

A constructive fixed point approach to the existence of a triangle with prescribed angle bisector lengths

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Dedicated to the memory of Professor Laurentiu Panaitopol

Abstract

We show that the use of Brouwer fixed point theorem in Mironescu-Panaitopol's approach to the existence of a triangle with prescribed interior bisector lengths can be replaced by that of Banach fixed point theorem followed by an elementary limiting argument.

1 Introduction

Given a triangle ABC , let $a = |BC|$, $b = |AC|$, $c = |AB|$ (where say $|BC|$ denotes the length of the segment joining B to C), let h_A, h_B, h_C denote the lengths of the altitudes of the triangle, m_A, m_B, m_C the lengths of the medians of the triangle, and b_A, b_B, b_C the lengths of the bisectors of the triangle respectively traced from A, B, C . The *three altitudes problem* consists in, given three positive numbers m, n and p , finding a triangle with altitudes $h_a = m, h_b = n, h_c = p$, and if possible, constructing it with the ruler and the compass. In the *three medians problem*, the altitudes are replaced by the medians, and in the *three bisectors problem* the altitudes are replaced by the bisectors.

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Elementary geometric considerations show that, in any triangle, the lengths of the altitudes satisfy the relations $ah_a = bh_b = ch_c$, so that, by the triangle inequality, the requested triangle can exist only if conditions

$$\frac{1}{m} < \frac{1}{n} + \frac{1}{p}, \quad \frac{1}{n} < \frac{1}{p} + \frac{1}{m}, \quad \frac{1}{p} < \frac{1}{m} + \frac{1}{n}$$

hold. In the case of medians, it is easy to show that the medians of a triangle are always the sides of some triangle, so that a necessary conditions for the solvability of the problem of the three medians is that inequalities

$$m < n + p, \quad n < p + m, \quad p < m + n$$

hold. Simple geometric considerations and constructions show that under the respective necessary conditions above, the three altitudes problem and the three medians problem have a solution, which can be constructed with the ruler and the compass, so that the two problems usually take place in elementary Euclidian geometry textbooks.

For the corresponding *three bisectors problem*, there are two cases, according to one consider the *internal* or the *external* bisectors (see Fig. 1 and 2). In the first case, the bisectors are concurrent, in the second one they form a triangle.

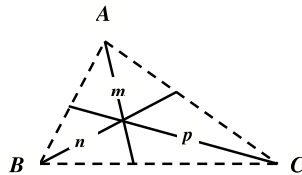


Figure 1

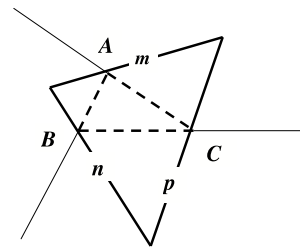


Figure 2

In 1842, Terquem [15] computed the length of the internal bisectors in terms of the side of the triangle, namely

$$\begin{aligned} b_A &= \frac{1}{b+c} \sqrt{bc[(b+c)^2 - a^2]}, \\ b_B &= \frac{1}{c+a} \sqrt{ac[(c+a)^2 - b^2]}, \\ b_C &= \frac{1}{a+b} \sqrt{ab[(a+b)^2 - c^2]}, \end{aligned} \tag{1}$$

and concluded that

given the three internal bisectors, there is an analytical possibility to determine the sides of the triangle. But the elimination leads to an equation of a very high degree, probably because it includes also the solutions of the external bisectors.

To obtain, say, the first formula in (1), it suffices, if M denotes the intersection of the bisector of angle A with side a , to apply the formula giving the area of a triangle in terms of the lengths of two sides and the sine of their angle in the obvious equality

$$\text{area } ABC = \text{area } ABM + \text{area } AMC,$$

to obtain

$$m = \frac{2bc}{b+c} \cos(A/2),$$

and then to express $\cos(A/2)$ in terms of the lengths of the sides from the relation

$$a^2 = b^2 + c^2 - 2bc[2 \cos^2(A/2) - 1].$$

One year later, answering a question raised in 1830 in volume 6 of the *Journal für reine und angewandte Mathematik* (Question 12, p. 213-214), von Renthe-Finke computed the area of a triangle in terms of its internal bisectors, concluded that the equation for the radius of the inscribed circle should be of the 16th degree, but refrained from deriving that equation explicitly.

The problem : "To construct a triangle given the three internal bisectors" was proposed again by Brocard in 1875 [6], and the question was repeated two years later in volume 3 of the same journal (Question 222, p. 32). Given three positive numbers m, n, p , this problem therefore consists in finding a positive solution (a, b, c) of the system of algebraic equations obtained by squaring both members of equations in (1) with b_A, b_B, b_C respectively replaced by m, n, p , namely

$$\begin{aligned} bc[(b+c)^2 - a^2] - (b+c)^2 m^2 &= 0, \\ ca[(c+a)^2 - b^2] - (c+a)^2 n^2 &= 0, \\ ab[(a+b)^2 - c^2] - (a+b)^2 p^2 &= 0. \end{aligned} \tag{2}$$

Until very recently, all the papers devoted to this problem used elimination theory to reduce the system to a single algebraic equation and/or graphical considerations.

Two contributions appeared in 1889, namely Bütberger's doctoral thesis in Bern [7] and a paper of Van den Berg [16]. This last author discussed systematically the corresponding three quartic equations, obtained a unique equation of degree 16 for the radius of the circumcircle by elimination and squaring, but did not conclude about the possibility of constructing by ruler and compass. One year later, Heymann [9] claimed that the problem lead finally to an equation of the 10th order, without giving neither the equation, nor the method, not any references. He also did not discuss the possibility of constructing the solution by ruler and compass.

The same year, Van den Berg [17] showed that, given three positive numbers m, n, p the necessary and sufficient conditions for the existence of a triangle with exterior bisectors of lengths m, n, p is that the largest of the three numbers mn, np, pm be larger than the sum of the other two. If this condition is satisfied, he proved that the problem has two solutions, leading to triangles having the same

perimeter. Notice that since m, n, p are finite, the triangle is not isosceles. Van den Berg's proof reduced the problem to the solution of a cubic equation. This paper is clearly described in [5]. A similar result was obtained independently by Heymann [10] in 1897. Hence the external bisectors problem was solved positively.

Returning to the internal bisectors problem, Korselt [11] used in 1895 the resultant of two cubic equations to derive "nach einer allerdings mühsamen Rechnung" an explicit equation of tenth degree for the ratio of two sides and discussed its solvability. Two years later, the same author [12] considered the special case where two bisectors are equal, which implies, according to a conjecture of Lhe-mus proved by Steiner, that the triangle is isosceles. He reduced the problem to a cubic equation, showed that a ruler and compass construction is impossible in this special case and concluded, *a fortiori*, to the impossibility for the general one.

In 1896, in a work [4] announced in [3], Barbarin, apparently unaware of [17] and [10], gave an elaborate and detailed discussion of both the interior and the exterior problems, with graphs of some of the curves involved. He reduced the problem to an equation of the 14th degree for the interior problem, and of the 16th degree for the exterior one. He observed that these equations are in general irreducible and that a ruler and compass construction is out of question. A PhD thesis of Baker [2] in 1911, devoted to the interior and exterior problems, referred only to Barbarin [3, 4], discussed the interior and exterior bisectors problems in a more direct fashion (without solving first, like Barbarin, the problem with two bisectors and an angle given), and studied the irreducibility and the group of the reduced equation. He also considered various special cases.

The internal bisector problem seems to have been forgotten until 1937, when Neiss [14], in a way somewhat similar to Korselt's one, showed that it is in general impossible to construct the triangle with ruler and compass. The same year, Wolff [19] derived in an elegant and explicit way an equation of degree ten for the reciprocal value of the radius of the inscribed circle in terms of the given bisectors, and showed that the domain of rationality of this equation was irreducible. His result was used one year later by van der Waerden [18] to show that the Galois group of Wolff equation with respect to the domain of rationality was the symmetric group S_{10} .

This did not prevent to see the question of the possibility of the geometric construction to be raised again one year later in *Mathematics Student*, vol. 7, p. 40, Question 1768! This long history, partly traced in [1] and [5], shows how a simple looking classical geometric problem can lead to deep researches in classical and modern algebra.

The impossibility of the elementary geometric construction and the difficulty of deducing rigorously existence from the high order algebraic equations has led Mironescu and Panaitopol [13] in 1995 to formulate the three interior bisectors problem as a *fixed point problem* in three new variables linearly related to the side of the triangle. They gave in this way a topological proof (via Brouwer fixed point theorem) of the existence of a solution of the interior bisectors problem for any data m, n, p , and furthermore showed its uniqueness (up to an isometry). Notice that the uniqueness conclusion contrasts with Van der Berg's result [17] for the external bisectors problem, and that the absence of restrictions upon m, p, n contrasts with the existence results for the altitudes, medians and exterior bisec-

tors problems. We recall this fixed point formulation in Section 2 before giving, in Section 3, an elementary and analytical existence proof for the fixed point of Mironescu-Panaitopol's operator, as well as a method of approximation.

2 Reduction to a fixed point problem

The main idea of Mironescu-Panaitopol's proof consists in an elegant reduction of the three bisectors problem to a fixed point problem in three new variables, linear combinations of a, b, c . We repeat this clever reduction for reader's convenience. The first relation in (2) can be written equivalently

$$4m^2 = \frac{(b+c)^2 - (b-c)^2}{(b+c)^2} [(b+c)^2 - a^2]$$

or

$$\begin{aligned} 4m^2 &= (b+c)^2 + \frac{(b-c)^2 a^2}{(b+c)^2} - [a^2 + (b-c)^2] \\ &= \left[b+c \pm \frac{(b-c)a}{b+c} \right]^2 - [a \pm (b-c)]^2. \end{aligned}$$

Eliminating $\frac{(b-c)a}{b+c}$ between those two equations one gets

$$2(b+c) = \sqrt{4m^2 + (c+a-b)^2} + \sqrt{4m^2 + (a+b-c)^2}. \quad (3)$$

Now, letting

$$a = y + z, \quad b = z + x, \quad c = x + y, \quad (4)$$

so that

$$x = \frac{b+c-a}{2}, \quad y = \frac{c+a-b}{2}, \quad z = \frac{a+b-c}{2},$$

are positive, one can write (3) as

$$x = \frac{1}{2} \left[\sqrt{m^2 + y^2} - y \right] + \frac{1}{2} \left[\sqrt{m^2 + z^2} - z \right]. \quad (5)$$

One deduces in a similar way, from the second and third equations in (2)

$$y = \frac{1}{2} \left[\sqrt{m^2 + z^2} - z \right] + \frac{1}{2} \left[\sqrt{m^2 + x^2} - x \right] \quad (6)$$

$$z = \frac{1}{2} \left[\sqrt{m^2 + x^2} - x \right] + \frac{1}{2} \left[\sqrt{m^2 + y^2} - y \right]. \quad (7)$$

For any $\alpha > 0$, define the continuous function $f_\alpha : \mathbb{R}_+ \rightarrow (0, \alpha/2]$ by

$$f_\alpha(t) = \frac{1}{2} \left[\sqrt{\alpha^2 + t^2} - t \right].$$

It is immediate to check that $f'_\alpha(t) = \frac{1}{2} \left[\frac{t}{\sqrt{\alpha^2+t^2}} - 1 \right] < 0$ for $t \geq 0$, so that f_α is decreasing on \mathbb{R}_+ , $|f'_\alpha(t)| < \frac{1}{2}$ for $t > 0$,

$$|f_\alpha(t) - f_\alpha(t')| < \frac{1}{2}|t - t'| \quad \text{for all } t \neq t' \quad \text{in } \mathbb{R}_+. \quad (8)$$

If $C := [0, m] \times [0, n] \times [0, p]$, define the continuous mapping $F : C \rightarrow C$ by

$$F(x, y, z) = (f_m(y) + f_m(z), f_n(z) + f_n(x), f_p(x) + f_p(y)). \quad (9)$$

The discussion above and equations (5)-(6)-(7) show that *the three internal bisectors problem has a solution if and only if F has a fixed point in C* . This is the first remarkable result of Mironescu-Panaitopol [13], who further deduced the existence of a fixed point of F from Brouwer's fixed point theorem (see e.g. [20]), and its uniqueness from the relation

$$\|F(P) - F(P')\| < \|P - P'\| \quad \text{for all } P \neq P' \quad \text{in } C, \quad (10)$$

which easily follows from (8) ($\|\cdot\|$ denotes the Euclidian norm in \mathbb{R}^3 and we have written $P = (x, y, z)$). Thus, *the existence and uniqueness (up to an isometry) of the triangle solving the internal three bisectors problem follows, and its sides lengths a, b, c are given by (4) with (x, y, z) the fixed point of F* .

3 An elementary constructive approach

Our remark originates from (10), which indeed allows to replace the use of the nonconstructive topological Brouwer fixed point theorem by an elementary, analytical and constructive fixed point argument. In doing so, a method of approximation is obtained to compute the unique fixed point of F , and hence to compute the sides lengths a, b, c of the triangle, according to (4).

Theorem 1. *The mapping F defined in (9) has a unique fixed point P^* in C , and given any sequence (λ_k) in $(0, 1)$ converging to 1, $P^* = \lim_{k \rightarrow \infty} P_k$, where P_k is the unique fixed point in C of $\lambda_k F$.*

Proof. Uniqueness is a direct consequence of inequality (10). For the existence, let (λ_k) be a sequence contained in $(0, 1)$ and converging to 1. For each $k \in \mathbb{N}$, define the operator $T_k := \lambda_k F$, so that $T_k : C \rightarrow C$, and

$$\|T_k(P) - T_k(P')\| < \lambda_k \|P - P'\| \quad \text{for all } P \neq P' \quad \text{in } C,$$

i.e. T_k is a contraction with constant $\lambda_k \in (0, 1)$. Using Banach fixed point theorem (more precisely its version in \mathbb{R}^n already proved by Goursat [8] in 1906), we obtain, for each $k \in \mathbb{N}$, a unique fixed point $P_k \in C$ of T_k . Now the sequence (P_k) contained in C has, using Bolzano-Weierstrass theorem, a subsequence (P_{k_n}) converging to some $P^* \in C$. The relations $P_{k_n} = \lambda_{k_n} F(P_{k_n})$ ($n \in \mathbb{N}$) and the continuity of F imply that $P^* = F(P^*)$, so that all convergent subsequences of (P_k) converge to the unique fixed point P^* of F . This implies that the sequence (P_k) itself converges to P^* . ■

After having obtained the unique fixed point P^* of F from elementary considerations, we show how it can be approximated. For any mapping $T : C \rightarrow C$, we denote as usual by T^q the q^{th} iterate $T \circ \dots \circ T$ (q times) of T .

Theorem 2. *Given $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $K(\varepsilon) \in \mathbb{N}$ and $Q(\varepsilon, k) \in \mathbb{N}$ such that, for any $k \geq K(\varepsilon)$ and $q \geq Q(\varepsilon, k)$, one has*

$$\left\| \left[\frac{k}{k+1} F \right]^q (0) - P^* \right\| < \varepsilon. \quad (11)$$

Proof. Let $\varepsilon > 0$. Take $\lambda_k := \frac{k}{k+1}$, so that $T_k := \frac{k}{k+1} F$ ($k \in \mathbb{N}$) and let P_k be the unique fixed point of T_k given by Theorem 1. Since $P^* = \lim_{k \rightarrow \infty} P_k$, there exists $K(\varepsilon) \in \mathbb{N}$ such that

$$\|P_k - P^*\| < \frac{\varepsilon}{2} \quad \text{for all } k \geq K(\varepsilon). \quad (12)$$

On the other hand, each P_k is the unique fixed point of $\frac{k}{k+1} F$, so that (see e.g. [20]) Banach fixed point theorem for contraction mappings implies that

$$P_k = \lim_{q \rightarrow \infty} \left[\frac{k}{k+1} F \right]^q (0).$$

and provides the error estimate

$$\begin{aligned} \left\| \left[\frac{k}{k+1} F \right]^q (0) - P_k \right\| &\leq \left(\frac{k}{k+1} \right)^q k \|F(0)\| \\ &= \left(\frac{k}{k+1} \right)^q k (m^2 + n^2 + p^2)^{1/2}. \end{aligned} \quad (13)$$

Since $\frac{k}{k+1} \in (0, 1)$, it is clear that there exists $Q(\varepsilon, k) \in \mathbb{N}$ such that

$$\left(\frac{k}{k+1} \right)^q k (m^2 + n^2 + p^2)^{1/2} < \frac{\varepsilon}{2} \quad \text{whenever } q \geq Q(\varepsilon, k). \quad (14)$$

Consequently, taking $k \geq K(\varepsilon)$ and $q \geq Q(\varepsilon, k)$, we deduce from (12), (13) and (14) that (11) holds. \blacksquare

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