cauchy. By the completeness of $Lip^\ell(X, \alpha)$, this implies that (Tf_{n_i}) is convergent in $Lip^\ell(X, \alpha)$.

Let $Y \subseteq \mathrm{int}X$ be the compact set containing $\varphi(X)$ obtained from [3, Lemma 1.5]. Then there exists a positive number d such that $Y_d = \{z \in \mathbb{C} : \mathrm{dist}(z, Y) \leq d\}$ is in $\mathrm{int}X$. Since every $f \in Lip^m(X, \alpha)$ is analytic in $\mathrm{int}X$, it follows from Cauchy's Estimate that $\|f^{(k)}\|_Y \leq \frac{k!}{d^k}\|f\|_X$ for all $k = 0, 1, 2, \ldots$. Therefore, using this estimate, Faà di Bruno's formula (2.1) and (2.4), we have

$$\begin{aligned}
\|Tf_{n_i} - Tf_{n_j}\|_{\ell,\alpha} &= \sum_{k=0}^{\ell} \frac{1}{k!} \|((f_{n_i} - f_{n_j}) \circ \varphi)^{(k)}\|_\alpha \\
&\leq \sum_{k=0}^{\ell} \frac{1}{k!} \sum_{q=0}^{k} \|(f_{n_i} - f_{n_j})^{(q)} \circ \varphi\|_\alpha \|h_{q,k}\|_\alpha \\
&\leq \sum_{k=0}^{\ell} \frac{1}{k!} \sum_{q=0}^{k} c_1(\|(f_{n_i}^{(q)} - f_{n_j}^{(q)}\|_Y + \|(f_{n_i}^{(q+1)} - f_{n_j}^{(q+1)}\|_Y)\|h_{q,k}\|_\alpha \\
&\leq \sum_{k=0}^{\ell} \frac{c_1}{k!} \sum_{q=0}^{k} (\frac{q!}{d^q} + \frac{(q+1)!}{d^{q+1}})\|f_{n_i} - f_{n_j}\|_X \|h_{q,k}\|_\alpha \\
&= c_1 \left(\sum_{k=0}^{\ell} \frac{1}{k!} \sum_{q=0}^{k} \frac{q!}{d^q}(1 + \frac{q+1}{d})\|h_{q,k}\|_\alpha \right) \|f_{n_i} - f_{n_j}\|_X.
\end{aligned}$$

Whence, $\|Tf_{n_i} - Tf_{n_j}\|_{\ell,\alpha} \to 0$ as $i, j \to \infty$ and this ends the proof. ∎

Remark 2.3. Using the same arguments as in the proof of theorems 2.1 and 2.2, one can show that the results of these theorems also hold true by replacing the big Lipschitz $Lip^n(X, \alpha)$ with the little Lipschitz $lip^n(X, \alpha)$, when $0 < \alpha < 1$.

3 Quasicompact homomorphisms between differentiable Lipschitz algebras

We begin this section by stating some facts about quasicompact homomorphisms. Let E and F be Banach spaces. The Banach space of all bounded linear operators $T : E \to F$ is denoted by $\mathcal{B}(E, F)$ and we denote by $\mathcal{K}(E, F)$ the Banach space of all compact operators $T : E \to F$. Note that $\mathcal{K}(E, F) \subseteq \mathcal{B}(E, F)$. For simplicity, $\mathcal{B}(E, E)$ and $\mathcal{K}(E, E)$ are abbreviated to $\mathcal{B}(E)$ and $\mathcal{K}(E)$, respectively.

Definition 3.1. Let $(E, \|.\|_E)$ and $(F, \|.\|_F)$ be Banach spaces such that $F \subseteq E$ and $\|.\|_E \leq \|.\|_F$ on F. Then $\mathcal{B}(E, F)$ is an algebra with the composition as multiplication. For every $T \in \mathcal{B}(E, F)$ we denote by $\|T\|_e$ the norm of $T + \mathcal{K}(E, F)$ in the Calkin algebra $\mathcal{B}(E, F)/\mathcal{K}(E, F)$, i.e., $\|T\|_e = \text{dist}(T, \mathcal{K}(E, F))$. The essential radius of T is defined as $r_e(T) = \lim_{n\to\infty} \|T^n\|_e^{\frac{1}{n}}$. We say that $T \in \mathcal{B}(E, F)$ is *Riesz* or *quasicompact* if $r_e(T) = 0$ or $r_e(T) < 1$, respectively. Also T is called *power compact*, if T^N is compact for some positive integer N.

For considering the concept of quasicompact homomorphisms between algebras of the type $Lip^n(X, \alpha)$ we note that $Lip^\ell(X, \alpha) \subseteq Lip^m(X, \alpha)$ and $\|.\|_{m,\alpha} \leq \|.\|_{\ell,\alpha}$, if $m \leq \ell$. Consequently, $\mathcal{B}(Lip^m(X, \alpha), Lip^\ell(X, \alpha))$ is an algebra and by Definition 3.1, quasicompact homomorphisms $T : Lip^m(X, \alpha) \to Lip^\ell(X, \alpha)$ make sense when $m \leq \ell$. We note that if $T : Lip^m(X, \alpha) \to Lip^\ell(X, \alpha)$ $(m \leq \ell)$ is a homomorphism induced by the selfmap $\varphi : X \to X$, then for each positive integer n the homomorphism $T^n : Lip^m(X, \alpha) \to Lip^\ell(X, \alpha)$ is induced by the selfmap $\varphi_n : X \to X$, where φ_n denotes the n-th iterate of φ.

We give some notations and a theorem due to Feinstein and Kamowitz [9] that we need in the sequel. Let A be a unital commutative semi-simple Banach algebra. For each a, b in M_A we set $\|a-b\| = \sup\{|\widehat{f}(a) - \widehat{f}(b)| : f \in A, \|f\| \leq 1\}$ (the norm of $a-b$ in the dual space A^* of A). Furthermore, we set $B^*(a, \varepsilon) = \{b \in M_A : \|a-b\| < \varepsilon\}$ for $\varepsilon > 0$ and $a \in M_A$.

Theorem 3.2. *[9, Theorem 1.2] Let A be a unital commutative semi-simple Banach algebra with connected maximal ideal space M_A. Suppose that T is a unital quasicompact endomorphism of A induced by the selfmap φ of M_A. Then the following hold:*

(i) The operators T^n converge in operator norm to a rank-one unital endomorphism S of A, and there exists $a \in M_A$ such that $Sf = \widehat{f}(a)1$ for all $f \in A$. This point a is the unique fixed point of φ.

(ii) For each $\varepsilon > 0$, there exists a positive integer N such that $\varphi_N(M_A) \subseteq B^*(a, \varepsilon)$.

(iii) $\bigcap \varphi_n(M_A) = \{a\}$.

In the remainder of this section, all the differentiable Lipschitz algebras of the type $Lip^n(X, \alpha)$ and $lip^n(X, \alpha)$ are assumed to be complete. In fact, they are assumed to be natural Banach function algebras on a perfect compact plane set X.

Let $T : Lip^m(X, \alpha) \to Lip^\ell(X, \alpha)$ be a homomorphism induced by the selfmap $\varphi : X \to X$. Note that when $m \leq \ell$ one can consider T as an endomorphism of $Lip^m(X, \alpha)$ induced by φ, since $Lip^\ell(X, \alpha) \subseteq Lip^m(X, \alpha)$. Also, we have

$$\mathcal{K}\left(Lip^m(X, \alpha), Lip^\ell(X, \alpha)\right) \subseteq \mathcal{K}\left(Lip^m(X, \alpha)\right), \tag{3.1}$$

since $\|.\|_{m,\alpha} \leq \|.\|_{\ell,\alpha}$. Consequently,

$$\operatorname{dist}\left(T, \mathcal{K}(Lip^m(X, \alpha))\right) \leq \operatorname{dist}\left(T, \mathcal{K}(Lip^m(X, \alpha), Lip^\ell(X, \alpha))\right).$$

Therefore, if $T : Lip^m(X, \alpha) \to Lip^\ell(X, \alpha)$ is a quasicompact homomorphism induced by the selfmap $\varphi : X \to X$, then T is also a quasicompact endomorphism of $Lip^m(X, \alpha)$ induced by φ. So, when X is connected, Theorem 3.2 yields to the existence of a unique fixed point z_0 of φ. In the next theorem, we give a necessary condition for the homomorphism T to be quasicompact.

Theorem 3.3. *Let X be a connected compact plane set and m, ℓ be two positive integers with $m \leq \ell$. Assume that $T : Lip^m(X, \alpha) \to Lip^\ell(X, \alpha)$ is a quasicompact homomorphism induced by the selfmap $\varphi : X \to X$. If z_0 is the fixed point of φ, then $|\varphi'(z_0)| < 1$.*

Proof. Since T is quasicompact, one has $\|T^n f - f(z_0).1\|_{\ell,\alpha} \to 0$ as $n \to \infty$, for every $f \in Lip^m(X, \alpha)$, by using Theorem 3.2(i). Employing the coordinate function Z we have

$$\|\varphi_n - z_0.1\|_{\ell,\alpha} \to 0 \qquad \text{as } n \to \infty. \tag{3.2}$$

On the other hand,

$$\begin{aligned}\|\varphi_n - z_0.1\|_{\ell,\alpha} \geq & \|(\varphi_n - z_0.1)'\|_\alpha \\ \geq & \|(\varphi_n - z_0.1)'\|_X \\ \geq & |\varphi_n'(z_0)|. \end{aligned} \tag{3.3}$$

Since z_0 is the fixed point of φ, one has $\varphi_n'(z_0) = (\varphi'(z_0))^n$ for all $n \in \mathbb{N}$. Consequently, employing (3.2) and (3.3) we have $|\varphi'(z_0)| < 1$. ∎

Remark 3.4. It is worth mentioning that, in general, the inclusion in (3.1) is proper. That is, a compact linear operator from the large space $Lip^m(X, \alpha)$ to itself whose image is contained in the smaller space $Lip^\ell(X, \alpha)$ ($m \leq \ell$), is not necessarily a compact linear operator between the spaces. For example, one can consider the Volterra operator $T : Lip^m(\overline{\mathbb{D}}, \alpha) \to Lip^m(\overline{\mathbb{D}}, \alpha)$ given by $(Tf)(z) = \int_0^z f(\xi)d\xi$, $f \in Lip^m(\overline{\mathbb{D}}, \alpha)$ where $\mathbb{D}$ denotes the open unit disc in the complex plane. Then $T \in \mathcal{K}(Lip^m(\overline{\mathbb{D}}, \alpha))$, since $(Tf)' = f$ and $p_\alpha(Tf) \leq 2\|f\|_{\overline{\mathbb{D}}}$. Also, the range of T is contained in $Lip^{m+1}(\overline{\mathbb{D}}, \alpha)$ and $T \notin \mathcal{K}(Lip^m(\overline{\mathbb{D}}, \alpha), Lip^{m+1}(\overline{\mathbb{D}}, \alpha))$, since $Lip^m(\overline{\mathbb{D}}, \alpha)$ is not finite dimensional.

Clearly every power compact operator is Riesz and hence quasicompact, but in general the converse of these implications are not true (see [9]). Feinstein and Kamowitz in [9] proved several general theorems about quasicompact endomorphisms and applied these results to the question of when quasicompact or Riesz endomorphisms of certain algebras are necessarily power compact. For example, they showed that for $C^1[0,1]$, the Banach algebra of continuously differentiable functions on $[0,1]$, there exist quasicompact endomorphisms which are not power compact. The conditions under which every quasicompact endomorphism of analytic Lipschitz algebras ($Lip_A(X, \alpha) = Lip(X, \alpha) \cap A(X)$) is power compact have been studied in [16]. In the next theorems we investigate this problem for the homomorphisms between Lipschitz algebras of the type $Lip^n(X, \alpha)$.

Theorem 3.5. *Let X be a connected compact plane set with nonempty interior and m, ℓ be two positive integers with $m \leq \ell$. Let $T : Lip^m(X, \alpha) \to Lip^\ell(X, \alpha)$ be a quasicompact homomorphism induced by the selfmap $\varphi : X \to X$, with the fixed point z_0. If $z_0 \in intX$, then T is power compact.*

Proof. By the hypothesis, $B(z_0, \delta) \subseteq \text{int}X$ for some positive number δ. As mentioned before, one can consider T as a quasicompact endomorphism of $Lip^m(X, \alpha)$ induced by the selfmap φ. Therefore by Theorem 3.2(ii), there exists a positive integer N such that

$$\varphi_n(X) \subseteq B^*(z_0, \frac{\delta}{\|Z\|_{m,\alpha}}), \tag{3.4}$$

for all $n \geq N$. On the other hand, for each $z, w \in X$ we have $|z - w| \leq \|Z\|_{m,\alpha}\|z - w\|$, since $Lip^m(X, \alpha)$ contains the coordinate function Z. Consequently, $B^*(z, r) \subseteq B(z, r\|Z\|_{m,\alpha})$ for each $z \in X$ and $r > 0$. This result along with (3.4) implies that $\varphi_n(X) \subseteq B(z_0, \delta)$ for all $n \geq N$. Since $T^N : Lip^m(X, \alpha) \to Lip^\ell(X, \alpha)$ is a homomorphism induced by φ_N and $\varphi_N(X) \subseteq \text{int}X$, Theorem 2.2 implies that T^N is compact and therefore T is power compact. ∎

Remark 3.6. One can show that the result of Theorem 3.5 also holds for the little Lipschitz $lip^n(X,\alpha)$, when $0<\alpha<1$.

We now consider the special case of $\alpha = 1$ and state some results for the algebras $Lip^n(X,1)$. It is worth mentioning that the fixed point z_0 of the inducing selfmap $\varphi : X \to X$ plays an important role in the previous theorems of this section. The desired properties of this fixed point were obtained from Theorem 3.2. However, we show that if a selfmap φ belongs to $Lip(X,1)$, then parts (ii) and (iii) of Theorem 3.2 hold when $p_1(\varphi) < 1$. More precisely, we have the following.

Lemma 3.7. *Let X be a compact plane set, $\varphi : X \to X$ belong to $Lip(X,1)$, and $p_1(\varphi) < 1$. Then*

(i) There exists $z_0 \in X$ such that $\bigcap \varphi_n(X) = \{z_0\}$. This point z_0 is the unique fixed point of φ.

(ii) For each $\varepsilon > 0$, there exists a positive integer N such that $\varphi_N(X) \subseteq B(z_0,\varepsilon)$.

Proof. First note that for each positive integer $n \geq 2$ by the definition of p_1, we have

$$|\varphi_n(z)-\varphi_n(w)| \leq p_1(\varphi)|\varphi_{n-1}(z)-\varphi_{n-1}(w)|,$$

for every $z,w \in X$. Consequently,

$$|\varphi_n(z)-\varphi_n(w)| \leq p_1^n(\varphi)\mathrm{diam}X, \tag{3.5}$$

for every $z,w \in X$ and $n \in \mathbb{N}$. Therefore we have $\mathrm{diam}\varphi_n(X) \leq p_1^n(\varphi)\mathrm{diam}X$. Since $p_1(\varphi) < 1$, it follows that $\lim \mathrm{diam}\varphi_n(X) = 0$ as $n \to \infty$ and hence $\bigcap \varphi_n(X) = \{z_0\}$ for some $z_0 \in X$. Obviously z_0 is the unique fixed point of φ. Finally, for each $\varepsilon > 0$ using (3.5), one can find a positive integer N with $|z_0 - \varphi_N(w)| \leq \varepsilon$ for every $w \in X$. This ends the proof. ∎

Note that part (i) of Lemma 3.7 is the classical contraction mapping theorem for complete metric spaces. By the above lemma and using the same technique as in the proof of Theorem 3.5, we have the following result.

Theorem 3.8. *Let X be a compact plane set with nonempty interior and m,ℓ be two positive integers with $m \leq \ell$. Let $T : Lip^m(X,1) \to Lip^\ell(X,1)$ be a homomorphism induced by the selfmap $\varphi : X \to X$. Assume that $p_1(\varphi) < 1$ and z_0 is the fixed point of φ. If $z_0 \in \mathrm{int}X$, then T is power compact.*

In [16], it was shown that if T is a nonzero power compact endomorphism of $Lip_A(X,1)$ induced by a non-constant selfmap φ on a certain compact plane set X, then the fixed point of φ is an interior point of X. We show that the same condition is necessary for a nonzero homomorphism $T : Lip^m(X,1) \to Lip^\ell(X,1)$ ($m \leq \ell$) to be power compact. Before proving this fact we state the definition of the type of plane sets which we shall consider (see also [3]).

Definition 3.9. A plane set X has an internal circular tangent at $c \in \partial X$ if there exists an open disc Δ contained in X with $\overline{\Delta} \cap X = \{c\}$. A plane set X *is strongly accessible from the interior* if it has an internal circular tangent at each point of its boundary.

We remark that the closed unit disc $\overline{\mathbb{D}}$ and $\overline{B}(z_0, r) \setminus \cup_{k=1}^{n} B(z_k, r_k)$ are examples of sets in Definition 3.9, where closed discs $\overline{B}(z_k, r_k)$ are mutually disjoint in $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

A compact plane set X is said to *have peak boundary with respect to* $B \subseteq C(X)$ if for each $c \in \partial X$ there exists a non-constant function $h \in B$ such that $\|h\|_X = h(c) = 1$ (see [3]).

Theorem 3.10. *Let Ω be a bounded domain in the plane and $X = \overline{\Omega}$ satisfy the $(*)$-condition, be strongly accessible from the interior and have peak boundary with respect to $Lip^m(X, 1)$. Assume that $T : Lip^m(X, 1) \to Lip^{\ell}(X, 1)$ ($m \leq \ell$) is a nonzero power compact homomorphism induced by the non-constant selfmap $\varphi : X \to X$. If z_0 is the fixed point of φ, then $z_0 \in intX$.*

Proof. If $T : Lip^m(X, 1) \to Lip^{\ell}(X, 1)$ is a power compact homomorphism, then by a slight modification of the proof of [3, Theorem 3.4], as it was mentioned in [3, Remark 3.5], one has $\varphi_N(X) \subseteq \mathrm{int}X$ for some positive integer N. Since z_0 is the fixed point of φ, we have $\varphi_N(z_0) = z_0$ and therefore $z_0 \in \mathrm{int}X$. ■

Using Theorem 3.10 in the following example which is similar to an example in [16], we show that there exists a quasicompact endomorphism of $Lip^m(\overline{\mathbb{D}}, 1)$ which is not power compact.

Example 3.11. For $c > 1$, consider the selfmap $\varphi(z) = \frac{z+(c-1)}{c}$ for every $z \in \overline{\mathbb{D}}$. For each positive integer n one has $\varphi_n(z) = \frac{z+(c^n-1)}{c^n}$. Therefore φ_n takes $\overline{\mathbb{D}}$ onto the closed disc with radius $\frac{1}{c^n}$ centered at $1 - \frac{1}{c^n}$ in $\overline{\mathbb{D}}$.

For each $0 < \alpha \leq 1$, let T_c be the endomorphism of $Lip^m(\overline{\mathbb{D}}, \alpha)$ induced by φ. Note that since $\varphi \in Lip^m(\overline{\mathbb{D}}, \alpha)$, the endomorphism T_c is well-defined. Let $Lf = f(1).1$ for every $f \in Lip^m(\overline{\mathbb{D}}, \alpha)$, then L is a compact (rank-one) endomorphism of $Lip^m(\overline{\mathbb{D}}, \alpha)$. For each $f \in Lip^m(\overline{\mathbb{D}}, \alpha)$ we have

$$\|T_c^n f - Lf\|_\alpha \leq \|f\|_\alpha \frac{3}{(c^n)^\alpha}. \tag{3.6}$$

Also by k times differentiation, we have $(T_c^n f - Lf)^{(k)} = c^{-nk} f^{(k)} \circ \varphi_n$ for each k ($1 \leq k \leq m$). Therefore

$$\|(T_c^n f - Lf)^{(k)}\|_\alpha \leq c^{-nk}(\|f^{(k)}\|_{\overline{\mathbb{D}}} + c^{-n\alpha} p_\alpha(f^{(k)})) \leq c^{-n\alpha}\|f^{(k)}\|_\alpha, \tag{3.7}$$

for each k ($1 \leq k \leq m$). Now using (3.6) and (3.7), we get

$$\begin{aligned}
\|T_c^n f - Lf\|_{m,\alpha} =& \sum_{k=0}^{m} \frac{1}{k!} \|(T_c^n f - Lf)^{(k)}\|_\alpha \\
\leq& 3c^{-n\alpha}\|f\|_\alpha + \sum_{k=1}^{m} c^{-n\alpha} \frac{\|f^{(k)}\|_\alpha}{k!} \\
\leq& 3c^{-n\alpha}\|f\|_{m,\alpha}.
\end{aligned}$$

Therefore $\|T_c^n - L\| \leq \frac{3}{c^{n\alpha}}$ and consequently, one has $\mathrm{dist}(T_c^n, \mathcal{K}) \leq \frac{3}{c^{n\alpha}}$ where $\mathcal{K} = \mathcal{K}(Lip^m(\overline{\mathbb{D}}, \alpha))$. This implies that $\mathrm{dist}(T_c^n, \mathcal{K})^{\frac{1}{n}} \leq \frac{3^{\frac{1}{n}}}{c^\alpha}$ and therefore $r_e(T_c) \leq \frac{1}{c^\alpha} < 1$.

It follows that for each $0 < \alpha \leq 1$ the endomorphisms T_c of $Lip^m(\overline{\mathbb{D}}, \alpha)$ are quasicompact. Note that the fixed point of φ, $z_0 = 1$, does not belong to $\mathbb{D}$. Therefore Theorem 3.10 implies that the endomorphisms T_c are not power compact when $\alpha = 1$.

Acknowledgement. We would like to thank the referee for valuable comments and for calling our attention to the references [5], [8]. Also the fact that the differentiable Lipschitz algebras $Lip^n(X, \alpha)$ and $lip^n(X, \alpha)$ are always natural (even when they are incomplete) was drawn to our attention by the referee.

References

[1] M. Abramowitz and I. A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, U.S. Department of Commerce, Washington, D.C., 1964.

[2] W. G. Bade, P. C. Curtis and H. G. Dales, Amenability and weak amenability for Beurling and Lipschitz algebras, Proc. Lond. Math. Soc., III. Ser. **55** (1987), 359-377.

[3] F. Behrouzi and H. Mahyar, Compact endomorphisms of certain analytic Lipschitz algebras, Bull. Belg. Math. Soc. Simon Stevin **12** (2005), No. 2, 301-312.

[4] W. J. Bland and J. F. Feinstein, Completions of normed algebras of differentiable functions, Stud. Math. **170** (2005), No. 1, 89-111.

[5] H. G. Dales, PhD Thesis, University of Newcastle-upon-Tyne, UK, 1970.

[6] H. G. Dales, Banach algebras and automatic continuity, London Mathematical Society Monographs, New Series, Volume 24, The Clarendon Press, Oxford, 2000.

[7] H. G. Dales and A. M. Davie, Quasianalytic Banach function algebras, J. Funct. Anal. **13** (1973), 28-50.

[8] H. G. Dales and J. F. Feinstein, Normed algebras of differentiable functions on compact plane sets, preprint.

[9] J. F. Feinstein and H. Kamowitz, Quasicompact and Riesz endomorphisms of Banach algebras, J. Funct. Anal. **225** (2005), No. 2, 427-438.

[10] T. G. Honary and H. Mahyar, Approximation in Lipschitz algebras of infinitely differentiable functions, Bull. Korean Math. Soc. **36** (1999), No. 4, 629-636.

[11] T. G. Honary and H. Mahyar, Approximation in Lipschitz algebras, Quaest. Math. **23** (2000), No.1, 13-19.

[12] K. Jarosz, $\mathrm{Lip}_{Hol}(X, \alpha)$, Proc. Amer. Math. Soc. **125** (1997), No. 10, 3129-3130.

[13] G. M. Leibowitz, Lectures on complex function algebras, Scott, Foresman and Co., Glenview, Illinois, 1970.

[14] H. Mahyar, Approximation in Lipschitz algebras and their maximal ideal spaces, PhD Thesis, Tarbiat Moallem University, 1994, Tehran, Iran.

[15] H. Mahyar, Compact endomorphisms of infinitely differentiable Lipschitz algebras, Rocky Mt. J. Math. **39** (2009), No. 1, 193-217.

[16] H. Mahyar and A. H. Sanatpour, Quasicompact endomorphisms of Lipschitz algebras of analytic functions, Publ. Math. Debrecen, **76**/1-2 (2010), 135-145.

[17] D. R. Sherbert, Banach algebras of Lipschitz functions, Pacific J. Math. **13** (1963), 1387-1399.

[18] D. R. Sherbert, The structure of ideals and point derivations in Banach algebras of Lipschitz functions, Trans. Amer. Math. Soc. **111** (1964), 240-272.

Department of Mathematics,
Tarbiat Moallem University,
49, Mofateh Ave.,
Tehran 15719, Iran.
email:mahyar@saba.tmu.ac.ir, a_sanatpour@tmu.ac.ir