Mixed finite element-finite volume methods

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Abstract

This paper is devoted to present a numerical methods for a model of incompressible and miscible flow in porous media. We analyze a numerical scheme combining a mixed finite element method (MFE) and finite volume scheme (FV) for solving a coupled system includes an elliptic equation (pressure and velocity) and a linear convection-diffusion equation (concentration). The (FV) scheme considered is "vertex centered" type semi implicit. We show that this scheme is L^∞, BV stable under a CFL condition and satisfies a discrete maximum principle. We prove also the convergence of the approximate solution obtained by the combined scheme (MFE)-(FV) to the solution of the coupled system. Finally the numerical results are presented for two spaces dimensions problem in a homogenous isotropic medium.

1 Introduction

Fluid flow and contaminant transport in porous media play an important place in the oil recovery and in the environmental problems. It is expensive to monitor the pollutant through a physical observation. Thus the numerical simulations are an indispensable tools for studying such problems.
The fluid flow and solute transport through porous media are modelled by a coupled system of partial differential equations. The model under consideration includes the Darcy flow equation coupled to a convection-diffusion equation.
The Darcy flow equation is an elliptic equation coupling a conservation equation with Darcy's law which described saturated flow. The several numerical methods have been proposed for the discretization of this equation. The most popular are

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the mixed finite element methods (MFE). The reasons for their popularity are that they provide the local mass conservation property and they produce a accurate approximation of the two variables pressure and velocity (see [13], [17], [39], [11] and the references therein).

The (MFE) method does not have an explicit flux expression, which is a favorable property when extensions to multiphase flow. For this reason many methods have been developed. The multipoint flux approximation (MPFA) method is a control volume method which provides a local explicit flux and it is locally mass conservative (see [1], [2], [16], [42] and the references therein).

The equivalence between the (MPFA) method and the mixed finite element method for quadrilateral grids was established in [31] using a specific numerical quadrature.

For a nonlinear parabolic equation, two versions (a semi discrete and full discrete schemes) of a mixed finite element method have been analyzed and developed for the degenerate and non degenerate cases in [10] with a superconvergence results for the nondegenerate case.

The extension of the Darcy equation in unsaturated flow gives Richards' equation, a nonlinear degenerate parabolic equation (see [25]). Several papers are considering numerical schemes for Richards' equation. A finite element and finite difference schemes are developed in [9] for a root-soil systems. An Euler implicit-mixed finite element scheme is analyzed in [36, 37] and a multipoint flux approximation method (MPFA) is considered in [30].

For the convection-diffusion equation, the finite volume (FV) methods are well suited for the discretization of this type equation that result of conservation laws. The advantages of these methods are locally conservative, can be applicable on arbitrary geometries (structured or unstructured meshes) and produce the solutions satisfying the maximum principle which guarantees that the mathematical model produces physically meaningful solutions. There is an extensive literature on this subject (see for example [7], [29], [33], [32], [19] and the references therein).

A large variety of methods have been proposed for the discretization of the convection-diffusion equations. We mention [8] for a combined finite volume-finite element method, [22] for a combined finite volume-mixed hybrid finite elements methods and [26, 40] for a discontinuous Galerkin method. Moreover, an Euler implicit-mixed finite element scheme is analyzed for a sub-surface fluid flow in [38].

The (FV) methods for elliptic, parabolic and hyperbolic problems have been studied and analyzed in [19].

A three families of vertex centered finite volumes schemes (explicit, implicit and semi-implicit) for immiscible two-phase flow were developed and analyzed in [3, 4, 5]. These authors have showed that the approximation obtained by the semi-implicit finite volume scheme is better.

The aim of this paper is to investigate a numerical scheme for a coupled system which includes an elliptic equation (pressure-velocity) and a linear convection-diffusion equation (concentration) for an incompressible miscible flow in porous media. A mixed finite element (MFE) method is employed to approximate the flow equation combined with a finite volume (FV) method for the transport equation on unstructured grids. The discretization scheme used in this study is based

on a vertex-centered finite volume.
The paper is organized as follows. In section 2 we define the mathematical model describing the flow and the transport in porous media. In section 3 we give some notations for the time and space discretizations. In section 4 we present the mixed finite element method and the finite volume scheme. The finite volume scheme considered is "vertex centered" type semi-implicit: explicit for the convection and implicit for the diffusion. In section 5 we prove that the (FV) scheme is L^∞, BV stable under a CFL condition and satisfies a discrete maximum principle. Section 6 is devoted to show the convergence of the approximate solution of the combined scheme to weak solution of the coupled problem. In section 7 we present the results of a selected numerical test. Numerical simulations are given for a 2D miscible flow problem in a homogenous isotropic medium.

2 Mathematical problem

Let Ω be a bounded open subset with a smooth boundary $\partial\Omega = \Gamma$, $J =]0,T[$ $(T > 0)$ a time interval and $Q = \Omega\times]0,T[$. We consider the system of equations describing the flow of miscible incompressible fluid through a porous media. We will neglect the effect of gravity. This system is then the following:
Velocity-pressure equation:

$$\begin{cases} \overrightarrow{q} = -\mathbf{K}\nabla p & in\ \ Q, \\ div\,\overrightarrow{q} = 0 & in\ \ Q, \\ (\overrightarrow{q}.\overrightarrow{n})|\Gamma_1 \cup \Gamma_4 = 0,\ p|\Gamma_2 = p_0,\ p|\Gamma_3 = p_1 & on\ ]0,T[. \end{cases} \tag{2.1}$$

Concentration equation:

$$\begin{cases} 0 \le c(x,t) \le 1 & in\ \ Q, \\ \Phi(x)\frac{\partial c}{\partial t} - div(D\nabla c - c\,\overrightarrow{q}) = 0 & in\ \ Q, \\ c|\Gamma_1 = 0,\ (D\nabla c.\overrightarrow{n})|\Gamma_2 \cup \Gamma_4 = 0,\ c|\Gamma_3 = 1 & on\ ]0,T[, \\ c(x,0) = c^0 & in\ \ \Omega. \end{cases} \tag{2.2}$$

Where p is the pressure and $\overrightarrow{q}$ is the Darcy velocity of the fluid, Φ and $\mathbf{K}$ are the porosity and the absolute permeability tensor of the porous media and c is the concentration.
The diffusion-dispersion tensor D is given by:

$$D(x, \overrightarrow{q}) = d_e I + |\overrightarrow{q}|[\alpha_l E(\overrightarrow{q}) + \alpha_t(I - E(\overrightarrow{q}))],$$

with $E(\overrightarrow{q}) = \frac{q_i q_j}{|\overrightarrow{q}|^2}$, $|\overrightarrow{q}|$ is the Euclidian norm of $\overrightarrow{q}$, d_e is the effective diffusion coefficient, α_l and α_t are the magnitudes of longitudinal and transverse dispersion respectively.
The boundary Γ splits up into four parts such that $\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3} \cup \overline{\Gamma_4}$, p_0 is a given pressure at Γ_2, p_1 is a given pressure at Γ_3 and c^0 is the initial condition.
For the problem (2.1)-(2.2) the following hypotheses are made on data:
(H_1) Ω is a bounded open polygonal subset of $\mathbb{R}^d$ $(d \le 3)$, with a smooth boundary Γ.

(H_2) $\Phi \in L^\infty(\Omega)$ such that $0 < \Phi_- < \Phi(x) < \Phi^+$ a.e. in Ω.
(H_3) $\mathbf{K}$ is a bounded, symmetric and uniformly positive definite tensor
$(i.e.\ \exists\, k^-, k^+ \in \mathbb{R}_+^*$ such that $0 < k^- |\overrightarrow{\xi}|^2 \le \mathbf{K}\,\overrightarrow{\xi}.\overrightarrow{\xi} \le k^+ |\overrightarrow{\xi}|^2 < \infty,\ \forall \overrightarrow{\xi} \in \mathbb{R}^d)$.
(H_4) D is a bounded, symmetric and uniformly positive definite tensor
$(i.e.\ \exists\, d^-, d^+ \in \mathbb{R}_+^*$ such that $0 < d^- |\overrightarrow{\xi}|^2 \le D\overrightarrow{\xi}.\overrightarrow{\xi} \le d^+ |\overrightarrow{\xi}|^2 < \infty,$ $\forall \overrightarrow{\xi} \in \mathbb{R}^d)$.
(H_5) $c^0 \in L^\infty(\Omega)$ such that $0 \le c^0 \le 1$ *a.e. in* Ω.
(H_6) $\overrightarrow{q} \in (L^\infty(Q))^d$.
(H_7) $p_0 \in L^\infty[J, H^{1/2}(\Gamma_2)]$, $p_1 \in L^\infty[J, H^{1/2}(\Gamma_3)]$.
Now, we introduce a weak formulation of the coupled problem (2.1)-(2.2).
First let the spaces X_0, M and W be defined as:

$$X_0 = \{\overrightarrow{s} \in H(div,\Omega),\ \overrightarrow{s}.\overrightarrow{n} = 0 \quad on\ \ \Gamma_1 \cup \Gamma_4\}, \quad M = L^2(\Omega)$$

where

$$H(div,\Omega) = \{\overrightarrow{s} \in (L^2(\Omega))^d, div\overrightarrow{s} \in L^2(\Omega)\}$$

and

$$W = \{\varphi \in C^1(]0,T[, C^2(\overline{\Omega})); \varphi(.,T) = 0\ and\ \varphi = 0\ on\ \Gamma_2\}.$$

Definition 2.1. *Under the assumptions* $(H_1) - (H_7)$*, the weak formulation of the coupled problem (2.1)-(2.2) can be written as follows:*
Find $(\overrightarrow{q}, p, c) : J \mapsto X_0 \times M \times W$ *such that:*

$$\begin{aligned}
&\int_\Omega \mathbf{K^{-1}}\overrightarrow{q}.\overrightarrow{s}\,d\Omega - \int_\Omega p\,div\overrightarrow{s}\,d\Omega = -\int_{\Gamma_2} p_0\,\overrightarrow{s}.\overrightarrow{n}\,ds - \int_{\Gamma_3} p_1\,\overrightarrow{s}.\overrightarrow{n}\,ds \qquad \forall \overrightarrow{s} \in X_0\\
&\int_\Omega v\,div\overrightarrow{q}\,d\Omega = 0 \qquad \forall v \in M
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \left(\Phi c\frac{\partial\varphi}{\partial t} + [c\overrightarrow{q} - D\nabla c].\nabla\varphi\right) dxdt + \int_\Omega c^0\varphi(x,0)dx\\
&- \int_0^T\!\!\int_{\Gamma_3} (\overrightarrow{q}.\overrightarrow{n})\varphi\,dsdt = 0 \qquad \forall\varphi \in W.
\end{aligned} \tag{2.4}$$

For the existence and uniqueness of the weak solution of the coupled system (2.1)-(2.2) we refer to [15] and the references therein.

3 Notations and discretizations

In this section we will describe the time and space discretizations.
For the time discretization, we split up the time interval $J =]0,T[$ such that $0 = t_0 < ... < t_n < .... < t_N = T$. We define the time step by $\Delta t^n = t_n - t_{n-1}$ and $\Delta t = max_{1\le n\le N}\Delta t^n$.
Furthermore, we define the following notations for the space discretization:
• $\Lambda_h := (K_i)_{i=1,...,Ne}$ an admissible triangulation such that $\overline{\Omega} = \cup_{K\in(\Lambda_h)}\overline{K}$.
• $\{x_j, j \in I\}$: the set of vertices of all K with I is an index set and N_v is the number of vertices in $(\Lambda_h)_{h>0}$.

• $|K|$: the area of K.
• M_j : the vertex centered control volume associated with each vertex $x_j, j \in I$ (see Figure 1) and $|M_j|$: the measure of M_j.
• $\Sigma_h := (M_i)_{i=1,\dots,N_s}$ the dual mesh such that $\overline{\Omega} = \cup_{M_i \in \Sigma_h} M_i$.
• $\overrightarrow{n}_{jl}$: the outward unit normal to $l \in M_j$.
• ξ_h : the set of the all edges [resp. faces] in 2D [resp. 3D] of Λ_h.
• $h = min\{(|l|)^{\frac{1}{d-1}};\ l \in M_j,\ M_j \in \Sigma_h\}$.
• $H = max\{(|L|)^{\frac{1}{d-1}};\ L \in \partial K,\ K \in \Lambda_h\}$.
• $\delta_{jl} = \delta(x_j, x_l)$ is the euclidian distance between x_j and x_l.
• $\Phi_j = \frac{1}{|M_j|} \int_{M_j} \Phi(x)dx$.
• $c_j^0 = \frac{1}{|M_j|} \int_{M_j} c^0(x)dx$.
• c_j^n (resp c_l^n) is an approximation of $c(x_j, t_n)$ assumed constant in $|M_j|$ (resp $c(x_l, t_n)$ assumed constant on $l \in M_j$).

We also need of the following hypotheses on the regularity of the mesh:
(H_8) For (d=2) the triangulation is weakly acute (no triangle with an angle greater than $\pi/2$).
(H_9) $h^d \leq |K| \leq H^d, \quad \forall\ K \in (\Lambda_h)_{h>0}$.
(H_{10}) $\gamma H \leq |M_j|^{1/d} \leq \beta h, \quad \forall j \in I$, where β and γ are constants independent of h.

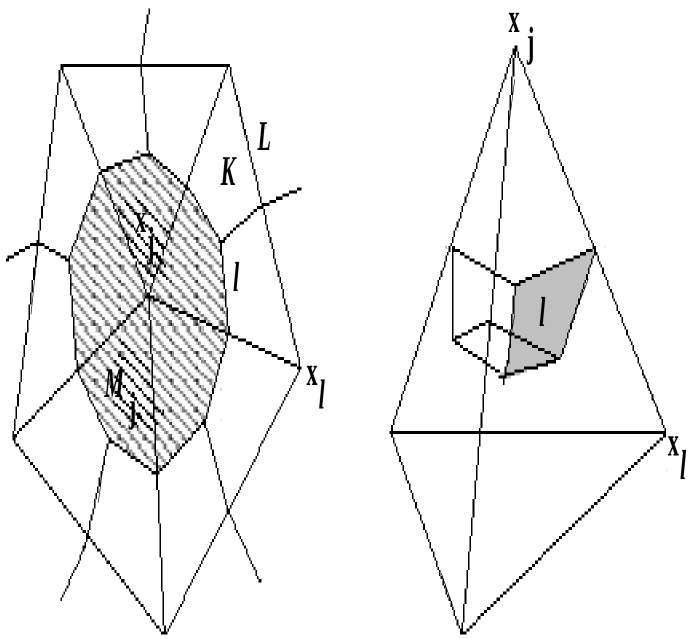

Figure 1: A vertex centered control volume in $\mathbb{R}^2$ and in $\mathbb{R}^3$.

4 Definition of the scheme

In this section the numerical scheme is presented. The Mixed finite element method is employed for approximated the pressure-velocity equation and the finite volume scheme is used for discretized the concentration equation.

4.1 Mixed Finite Element method

The pressure-velocity equation and the concentration are coupled via the velocity term, an accurate approximation to the Darcy velocity $\overrightarrow{q}$ is essential in order to have an accurate approximation of the concentration c. For this reason the mixed finite element method provides a good approximation of the velocity field.

For fixed $t \in J$, the mixed finite element scheme is defined as:

$$\left\{\begin{array}{ll} \text{Find } (\overrightarrow{q}, p) \in X_0 \times M \text{ such that:} & \\ \displaystyle\int_\Omega \mathbf{K^{-1}}\overrightarrow{q}.\overrightarrow{s}d\Omega - \int_\Omega p div \overrightarrow{s} d\Omega = -\int_{\Gamma_2} p_0 \overrightarrow{s}.\overrightarrow{n}ds - \int_{\Gamma_3} p_1 \overrightarrow{s}.\overrightarrow{n}ds & \\ & \forall \overrightarrow{s} \in X_0, \\ \displaystyle\int_\Omega v div \overrightarrow{q} d\Omega = 0 & \forall v \in M. \end{array}\right. \tag{4.1}$$

Theorem 4.1. *Under the assumptions $(H_1) - (H_2)$, the problem (4.1) admits a unique weak solution $(p, \overrightarrow{q})$.*

Proof. (see [13] and [41]). ∎

Now, we define the following finite dimensional subspaces of X_0 and M:

$$X_0^h = \{\overrightarrow{s}_h \in H(div, \Omega), \overrightarrow{s}_h.\overrightarrow{n} = 0 \;\; on \;\; \Gamma_1 \cup \Gamma_4, \;\; \overrightarrow{s}_h | K \in RT_0(K); \forall K \in \Lambda_h\}$$

and

$$M^h = \{v^h \in L^2(\Omega), v^h | K = Constant, \;\; \forall K \in (\Lambda_h)_{h>0}\}.$$

where $RT_0(K)$ is the lowest order Raviart-Thomas space defined as follows (see [13] and [41]):

$$\begin{aligned} RT_0(K) &= (\mathbf{P}_0(K))^d \oplus \mathbf{P}_0(K)x \\ &= \{\overrightarrow{r} + \beta x; \;\; \overrightarrow{r} \in \mathbb{R}^d, \beta \in \mathbb{R}\}, \end{aligned}$$

with $x = (x_1, ..., x_d)^t$ and $\mathbf{P}_k$ denotes the space of polynomials of degree $\leq k$.
The discrete mixed finite element approximation of the problem (4.1) is given by:

$$\left\{\begin{array}{ll} \text{Find } (\overrightarrow{q}_h, p_h) \in X_0^h \times M_h \text{ such that:} & \\ \displaystyle\int_\Omega \mathbf{K^{-1}}\overrightarrow{q}.\overrightarrow{s}d\Omega - \int_\Omega p_h div \overrightarrow{s}_h d\Omega = -\int_{\Gamma_2} p_0 \overrightarrow{s}_h.\overrightarrow{n}ds - \int_{\Gamma_3} p_1 \overrightarrow{s}_h.\overrightarrow{n}ds & \\ & \forall \overrightarrow{s}_h \in X_0^h, \\ \displaystyle\int_\Omega v_h div \overrightarrow{q}_h d\Omega = 0 & \forall v_h \in M^h. \end{array}\right. \tag{4.2}$$

Theorem 4.2. *Under the assumptions $(H_1) - (H_2)$, the problem (4.2) admits a unique solution (see [13]).*

4.2 The finite volume scheme

In this section, we describe the finite volume method used for the approximation of the transport equation. The (FV) scheme considered here is "vertex centered" type semi-implicit. We use a Godunov scheme (because this scheme is consistent and conservative) to approach the convection term (see [28]) and a P_1 finite element approximation for the diffusion term. This approximation leads to a robust scheme which satisfy a discrete maximum principle.
Integrating (2.2) over the set $M_j \times [t_n, t_{n+1}]$, we obtain:

$$\Phi_j|M_j|(c_j^{n+1} - c_j^n) = -\sum_{l \in \partial M_j} \int_{t_n}^{t_{n+1}} \int_l c\,\overrightarrow{q}.\overrightarrow{n}_{jl} dl dt + \sum_{l \in \partial M_j \setminus \Gamma} \int_{t_n}^{t_{n+1}} \int_l D\nabla c.\overrightarrow{n}_{jl} dl dt. \tag{4.3}$$

Using an explicit approximation of the convection term and an implicit approximation of the diffusion term, we obtain

$$\Phi_j|M_j|(c_j^{n+1} - c_j^n) = \Delta t^n \sum_{l \in \partial M_j} \int_l c^n(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl}) dl + \Delta t^n \sum_{l \in \partial M_j \setminus \Gamma} \int_l D(\nabla c^{n+1}).\overrightarrow{n}_{jl} dl. \tag{4.4}$$

This scheme can be written in the following form:

$$\begin{aligned}\Phi_j(c_j^{n+1} - c_j^n)|M_j| = &- \Delta t^n \sum_{l \in \partial M_j} c_{jl}^n(\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})|l| \\ &+ \Delta t^n \sum_{K \cap M_j \neq 0} \sum_{l \in \partial M_j \setminus \Gamma} D_K(\nabla c^{n+1})_K.\overrightarrow{n}_{jl}|l|,\end{aligned} \tag{4.5}$$

with $D_K = D_{|K}$ and $c_{jl}^n = c_j^n$ if $\overrightarrow{q}_l^n.\overrightarrow{n}_{jl} \geq 0$ and $c_{jl}^n = c_l^n$ otherwise.
The convection term is approximated by an upwind Godunov scheme and the diffusion term is approximated in the following way.
For $K \cap M_j \neq \emptyset$, we have

$$\begin{aligned}\sum_{l \in \partial M_j \cap K \setminus \Gamma} D_K(\nabla c^{n+1})_K.\overrightarrow{n}_{jl}|l| &= \sum_{l \in \partial M_j \cap K \setminus \Gamma} D_K(\nabla c^{n+1})_K.\overrightarrow{n}_{jl}|l| \\ &= (\nabla c^{n+1})_K.D_K \sum_{l \in \partial M_j \cap K \setminus \Gamma} \overrightarrow{n}_{jl}|l| \\ &= (\nabla c^{n+1})_K.D_K \sum_{l \in \partial K \cap M_j} \overrightarrow{n}_{K,l}|l| \\ &= \frac{(\nabla c^{n+1})_K}{2}.D_K \overrightarrow{n}_{K,L}|L|,\end{aligned}$$

where $L \in \partial K$ such that $L \cap M_j = \emptyset$.
Let the standard P_1 finite element basis functions satisfying $N_{M_i}(x_{M_j}) = \delta_{ij}$.
For $K \cap M_j \neq \emptyset$, we have

$$\nabla N_{M_j,K} = \nabla N_{M_j}|K = -\frac{|L|}{2|K|}\overrightarrow{n}_{K,L}.$$

A P_1 approximation for $(\nabla c^{n+1})_K$ leads to:

$$\begin{aligned}(\nabla c^{n+1})_K &= \sum_{l\in\partial M_l\cap K\neq\emptyset} c^{n+1}_{M_l}\nabla N_{M_l,K}\\ &= \sum_{l\in\partial M_l\cap K\neq\emptyset}(c^{n+1}_{M_l}-c^{n+1}_{M_j})\nabla N_{M_l,K}\\ &= \sum_{l\in\partial M_j\cap K\backslash\Gamma}(c^{n+1}_{M_l}-c^{n+1}_{M_j})\nabla N_{M_l,K}.\end{aligned}$$

Then

$$\begin{aligned}\sum_{l\in\partial M_j\cap K} D_K(\nabla c^{n+1})_K.\overrightarrow{n}_{jl}|l| &= \frac{(\nabla c^{n+1})_K}{2}.D_K\overrightarrow{n}_{K,L}|L|\\ &= \sum_{l\in\partial M_j\cap K}(c^{n+1}_{M_l}-c^{n+1}_{M_j})\frac{D^n_{jl}}{\delta_{jl}}|l|,\end{aligned}$$

where

$$D^n_{jl} = -\frac{|K|}{|l|}\delta_{jl}\nabla N_{M_l,K}.D_K\nabla N_{M_j,K}. \tag{4.6}$$

Hence the semi-implicit finite volume scheme for the equation (2.2) is given by:

$$\begin{aligned}c^{n+1}_j-\frac{\Delta t^n}{|M_j|\Phi_j}&\sum_{l\in\partial M_j\backslash\Gamma}(c^{n+1}_l-c^{n+1}_j)\frac{D^n_{jl}}{\delta_{jl}}|l|\\ &= c^n_j+\frac{\Delta t^n}{\Phi_j|M_j|}\sum_{l\in\partial M_j}c^n_l(-\overrightarrow{q}^n_l.\overrightarrow{n}_{jl})^+|l|-\frac{\Delta t^n}{|M_j|\Phi_j}\sum_{l\in\partial M_j\backslash\Gamma}c^n_j(-\overrightarrow{q}^n_l.\overrightarrow{n}_{jl})^-|l|]\end{aligned}\tag{4.7}$$

with $(-\overrightarrow{q}.\overrightarrow{n}_{jl})^+$ and $(-\overrightarrow{q}.\overrightarrow{n}_{jl})^-$ denote the positive and negative parts of $(-\overrightarrow{q}.\overrightarrow{n}_{jl})$ (i.e $(-\overrightarrow{q}.\overrightarrow{n}_{jl})^+ = \max((-\overrightarrow{q}.\overrightarrow{n}_{jl}),0)$ *and* $(-\overrightarrow{q}.\overrightarrow{n}_{jl})^- = -\min((-\overrightarrow{q}.\overrightarrow{n}_{jl}),0)$.)
From the velocity-pressure equation, we have $div\overrightarrow{q}=0$ *i.e.*

$$\sum_{l\in\partial M_j\backslash\Gamma}(\overrightarrow{q}^n_l.\overrightarrow{n}_{jl})|l| = \sum_{l\in\partial M_j\backslash\Gamma}\left[(-\overrightarrow{q}^n_l.\overrightarrow{n}_{jl})^+|l|-(-\overrightarrow{q}^n_l.\overrightarrow{n}_{jl})^-|l|\right]=0.$$

This leads to

$$\sum_{l\in\partial M_j}\left[c^n_l(-\overrightarrow{q}^n_l.\overrightarrow{n}_{jl})^+|l|-c^n_j(-\overrightarrow{q}^n_l.\overrightarrow{n}_{jl})^-|l|\right]=\sum_{l\in\partial M_j}(c^n_l-c^n_j)(-\overrightarrow{q}^n_l.\overrightarrow{n}_{jl})^+|l|. \tag{4.8}$$

Finally, the scheme (4.7) becomes

$$\begin{aligned}c^{n+1}_j-\frac{\Delta t^n}{\Phi_j|M_j|}\sum_{l\in\partial M_j\backslash\Gamma}(c^{n+1}_l-c^{n+1}_j)\frac{D^n_{jl}}{\delta_{jl}}|l| &=\\ c^n_j+\frac{\Delta t^n}{\Phi_j|M_j|}\sum_{l\in\partial M_j}(c^n_l-c^n_j)(-\overrightarrow{q}^n_l.\overrightarrow{n}_{jl})^+|l|,\end{aligned}\tag{4.9}$$

with $c_l^n = 1$ if $l \in \partial M_l \cap \Gamma_3$.
The scheme (4.9) can be written in the following matrix form:

$$A^n C^{n+1} = \mathbb{S}^n, \tag{4.10}$$

with $\mathbf{C}^{n+1} = (c_j^{n+1})_{1 \le j \le N_v}$, $\mathbb{S}^n = (S_j^n)_{1 \le j \le N_v}$ and A^n is a band matrix with:

$$A_{jj}^n = 1 + \frac{\Delta t^n}{|\Phi_j|M_j|} \sum_{l \in \partial M_j \setminus \Gamma} \frac{D_{jl}^n}{\delta_{jl}} |l|,$$

$$A_{jl}^n = -\frac{\Delta t^n}{\Phi_j|M_j|} \sum_{l \in \partial M_j \cap \partial M_l} \frac{D_{jl}^n}{\delta_{jl}} |l| \;\; for\; j \neq l$$

and

$$S_j^n = c_j^n \Big(1 - \frac{\Delta t^n}{\Phi_j|M_j|} \sum_{l \in \partial M_j} (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|\Big) + \frac{\Delta t^n}{\Phi_j|M_j|} \sum_{l \in \partial M_j} c_l^n (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|.$$

Property 4.3. *Under the assumptions (H_1)-(H_{10}), the coefficients D_{jl}^n satisfying the following inequality:*

$$0 < D^- \le D_{jl}^n \le D^+ < \infty.$$

Proof. (Cf. [3] Property 4.4 and Property 4.5). ■

This property is important for the analysis of the finite volume scheme and depends only on the triangulation of the domain and the matrix D.

Remark 4.4. *Under the assumptions (H_1)-(H_{10}) and the Property* 4.1 *A^n is a monotone matrix (i.e. $A_{jj}^n > 0$, $A_{jl}^n \le 0 \quad \forall j \neq l$ and $(A^n)^{-1}$ is a positive matrix).*

5 L^∞ stability and BV estimates

In this section we state the properties and the estimations which are satisfied by the finite volume scheme.

5.1 L^∞ Stability

Let the CFL condition be defined as:

$$CFL = \frac{\Delta t}{h} C_q \le 1, \;\; where\; C_q = \max_{n,j} \sum_{l \in \partial M_j} \frac{h|l|}{\Phi_j|M_j|} (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+. \tag{5.1}$$

Definition 5.1. *The approximate solution (c_j^n) is L^∞ stable on Ω, if:*

$$\|c_j^n\|_\infty = \sup_j |c_j^n| \le C, \qquad for \quad n = 1, ..., N,$$

C is a constant independent of h and Δt.

Proposition 5.2. *Under the assumptions* (H_1)-(H_{10}) *and CFL condition, the approximate solution* (c_j^n) *defined by (4.9) satisfies a discrete maximum principle:*

$$0 \leq c_j^n \leq 1, \quad \forall 0 \leq n \leq N \textit{ and } j \in I. \tag{5.2}$$

Furthermore, there is a function c^* *in* $L^\infty(Q)$ *with* $0 \leq c^*(x,t) \leq 1$ *in* Q*, such that*

$$c_h \rightharpoonup c^* \ \ in \ \ L^\infty(Q) \ \ weak^*,$$

where $c_h(x,t)$ *is defined by:*

$$c_h(x,t) = c_j^n \ \ for \ \ x \in M_j \ \ and \ \ t \in [t_n, t_{n+1}[.$$

Proof. We prove the discrete maximum principle (5.2) by induction. From the assumption (H_6), we have $0 \leq c_j^0 \leq 1$. Let us suppose that $0 \leq c_j^n \leq 1$ $\forall n$ and $j \in I$.
It follows from (4.9) that $\mathbb{C}^{n+1} = (A^n)^{-1}\mathbb{S}^n$. Using the CFL condition (5.1), we get for $M_j \in \Sigma_h$ and $0 \leq n \leq N$:

$$(1 - \frac{\Delta t^n}{\Phi_j|M_j|} \sum_{l \in \partial M_j \setminus \Gamma} (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|) \geq 0,$$

thus $S_j^n \geq 0, \forall j \in I$ and since $(A^n)^{-1}$ is a positive matrix, we conclude that $c_j^{n+1} \geq 0$, $\forall j \in I$.
On the other hand the scheme (4.9) can be rewritten as:

$$\begin{aligned} c_j^{n+1}(1 + \frac{\Delta t^n}{\Phi_j|M_j|} \sum_{l \in \partial M_j \setminus \Gamma} \frac{D_{jl}^n}{\delta_{jl}}|l|) - \frac{\Delta t^n}{\Phi_j|M_j|} \sum_{l \in \partial M_j \setminus \Gamma} c_l^{n+1} \frac{D_{jl}^n}{\delta_{jl}}|l| \\ = c_j^n(1 + \frac{\Delta t^n}{\Phi_j|M_j|} \sum_{l \in \partial M_j} (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|) + \frac{\Delta t^n}{\Phi_j|M_j|} \sum_{l \in \partial M_j \setminus \Gamma} c_l^n (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|. \end{aligned}$$

Therefore we have

$$\begin{aligned} (1 - c_j^{n+1})(1 + \frac{\Delta t^n}{\Phi_j|M_j|} \sum_{l \in \partial M_j \setminus \Gamma} \frac{D_{jl}^n}{\delta_{jl}}|l|) - \frac{\Delta t^n}{\Phi_j|M_j|} \sum_{l \in \partial M_j \setminus \Gamma} (1 - c_l^{n+1}) \frac{D_{jl}^n}{\delta_{jl}}|l| \\ (1 - c_j^n)(1 + \frac{\Delta t^n}{\Phi_j|M_j|} \sum_{l \in \partial M_j} (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|) + \\ \frac{\Delta t^n}{\Phi_j|M_j|} \sum_{l \in \partial M_j \setminus \Gamma} (1 - c_l^n)(-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|. \end{aligned}$$

This relation can be written in the following matrix form:

$$A^n(\mathbb{I} - \mathbf{C}^n) = \mathbb{S}^{*,n},$$

where $\mathbb{I} = (1, ..., 1)^t$ *and* $\mathbb{S}^{*,n} = (S_j^{*,n})_{1\le j\le N_s}$,

$$S_j^{*,n} =(1-c_j^n)(1-\frac{\Delta t^n}{\Phi_j|M_j|}\sum_{l\in\partial M_j\backslash\Gamma}(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l|) + \Phi_j|M_j|\sum_{l\in\partial M_j\backslash\Gamma}(1-c_l^n)(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l|).$$

Then $S_j^{*,n} \ge 0, \forall j \in I$ and since $(A^n)^{-1}$ is a positive matrix, we deduce that $c_j^{n+1} \le 1, \forall \quad n, j$. This concludes the proof of (5.2).
Moreover using (5.2), we have $0 \le c_h = c_j^n \le 1$ then $(c_h)_{h>0}$ is a bounded sequence, thus we can extract a subsequence which will still be denoted by c_h, such that c_h converges to c^* for the weak star topology of $L^\infty(Q)$. This completes the proof of Proposition (5.2). ∎

We will now give some estimations which are used to obtain the strong convergence of the approximate solution c_h in $L^2(Q)$.

5.2 BV weak estimates

Lemma 5.3. *Under the assumptions* (H_1)*-*(H_{10}) *and the CFL condition with (CFL* $\le 1-\varepsilon$*), we have the following estimates for the scheme (4.9):*

$$\sum_{n=0}^{N}\Delta t^n\sum_{l\in\partial M_j}(c_l^n-c_j^n)^2(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l| \le \frac{C_0}{\varepsilon}, \tag{5.3}$$

$$\sum_{n=0}^{N}\Delta t^n\sum_{l\in\partial M_j}(c_l^{n+1}-c_j^{n+1})^2 D_{jl}^n|l| \le C_1 h, \tag{5.4}$$

where ε *is a small parameter,* C_0 *and* C_1 *are constants independent of* h *and* Δt*.*

Proof. The equation (4.9) can be written in the following form:

$$c_j^{n+1} = w_j^n + \frac{\Delta t^n}{\Phi_j|M_j|}\sum_{l\in\partial M_j\backslash\Gamma}(c_l^{n+1}-c_j^{n+1})\frac{D_{jl}^n}{\delta_{jl}}|l|, \tag{5.5}$$

with

$$w_j^n = c_j^n + \frac{\Delta t^n}{\Phi_j|M_j|}\sum_{l\in\partial M_j}(c_l^n-c_j^n)(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l|. \tag{5.6}$$

Using the Cauchy-Schwartz inequality, we obtain

$$(c_j^n-w_j^n)^2 \le \left(\frac{\Delta t^n}{\Phi_j|M_j|}\right)\left(\frac{\Delta t^n}{\Phi_j|M_j|}\sum_{l\in\partial M_j}(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl}^+)|l|\right)\left(\sum_{l\in\partial M_j}(c_l^n-c_j^n)^2(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l|\right).$$

The CFL condition with ($CFL \leq 1-\varepsilon$) implies

$$(c_j^n - w_j^n)^2 \leq \frac{(1-\varepsilon)\Delta t^n}{\Phi_j |M_j|} \left(\sum_{l\in\partial M_j} (c_l^n - c_j^n)^2 (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| \right). \tag{5.7}$$

On the other hand, using the following equality:

$$\frac{1}{2}(a-b)^2 = \frac{1}{2}(b^2-a^2) + a(a-b) \quad \forall a,b \in \mathbb{R}$$

we obtain

$$\begin{aligned}
&\frac{\Delta t^n}{2} \sum_{j\in I} \sum_{l\in\partial M_j} (c_l^n - c_j^n)^2 (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| \\
&\quad = \frac{\Delta t^n}{2} \sum_{j\in I} \sum_{l\in\partial M_j} ((c_l^n)^2 (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| - (c_j^n)^2 (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|) \\
&\quad + \Delta t^n \sum_{l\in\partial M_j} c_j^n (c_l^n - c_j^n)(-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| \\
&\quad = \frac{\Delta t^n}{2} \sum_{j\in I} \sum_{l\in\partial M_j \setminus \Gamma} ((c_l^n)^2 (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| - (c_j^n)^2 (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|) \\
&\quad + \frac{\Delta t^n}{2} \sum_{j\in I} \| \overrightarrow{q} \|_\infty |\partial M_j \cap \Gamma_3| + \Delta t^n \sum_{l\in\partial M_j} c_j^n (c_l^n - c_j^n)(-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|.
\end{aligned} \tag{5.8}$$

where $\| \overrightarrow{q} \|_\infty = \| \overrightarrow{q} \|_{L^\infty(\Gamma_3 \times]0,T[)}$.
We have

$$\begin{aligned}
&\sum_{j\in I} \sum_{l\in\partial M_j \setminus \Gamma} [(c_l^n)^2 (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| - (c_j^n)^2 (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|] \\
&\qquad = \sum_{j\in I} (c_j^n)^2 [\sum_{l\in\partial M_j \setminus \Gamma} \overrightarrow{q}_l^n . \overrightarrow{n}_{jl} |l|] = 0.
\end{aligned}$$

Then the equation (5.8) becomes

$$\begin{aligned}
&\frac{\Delta t^n}{2} \sum_{j\in I} \sum_{l\in\partial M_j} (c_l^n - c_j^n)^2 (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| \\
&= \frac{\Delta t^n}{2} \| \overrightarrow{q} \|_{L^\infty(\Gamma_3 \times]0,T[)} \sum_{j\in I} |\partial M_j \cap \Gamma_3| + \Delta t^n \sum_{l\in\partial M_j} c_j^n (c_l^n - c_j^n)(-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|.
\end{aligned} \tag{5.9}$$

Using equation (5.6) to replace the last term of equation (5.9), we get

$$\begin{aligned}
&\frac{\Delta t^n}{2} \sum_{j\in I} \sum_{l\in\partial M_j} (c_l^n - c_j^n)^2 (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| \\
&= \frac{\Delta t^n}{2} \| \overrightarrow{q} \|_{L^\infty(\Gamma_3 \times]0,T[)} \sum_{j\in I} |\partial M_j \cap \Gamma_3| + \sum_{l\in\partial M_j} \Phi_j |M_j| c_j^n (c_j^n - w_j^n).
\end{aligned} \tag{5.10}$$

Moreover, we have

$$\sum_{j\in I}\frac{\Phi_j|M_j|}{2}((w_j^n)^2-(c_j^n)^2)=\sum_{j\in I}\Phi_j|M_j|(c_j^n(w_j^n-c_j^n))+\sum_{j\in I}\frac{\Phi_j|M_j|}{2}(c_j^n-w_j^n)^2.$$

Using (5.7) and (5.10) we get

$$\begin{aligned}
&\sum_{j\in I}\frac{\Phi_j|M_j|}{2}((w_j^n)^2-(c_j^n)^2)\\
&\quad\le\sum_{j\in I}\Phi_j|M_j|c_j^n(w_j^n-c_j^n)+\frac{(1-\varepsilon)\Delta t^n}{2}\sum_{l\in\partial M_j}(c_l^n-c_j^n)^2(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l|\\
&\quad\le\sum_{j\in I}\Phi_j|M_j|c_j^n(w_j^n-c_j^n)+\frac{\Delta t^n}{2}\|\overrightarrow{q}\|_{L^\infty(\Gamma_3\times]0,T[)}|\sum_{j\in I}|\partial M_j\cap\Gamma_3|\\
&\quad+\sum_{j\in I}\Phi_j|M_j|c_j^n(c_j^n-w_j^n)-\frac{\varepsilon\Delta t^n}{2}(\sum_{l\in\partial M_j}(c_l^n-c_j^n)^2(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l|),
\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}
&\frac{\varepsilon\Delta t^n}{2}(c_l^n-c_j^n)^2(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l|)\le\\
&\qquad\qquad\sum_{j\in I}\frac{\Phi_j|M_j|}{2}((c_j^n)^2-(w_j^n)^2)+\frac{\Delta t^n}{2}\|\overrightarrow{q}\|_{L^\infty(\Gamma_3\times]0,T[)}|\Gamma_3|.
\end{aligned}$$

Similarly, we have

$$c_j^{n+1}(c_j^{n+1}-w_j^n)=\sum_{l\in\partial M_j}\frac{\Delta t^n}{\Phi_j|M_j|}c_j^{n+1}(c_l^{n+1}-c_j^{n+1})\frac{D_{jl}^n}{\delta_{jl}}|l|,$$

This gives

$$\begin{aligned}
&\sum_{j\in I}\frac{\Phi_j|M_j|}{2}((c_j^{n+1})^2-(w_j^n)^2)\\
&\quad=\sum_{j\in I}\Phi_j|M_j|(c_j^{n+1}(w_j^n-c_j^{n+1}))-\sum_{j\in I}\frac{\Phi_j|M_j|}{2}\sum_{j\in I}(c_j^{n+1}-w_j^n)^2\\
&\quad\le\sum_{l\in\partial M_j}\frac{\Delta t^n}{\Phi_j|M_j|}c_j^{n+1}(c_l^{n+1}-c_j^{n+1})\frac{D_{jl}^n}{\delta_{jl}}|l|.
\end{aligned}$$

In fact that

$$\begin{aligned}
&\Delta t^n\sum_{l\in\xi_h}(c_l^{n+1}-c_j^{n+1})^2\frac{D_{jl}^n}{\delta_{jl}}|l|\\
&\quad=\Delta t^n\sum_{l\in\xi_h}(c_l^{n+1}-c_j^{n+1})c_l^{n+1}\frac{D_{jl}^n}{\delta_{jl}}|l|+\Delta t^n\sum_{l\in\xi_h}(c_j^{n+1}-c_l^{n+1})c_j^{n+1}\frac{D_{jl}^n}{\delta_{jl}}|l|\\
&\quad=\sum_{j\in I}\sum_{l\in\partial M_j}\frac{\Delta t^n}{\Phi_j|M_j|}c_j^{n+1}(c_j^{n+1}-c_l^{n+1})\frac{D_{jl}^n}{\delta_{jl}}|l|.
\end{aligned}$$

It follows that

$$\Delta t^n \sum_{l\in\xi_h}(c_l^{n+1}-c_j^{n+1})^2\frac{D_{jl}^n}{\delta_{jl}}|l| \le \sum_{j\in I}\frac{\Phi_j|M_j|}{2}((w_j^n)^2-(c_j^{n+1})^2).$$

Summing over n=0,....,N, we obtain:

$$\begin{aligned}&\sum_{n=0}^{N}\frac{\varepsilon\Delta t^n}{2}\sum_{l\in\partial M_j}(c_l^n-c_j^n)^2(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l| + \sum_{n=0}^{N}\Delta t^n\sum_{l\in\xi_h}(c_l^{n+1}-c_j^{n+1})^2\frac{D_{jl}^n}{\delta_{jl}}|l|\\ &\le \sum_{n=0}^{N}\sum_{j\in I}\Phi_j\frac{|M_j|}{2}((c_j^n)^2-(c_j^{n+1})^2)+\sum_{n=0}^{N}\frac{\Delta t^n}{2}\|\overrightarrow{q}\|_{L^\infty(\Gamma_3\times]0,T[)}|\Gamma_3|\\ &\le \frac{\Phi_j|\Omega|}{2}\|c^0\|^2_{L^\infty(\Omega)}+\frac{T}{2}\|\overrightarrow{q}\|_{L^\infty(\Gamma_3\times]0,T[)}|\Gamma_3|.\end{aligned}$$

Consequently, we have the following estimates:

$$\sum_{n=0}^{N}\frac{\varepsilon\Delta t^n}{2}\sum_{l\in\partial M_j}(c_l^n-c_j^n)^2(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l| + \sum_{n=0}^{N}\Delta t^n\sum_{l\in\xi_h}(c_l^{n+1}-c_j^{n+1})^2\frac{D_{jl}^n}{\delta_{jl}}|l| \le C_0 \tag{5.11}$$

and

$$\sum_{n=0}^{N}\sum_{l\in\partial M_j}\Delta t^n(c_l^n-c_j^n)^2(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l| \le \frac{C_0}{\varepsilon}$$

Where

$$C_0=\frac{\Phi_j|\Omega|}{2}\|c^0\|^2_{L^\infty(\Omega)}+\frac{T}{2}\|\overrightarrow{q}\|_{L^\infty(\Gamma_3\times]0,T[)}|\Gamma_3|.$$

Finally, using the inequality $\delta_{jl}\le H\le\frac{\beta}{\gamma}h$, we conclude that

$$\sum_{n=0}^{N}\sum_{l\in\partial M_j}\Delta t^n(c_l^{n+1}-c_j^{n+1})^2D_{jl}^n|l|\le C_1h.$$

Where

$$C_1=(Tq_0|\Gamma_3|+\Phi|\Omega|\|c^0\|^2_{L^\infty(\Omega)})\frac{\beta}{2\gamma}.$$

This completes the proof of Lemma (5.3). ∎

Lemma 5.4. *Under the assumptions* (H_1)-(H_{10}) *and the CFL condition, we have the following estimates:*

$$\sum_{n=0}^{N}\sum_{l\in\xi_h}\Delta t^n|c_l^n-c_j^n|(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+\le C_2h^{\frac{-1}{2}}, \tag{5.12}$$

and

$$\sum_{n=0}^{N}\sum_{l\in\xi_h}\Delta t^n|c_l^n-c_j^n|D_{jl}^n|l|\le C_3 \tag{5.13}$$

where C_2 and C_3 are constants independent of h and Δt.

Proof. Using the Cauchy-Schwartz inequality, we obtain:

$$\sum_{n=0}^{N}\sum_{l\in\xi_h}\Delta t^n|c_l^n-c_j^n|(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l| \leq (\sum_{n=0}^{N}\sum_{l\in\xi_h}\Delta t^n(c_l^n-c_j^n)^2(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l|)^{1/2} \times (\sum_{n=0}^{N}\sum_{l\in\xi_h}\Delta t^n(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l|)^{1/2}.$$

The estimate (5.3) of Lemma (5.3) leads to:

$$\begin{aligned}\sum_{n=0}^{N}\sum_{l\in\xi_h}\Delta t^n(|c_l^n-c_j^n|(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l| &\leq \sqrt{C_0}(\|\overrightarrow{q}\|_{(L^\infty(Q))^d}T)^{\frac{1}{2}}\left(\frac{1}{2}\sum_{j\in I}|\partial M_j|\right)^{\frac{1}{2}}\\ &\leq \sqrt{C_0}(\|\overrightarrow{q}\|_{(L^\infty(Q))^d}T)^{\frac{1}{2}}\left(\frac{1}{2h}\sum_{j\in I}|M_j|\right)^{\frac{1}{2}}\\ &\leq \sqrt{C_0}(\|\overrightarrow{q}\|_{(L^\infty(Q))^d}T)^{\frac{1}{2}}\left(\frac{|\Omega|}{2}\right)^{\frac{1}{2}}h^{\frac{-1}{2}}.\end{aligned}$$

Hence

$$\sum_{n=0}^{N}\sum_{l\in\partial M_j}\Delta t^n(|c_l^n-c_j^n|(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l| \leq C_2h^{\frac{-1}{2}}.$$

Similarly, we get

$$\sum_{n=0}^{N}\sum_{l\in\xi_h}\Delta t^n|c_l^n-c_j^n|D_{jl}^n|l| \leq (\sum_{n=0}^{N}\sum_{l\in\xi_h}\Delta t^n(c_l^n-c_j^n)^2D_{jl}^n|l|)^{\frac{1}{2}}(\sum_{n=0}^{N}\sum_{l\in\xi_h}\Delta t^nD_{jl}^n|l|)^{\frac{1}{2}}.$$

The estimate (5.4) of Lemma (5.3) and the Property (4.3) imply

$$\begin{aligned}\sum_{n=0}^{N}\sum_{l\in\partial M_j}\Delta t^n|c_l^n-c_j^n|D_{jl}^n|l| &\leq \left(C_0\frac{\beta}{\gamma}h\right)^{\frac{1}{2}}(D^+T)^{\frac{1}{2}}\left(\frac{1}{2}\sum_{j\in I}|\partial M_j|\right)^{\frac{1}{2}}\\ &\leq \left(C_0\frac{\beta}{2\gamma}h\right)^{\frac{1}{2}}(D^+T)^{\frac{1}{2}}\left(\frac{1}{2h}\sum_{j\in I}|M_j|\right)^{\frac{1}{2}}\\ &\leq \left(C_0\frac{\beta}{2\gamma}\right)^{\frac{1}{2}}(D^+T)^{\frac{1}{2}}\left(\frac{1}{2}|\Omega|\right)^{\frac{1}{2}}.\end{aligned}$$

Finally, we get

$$\sum_{n=0}^{N}\sum_{l\in\partial M_j}\Delta t^n|c_l^n-c_j^n|D_{jl}^n|l| \leq C_3.$$

This completes the proof of Lemma (5.4). ∎

Theorem 5.5. *Under the assumptions* (H_1)-(H_{10}) *and the CFL condition, we have the following BV estimates:*

$$\sum_{n=0}^{N} \Delta t^n \sum_{j \in I} |M_j| (c_j^{n+1} - c_j^n)^2 \leq C_4 \Delta t. \tag{5.14}$$

and

$$\sum_{n=0}^{N} \Delta t^n \sum_{l \in \partial M_j} |l| (c_l^n - c_j^n)^2 \leq C_5 h \tag{5.15}$$

where the constants C_4 *and* C_5 *are independent of* h *and* Δt.

Proof. Multiplying (4.9) with $(c_j^{n+1} - c_j^n)$, we obtain:

$$\begin{aligned}
\sum_{j \in I} \Phi_j |M_j| (c_j^{n+1} - c_j^n)^2 &= \Delta t^n \sum_{j \in I} \sum_{l \in \partial M_j} (c_j^{n+1} - c_j^n)(c_l^n - c_j^n)(-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| \\
&+ \Delta t^n \sum_{j \in I} \sum_{l \in \partial M_j} (c_j^{n+1} - c_j^n)(c_l^{n+1} - c_j^{n+1}) \frac{D_{jl}^n}{\delta_{jl}} |l| \\
&= A^{n+1} - A^n + B^{n+1} - B^n,
\end{aligned}$$

where

$$\begin{aligned}
A^s &= \Delta t^n \sum_{j \in I} \sum_{l \in \partial M_j} c_j^s (c_l^n - c_j^n)(-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|, \quad s = n \text{ or } n+1, \\
B^s &= \Delta t^n \sum_{j \in I} \sum_{l \in \partial M_j \setminus \Gamma} c_j^s (c_l^{n+1} - c_j^{n+1}) \frac{D_{jl}^n}{\delta_{jl}} |l|, \qquad s = n \text{ or } n+1.
\end{aligned}$$

We have then

$$\begin{aligned}
\sum_{n=0}^{N} |A^s| &= \sum_n \left| \Delta t^n \sum_{j \in I} \sum_{l \in \partial M_j \cap \Gamma_3} c_j^s (c_l^n - c_j^n)(-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| \right| \\
&+ \sum_{n=0}^{N} \left| \Delta t^n \sum_{l \in \xi_h} c_j^s (c_l^n - c_j^n)(-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| \right| \\
&+ \sum_{n=0}^{N} \left| \Delta t^n \sum_{l \in \xi_h} c_l^s (c_j^n - c_l^n)(-\overrightarrow{q}_l^n . \overrightarrow{n}_{lj})^+ |l| \right| \\
&\leq T \| \overrightarrow{q} \|_\infty |\Gamma_3| + \sum_{n=0}^{N} \sum_{l \in \xi_h} \frac{\Delta t^n}{2} ((c_l^n - c_j^n)^2 + (c_j^s)^2)(-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| \\
&+ \sum_{n=0}^{N} \sum_{l \in \xi_h} \frac{\Delta t^n}{2} ((c_l^n - c_j^n)^2 + (c_j^s)^2)(-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l|
\end{aligned}$$

where

$$\| \overrightarrow{q} \|_\infty = \| \overrightarrow{q} \|_{L^\infty(\Gamma_3 \times]0,T[)}.$$

From the estimate (5.3) of Lemma (5.3) and the CFL condition, we deduce that

$$\sum_{n=0}^{N}|A^s| \leq T|\overrightarrow{q}\|_\infty|\Gamma_3| + \frac{C_0}{\varepsilon} + \|\overrightarrow{q}\|_\infty(N+1)\left(\frac{|\Omega|}{2C_q}\right).$$

In the same way, we have

$$\begin{aligned}\sum_{n=0}^{N}|B^s| &= \sum_{n=0}^{N}\left|\Delta t^n \sum_{l\in\xi_h}(c_j^s - c_l^s)(c_l^{n+1} - c_j^{n+1})\frac{D_{jl}^n}{\delta_{jl}}|l|\right| \\ &\leq \sum_{n=0}^{N}\sum_{l\in\xi_h}\frac{\Delta t^n}{2}((c_l^{n+1} - c_j^{n+1})^2 + (c_j^s - c_j^s)^2)\frac{D_{jl}^n}{\delta_{jl}}|l|.\end{aligned}$$

The estimate (5.11) yields

$$\sum_{n=0}^{N}|B^s| \leq C_0.$$

Hence

$$\begin{aligned}&\sum_{n=0}^{N}\sum_{j\in I}\Delta t^n\Phi_j|M_j|(c_j^{n+1} - c_j^n)^2 \\ &\qquad\leq \sum_{n=0}^{N}|A^s| + \sum_{n=0}^{N}|B^s| \\ &\qquad\leq \|\overrightarrow{q}\|_\infty + \frac{C_0}{\varepsilon} + (\|\overrightarrow{q}\|_{(L^\infty(Q))^d})(N+1)\left(\frac{|\Omega|}{2C_q}\right) + C_0.\end{aligned}$$

Using the assumption (H_2) we find

$$\sum_{n=0}^{N}\sum_{j\in I}\Delta t^n|M_j|(c_j^{n+1} - c_j^n)^2 \leq C_4\Delta t,$$

where

$$C_4 = \frac{\left[T\|\overrightarrow{q}\|_\infty + \frac{C_0}{\varepsilon} + (\|\overrightarrow{q}\|_{(L^\infty(Q))^d})(N+1)\left(\frac{|\Omega|}{2C_q}\right) + C_0\right]}{\Phi_-}.$$

From the estimate (5.4) of Lemma (5.3) and the Property (4.3) we conclude that

$$\sum_{n=0}^{N}\sum_{l\in\xi_h}\Delta t^n|c_l^n - c_j^n|D^-|l| \leq C_1 h,$$

This gives

$$\sum_{n=0}^{N}\sum_{l\in\xi_h}\Delta t^n|c_l^n - c_j^n||l| \leq C_5 h,$$

Where

$$C_5 = \frac{C_1}{D^-}.$$

This completes the proof of Theorem (5.5). ■

6 Convergence results

In this section we prove the convergence of the solution of the combined scheme (MFE)-(FV) to the coupled problem using the discrete maximum principle and the estimates obtained in the preceding section.

6.1 Convergence of the mixed finite element scheme

We give a convergence result for the mixed finite element scheme.

Theorem 6.1. *Let $(\overrightarrow{q}, p)$ be the solution of the problem (4.1) and $(\overrightarrow{q}_h, p_h) \in X_0^h \times M^h$ be the solution of the problem (4.2). If $\overrightarrow{q} \in (H^1(\Omega))^2$, $p \in H^1(\Omega)$ and $div\overrightarrow{q} \in H^1(\Omega)$ for any fixed time $t \in J$, then there exists constant C independent of h and Δt such that:*

$$\|\overrightarrow{q} - \overrightarrow{q}_h\|_{H(div,\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq Ch(\|\overrightarrow{q}\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)}).$$

Furthermore, the mixed finite element is convergent:

$$\overrightarrow{q}_h \longrightarrow \overrightarrow{q} \ \ strongly \ \ in \ H(div, \Omega)$$

and

$$p_h \longrightarrow p \ \ strongly \ in \ L^2(\Omega).$$

Proof. (Cf. [41]). ∎

This theorem gives also the first order convergence for the pressure and the velocity.

6.2 Convergence of the FV scheme of the transport equation

In this section we prove strong convergence of the approximate solution c_h to c^* in $L^2(Q)$, using L^∞ stability (Proposition (5.2)), BV estimates (Theorem 5.5) and the Kolmogorov relative compactness theorem.

Theorem 6.2. *Under the assumptions (H_1)-(H_{10}) and the CFL condition the approximate solution c_h given by the scheme (4.9) converge strongly to c^* in $L^2(Q)$ as h and Δt go 0.*

Proof. It follows from the proposition (5.2) and BV estimates (theorem 5.5) that c_h verifies the assumptions of the Kolmogorov theorem (see [12]) then c_h is relatively compact in $L^2(Q)$ (see for more details [19] and [21]). This implies the existence of a subsequence again denoted c_h such that

$$c_h \longrightarrow c^* \ strongly \ \ in \ L^2(Q).$$

This completes the proof of Theorem (6.2). ∎

6.3 Convergence of the combined scheme

Now we prove the convergence of the approximate solutions to a weak solution of the coupled system (2.1)-(2.2).

Theorem 6.3. *Under the assumptions* (H_1)-(H_{10}), *the approximate solution of the combined scheme (MFE)-(FV)* $(p_h, \overrightarrow{q}_h, c_h)$ *converges to* $(p, \overrightarrow{q}, c)$ *the solution of the coupled problem as* h *and* Δt *go* 0.

Proof. Let $\varphi \in C^1(\overline{\Omega} \times [0,T])$ with compact support contained in $\overline{\Omega} \times [0,T]$. Multiplying equation (4.9) by $\varphi(x_j, t_n) \in V$ and summing over n and j, we obtain

$$\begin{aligned}
&\sum_{n=0}^{N} \sum_{j\in I} \Phi_j |M_j| (c_j^{n+1} - c_j^n) \varphi(x_j, t_n) \\
&\qquad = \sum_{n=0}^{N} \sum_{j\in I} \Delta t^n \sum_{l\in\partial M_j} (c_l^n - c_j^n) \varphi(x_j, t_n) (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| \\
&\qquad + \sum_{n=0}^{N} \sum_{j\in I} \Delta t^n \sum_{l\in\partial M_j} (c_l^{n+1} - c_j^{n+1}) \varphi(x_j, t_n) \frac{D_{jl}^n}{\delta_{jl}} |l|.
\end{aligned} \tag{6.1}$$

We have

$$\begin{aligned}
&\sum_{n=0}^{N} \sum_{j\in I} \Phi_j |M_j| (c_j^{n+1} - c_j^n) \varphi(x_j, t_n) \\
&\quad = - \sum_{n=1}^{N} \sum_{j\in I} \Phi_j |M_j| c_j^n \Delta t^n \frac{\varphi(x_j, t_n) - \varphi(x_j, t_{n-1})}{\Delta t^n} - \sum_{j\in I} \Phi_j c_j^0 \varphi(x_j, 0).
\end{aligned}$$

Hence, as h and $\Delta t \to 0$ we find

$$\begin{aligned}
\sum_{j\in I} \Phi_j |M_j| (c_j^{n+1} - c_j^n) \varphi(x_j, t_n) \to & - \int_0^T \int_\Omega \Phi_j c(x,t) \varphi(x,t) dxdt \\
& - \int_\Omega \Phi c(x,0) \varphi(x,0) dxdt.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{n=0}^{N} \sum_{j\in I} \Delta t^n \sum_{l\in\partial M_j} (c_l^n - c_j^n) \varphi(x_j, t_n) (-\overrightarrow{q}_l^n . \overrightarrow{n}_{jl})^+ |l| \\
&\qquad = - \sum_{n=0}^{N} \sum_{j\in I} \Delta t^n \sum_{l\in\partial M_j} c_{jl}^n \overrightarrow{q}_l^n . \overrightarrow{n}_{jl} \varphi(x_j, t_n) |l|,
\end{aligned}$$

with $c_{jl}^n = c_j^n$ if $\overrightarrow{q}_l^n . \overrightarrow{n}_{jl} \geq 0$ and $c_{jl}^n = c_l^n$ otherwise.
Since

$$\sum_{l\in\partial M_j \setminus \Gamma} \varphi(x_j, t_n) c_j^n \overrightarrow{q}_l^n . \overrightarrow{n}_{jl} |l| = 0.$$

Therefore, we get

$$\sum_{n=0}^{N}\sum_{j\in I}\Delta t^n \sum_{l\in\partial M_j} c_{jl}^n \overrightarrow{q}_l^n.\overrightarrow{n}_{jl}\varphi(x_j,t_n)|l|$$

$$= \sum_{n=0}^{N}\sum_{j\in I}\Delta t^n \sum_{l\in\partial M_j\cap\Gamma_3} \varphi(x_l,t_n)c_l^n(\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})|l|$$

$$+ \sum_{n=0}^{N}\Delta t^n \sum_{l\in\xi_h} \varphi(x_j,t_n)(c_{jl}^n - c_j^n)\overrightarrow{q}_l^n.\overrightarrow{n}_{jl}|l|$$

$$= \sum_{n=0}^{N}\sum_{j\in I}\Delta t^n \sum_{l\in\partial M_j\cap\Gamma_3} \varphi(x_l,t_n)c_l^n(\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})|l|$$

$$+ \sum_{n=0}^{N}\sum_{l}\Delta t^n(c_{jl}^n - c_j^n)(\varphi(x_j,t_n)-\varphi(x_l,t_n))\overrightarrow{q}_l^n.\overrightarrow{n}_{jl}|l|$$

$$- \sum_{n=0}^{N}\sum_{j\in I}\Delta t^n \sum_{l\in\partial M_j} \varphi(x_l,t_n)c_j^n\overrightarrow{q}_l^n.\overrightarrow{n}_{jl}|l|.$$

The estimate (5.12) of lemma (5.4) yields

$$|\sum_{n=0}^{N}\sum_{l\in\partial M_j}\Delta t^n(c_{jl}^n - c_j^n)\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})||l| \leq Ch^{-1/2}.$$

Furthermore, we have

$$|\varphi(x_j,t_n)-\varphi(x_l,t_n)| \leq M\|x_j - x_l\| \leq M\delta_{jl} \leq M\frac{\beta}{\gamma}h,$$

where $M = \|\nabla\varphi\|_\infty$. This implies, as h and Δt go to 0

$$\sum_{n=0}^{N}\sum_{l\in\partial M_j}\Delta t^n|(\varphi(x_j,t_n)-\varphi(x_l,t_n))(c_{jl}^n - c_j^n)|\overrightarrow{q}_l^n.\overrightarrow{n}_{jl}||l| \leq Ch^{1/2} \to 0.$$

Consequently, we obtain

$$\sum_{n=0}^{N}\sum_{j\in I}\Delta t^n \sum_{l\in\partial M_j} \varphi(x_l,t_n)c_j^n\overrightarrow{q}_l^n.\overrightarrow{n}_{jl}|l| = \sum_{n=0}^{N}\sum_{j\in I}\Delta t^n c_j^n \sum_{l\in\partial M_j}\int_l \varphi(x,t_n)\overrightarrow{q}_l^n.\overrightarrow{n}_{jl}ds$$

$$= \sum_{n=0}^{N}\sum_{j\in I}\Delta t^n \int_{M_j} c_j^n div(\overrightarrow{q}_l^n\varphi(x,t_n))dx$$

$$= \sum_{n=0}^{N}\sum_{j\in I}\Delta t^n \int_{M_j} c_j^n \overrightarrow{q}^n.\nabla\varphi(x,t_n)dx.$$

Hence, as h and Δt go to 0, we have

$$\sum_{n=0}^{N}\sum_{j\in I}\Delta t^n \sum_{l\in\partial M_j}(c_l^n - c_j^n)\varphi(x_j,t^n)(-\overrightarrow{q}_l^n.\overrightarrow{n}_{jl})^+|l|$$

$$\to \int_0^T\int_{\Gamma_3}(\overrightarrow{q}.\overrightarrow{n})\varphi ds dt - \int_Q c^*(x,t)\overrightarrow{q}.\nabla\varphi(x,t)dxdt.$$

Now the last term of (6.1) is rearranged in the following form:

$$
\begin{aligned}
-\sum_{n=0}^{N}\sum_{j\in I}&\Delta t^n c_j^{n+1}\sum_{l\in\partial M_j\backslash\Gamma}(\varphi(x_l,t_n)-\varphi(x_j,t_n))\frac{D_{jl}^n}{\delta_{jl}}|l| \\
&= -\sum_{n=0}^{N}\sum_{j\in I}\Delta t^n c_j^{n+1}\sum_{M_j\cap K\neq 0}\sum_{l\in\partial M_j\cap K\backslash\Gamma}(D_K^n\nabla\varphi_K^n.\overrightarrow{n}_{jl}|l|) \\
&= -\sum_{n=0}^{N}\sum_{j\in I}\Delta t^n c_j^{n+1}\sum_{l\in\partial M_j\backslash\Gamma}D_l^n\nabla\varphi_l^n.\overrightarrow{n}_{jl}|l|. \\
&= \sum_{n=0}^{N}\sum_{j\in I}\Delta t^n c_j^{n+1}\sum_{l\in\partial M_j\cap\Gamma}(D_l^n\nabla\varphi_l^n.\overrightarrow{n}_{jl}|l|) \\
&\quad -\sum_{n=0}^{N}\sum_{j\in I}\Delta t^n c_j^{n+1}\sum_{l\in\partial M_j}(D_l^n\nabla\varphi_l^n.\overrightarrow{n}_{jl}|l|)
\end{aligned}
$$

As h and Δt go to 0, it follows that

$$
\begin{aligned}
\sum_{n=0}^{N}\sum_{j\in I}&\Delta t^n c_j^{n+1}\sum_{l\in\partial M_j\cap\Gamma}(D_l^n\nabla\varphi_l^n.\overrightarrow{n}_{jl}|l|)-\sum_{n=0}^{N}\sum_{j\in I}\Delta t^n c_j^{n+1}\sum_{l\in\partial M_j}(D_l^n\nabla\varphi_l^n.\overrightarrow{n}_{jl}|l|) \\
&\to \int_0^T\int_\Gamma c^*(\eta,t)D\nabla\varphi.\overrightarrow{n}\,dsdt-\int_Q c^*(x,t)div(D\nabla\varphi(x,t))dxdt \\
&= \int_Q D\nabla c^*(x,t).\nabla\varphi dxdt.
\end{aligned}
$$

Finally, passing to the limit in (6.1), we obtain

$$
\begin{aligned}
\int_0^T\int_\Omega[\Phi c^*\frac{\partial\varphi}{\partial t}+(c^*\overrightarrow{q^*}-D\nabla c^*).\nabla\varphi]dxdt+\int_\Omega\Phi c(x,0)\varphi(x,0)dxdt \\
-\int_0^T\int_{\Gamma_3}(\overrightarrow{q}.\overrightarrow{n})\varphi(x,t)dsdt=0,
\end{aligned}
$$

then $(\overrightarrow{q},p,c^*)$ is a weak solution of the variational problem (2.3)-(2.4) which admits a unique solution ($\overrightarrow{q}$,p,c). Hence c_h converge to c, $\overrightarrow{q}_h$ converge to $\overrightarrow{q}$ and p_h converge to p as h and Δt go to 0. This completes the proof of Theorem (6.3). ∎

7 Numerical simulations

In this section, we present some numerical results in 2D based on the combined (MFE)-(FV) method presented in this paper.

In the test presented, we consider a homogenous isotropic medium $\Omega=]0,1[\times]0,1[$. The contaminant are situated at the top of medium Ω, i.e. on the boundary $\Gamma_3=]0,1[\times\{1\}$ and as boundary conditions we impose a pressure $p_0=0.2(1-cos(0.5\times\pi x_2))$ on the right $\Gamma_2=\{1\}\times]0,1[$, a constant pressure $p_1=0.2$ on Γ_3 and no-flow boundary at the bottom $\Gamma_1=]0,1[\times\{0\}$ and on the left $\Gamma_4=$

$\{0\}\times]0,1[$. Furthermore the following numerical values are chosen: $\Phi = 0.2$, $\mathbf{K} = 1m/s$, $d_e = 0.05$, $\alpha_l = 0$, $\alpha_t = 0$ and $c^0 = 0$.
The contours pressure and the velocity are illustrated in Figure 2 and Figure 3 for to get an idea of fluid flow through a homogenous isotropic medium. The concentration contours are presented in Figure 4. This Figure shows that the pollutant moves from the top to the bottom.

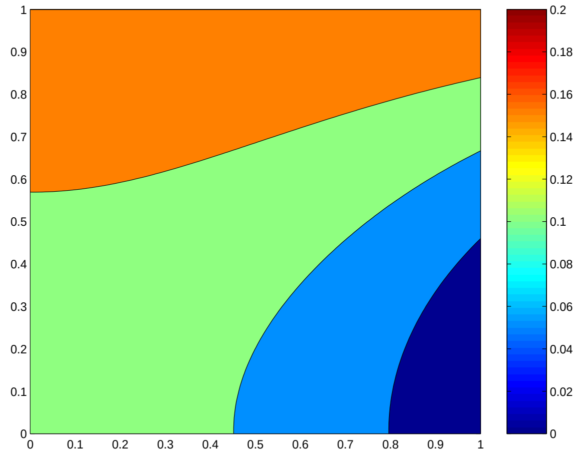

Figure 2: Contours pressure.

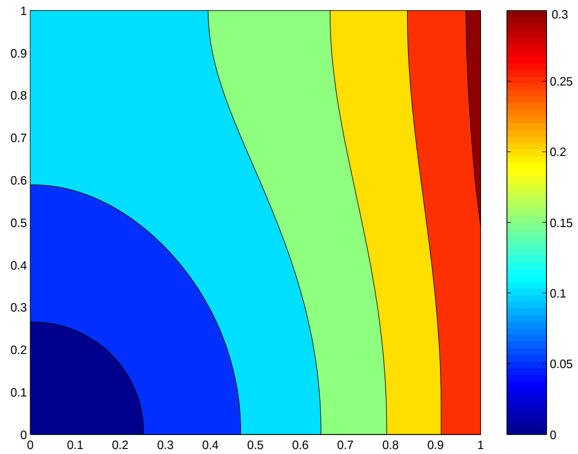

Figure 3: Velocity

8 Conclusion

In this paper, we study a numerical scheme combining a mixed finite element method (MFE) and finite volume scheme (FV) for the discretization of a system includes an elliptic pressure-velocity equation coupled to a linear convection-diffusion equation.
Numerical simulations in a homogenous isotropic medium were presented.

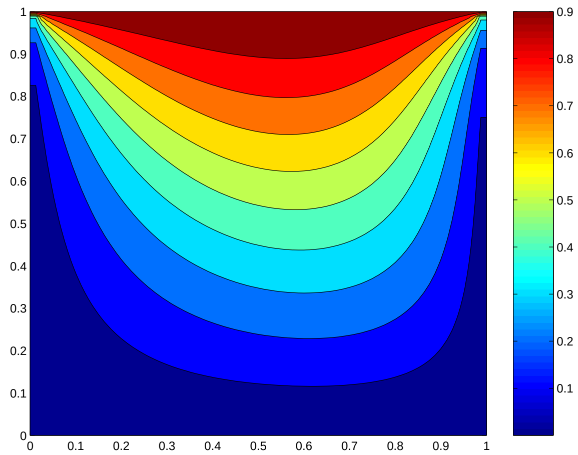

Figure 4: Concentration contours.

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