Application of duality techniques to starlikeness of weighted integral transforms

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Abstract

Let $\mathcal{A}$ be the class of normalized analytic functions in the unit disc and let $P_\gamma(\alpha, \beta)$ be the class of all functions $f \in \mathcal{A}$ satisfying the condition

$$\exists\, \eta \in \mathbb{R}, \quad \Re\left\{e^{i\eta}\left[(1-\gamma)\left(\frac{f(z)}{z}\right)^{\alpha} + \gamma\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} - \beta\right]\right\} 0.$$

We consider the integral transform

$$V_{\lambda,\alpha}(f)(z) = \left\{\int_0^1 \lambda(t)\left(\frac{f(tz)}{t}\right)^{\alpha} dt\right\}^{\frac{1}{\alpha}},$$

where $\lambda(t)$ is a real-valued nonnegative weight function normalized by $\int_0^1 \lambda(t)dt = 1$. In this paper we find conditions on the parameters $\alpha, \beta, \gamma, \mu$ such that $V_{\lambda,\alpha}(f)$ maps $P_\gamma(\alpha, \beta)$ into the class of starlike functions of order μ. We also provide a number of applications for various choices of $\lambda(t)$. Our results generalize known results on this topic.

1 Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

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which are analytic in the open unit disc $U = \{z \in \mathbb{C}; |z| < 1\}$. Let $S^*(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of all functions which are starlike of order μ in U. For functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{j,n} z^n, \quad (j = 1, 2),$$

the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ is an analytic function given by

$$(f_1 * f_2)(z) := z + \sum_{n=2}^{\infty} a_{1,n} a_{2,n} z^n, \quad (z \in U).$$

By applying the Pochhammer symbol (or the shifted factorial) given by

$$(a, 0) = 1 \quad < mboxand \quad (a, n) = a(a+1)(a+2)...(a+n-1), \quad (n = 1, 2, 3, ...),$$

it is known that the familiar Gaussian hypergeometric series defined by

$$F(a, b\,; c\,; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)} z^n, \quad (a, b, c \in \mathbb{C}, c \notin \{0, -1, -2, ...\}),$$

is analytic in the unit disc U.

Throughout this paper the function $\lambda : [0, 1] \mapsto \mathbb{R}$ is considered to be a non-negative function with $\int_0^1 \lambda(t)dt = 1$.

For $\alpha \geq 0$ and $f \in \mathcal{A}$, we define the weighted integral transform

$$V_{\lambda,\alpha}(f)(z) = \left(\int_0^1 \lambda(t) \left(\frac{f(tz)}{t} \right)^{\alpha} dt \right)^{\frac{1}{\alpha}}, \tag{1.1}$$

where for the power function the principle branch is taken.

We recall that the operator $V_{\lambda,\alpha}(f)$ reduces in some special cases to well-known operators such as the Libera, Bernardi, and Komatu operators. This operator $V_{\lambda,\alpha}(f)$ has been studied by a number of authors for various choices of $\lambda(t)$ (see e.g. [1,3,4,8,9]).

For $\beta < 1, \alpha \geq 0$ and $0 \leq \gamma \leq 1$, we introduce the class $P_\gamma(\alpha, \beta)$ of all functions $f \in \mathcal{A}$ such that

$$\Re \left\{ e^{i\eta} \left[(1-\gamma) \left(\frac{f(z)}{z} \right)^{\alpha} + \gamma \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^{\alpha} - \beta \right] \right\} 0, \quad (z \in U, \eta \in \mathbb{R}), \tag{1.2}$$

where for the power function the principle branch is taken.

If $\eta = 0$ in (1.2), then Liu [6] gave a univalent criterion for the class $P_\gamma(\alpha, \beta)$. Furthermore, if $f \in P_\gamma(1, \beta)$, many authors used duality theory for their discussion of the starlikeness of $V_{\lambda,1}(f)$. (e.g. [2,4,5,7]). By using similar methods for a function $f \in V_{\lambda,\alpha}(f)$, we here want to find conditions on α, β, γ and μ such that $V_{\lambda,\alpha}(f) \in S^*(\mu)$.

2 Main Results

In this section we start by considering the functions in $P_\gamma(\alpha,\beta)$ for which $\alpha > 0, 0 \le \beta \le 1$ and γ is any number in [0,1]. Let $\alpha > 0, \gamma > 0$ and define

$$\Lambda_\gamma(t) = \int_t^1 \frac{\lambda(s)}{s^{\frac{\alpha}{\gamma}}} ds.$$

Further let $g(t)$ be the solution of the initial value problem

$$\frac{d}{dt}\left(t^{\frac{\alpha}{\gamma}}(1+g(t))\right) = \frac{2}{\gamma}\frac{t^{\frac{\alpha}{\gamma}-1}[1-(1-\alpha+\alpha\mu)(1+t)]}{(1-\mu)(1+t)^2}, \quad g(0)=1. \tag{2.1}$$

Solving (2.1), we find

$$g(t) = \frac{2t^{\frac{-\alpha}{\gamma}}}{\gamma(1-\mu)}\int_{0^+}^{t} u^{\frac{\alpha}{\gamma}-1}\frac{[1-(1-\alpha+\alpha\mu)(1+u)]}{(1+u)^2}du - 1. \tag{2.2}$$

Theorem 2.1. *Let $\alpha \ge 1, \gamma > 0, 1-\frac{1}{\alpha} \le \mu \le 1-\frac{1}{2\alpha}$ and $\beta < 1$ be given by*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t)g(t)dt, \tag{2.3}$$

where $g(t)$ is given by (2.2). Assume that $\lim_{t\to 0^+} t^{\frac{\alpha}{\gamma}}\Lambda_\gamma(t) = 0$. Then $V_{\lambda,\alpha}(f) \in S^*(\mu)$ if and only if

$$\Re\int_0^1 \Lambda_\gamma(t)t^{\frac{\alpha}{\gamma}-1}\left[h(tz) - \frac{1-(1-\alpha+\alpha\mu)(1+t)}{\alpha(1-\mu)(1+t)^2}\right]dt \ge 0, \tag{2.4}$$

where

$$h(z) = \frac{1}{(1-z)^2}\left[1+\frac{\xi+1-2\alpha(1-\mu)}{2\alpha(1-\mu)}z\right], \quad |\xi|=1. \tag{2.5}$$

Proof. Let $f \in P_\gamma(\alpha,\beta)$ and define

$$H(z) = \frac{(1-\gamma)\left(\frac{f(z)}{z}\right)^\alpha + \gamma\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^\alpha - \beta}{1-\beta}.$$

Then for some $\eta \in \mathbb{R}$ we have $\Re(e^{i\eta}H(z))0$. Setting $F(z) = V_{\lambda,\alpha}(f)(z)$, we want to find conditions such that $\Re\frac{zF'(z)}{F(z)} > \mu$, $z \in U$. By the duality principle (see [10]) we may restrict our attention to the function $f \in P_\gamma(\alpha,\beta)$ for which $H(z) = \frac{1+xz}{1+yz}$, $|x| = |y| = 1$.
Set $G(z) = \left(\frac{F(z)}{z}\right)^\alpha$, then it is easy to see that G is analytic in U and $G(0) = 1$. Since $\frac{zF'(z)}{F(z)} = 1 + \frac{1}{\alpha}\frac{zG'(z)}{G(z)}$, it is well known (see [10]) that F is starlike of order μ if and only if

$$1+\frac{1}{\alpha}\frac{zG'(z)}{G(z)} = \frac{zF'(z)}{F(z)} \ne (1-\mu)\frac{1+t}{1-t}+\mu, \quad with \quad |t|=1, z\in U.$$

After some algebraic computation we find that it is equivalent with the following condition

$$F \in S^*(\mu) \Longleftrightarrow \left(\frac{F(z)}{z}\right)^{\alpha} * h(z) \neq 0, \quad z \in U,$$

where

$$h(z) = \frac{1}{(1-z)^2}\left[1 + \frac{\xi + 1 - 2\alpha(1-\mu)}{2\alpha(1-\mu)} z\right], \quad |\xi| = 1.$$

Hence $F \in S^*(\mu)$ if and only if $(\gamma \neq 0)$,

$$\begin{aligned}
0 &\neq \left(\frac{F(z)}{z}\right)^{\alpha} * h(z) \\
&= \int_0^1 \frac{\lambda(t)}{1-tz} dt * \left[(1-\beta)\frac{\alpha}{\gamma z^{\frac{\alpha}{\gamma}}} \int_0^z \frac{\omega^{\frac{\alpha}{\gamma}-1}(1+x\omega)}{1+y\omega} d\omega + \beta\right] * h(z) \\
&= (1-\beta)\int_0^1 \lambda(t)\left(h(tz) + \frac{\beta}{1-\beta}\right) dt * \frac{\alpha}{\gamma z^{\frac{\alpha}{\gamma}}} \int_0^z \frac{\omega^{\frac{\alpha}{\gamma}-1}(1+x\omega)}{1+y\omega} d\omega \\
&= (1-\beta)\int_0^1 \lambda(t)\left(h(tz) - g(t)\right) dt * \frac{\alpha}{\gamma z^{\frac{\alpha}{\gamma}}} \int_0^z \frac{\omega^{\frac{\alpha}{\gamma}-1}(1+x\omega)}{1+y\omega} d\omega \\
&= (1-\beta)\int_0^1 \lambda(t)\left[\frac{\alpha}{\gamma z^{\frac{\alpha}{\gamma}}} \int_0^z \omega^{\frac{\alpha}{\gamma}-1} h(t\omega) d\omega - g(t)\right] dt * \frac{1+x\omega}{1+y\omega}.
\end{aligned}$$

By the same method of proof as in [2], it is easily verified that this holds if and only if

$$\Re \int_0^1 \Lambda_\gamma(t) t^{\frac{\alpha}{\gamma}-1}\left[h(tz) - \frac{1-(1-\alpha+\alpha\mu)(1+t)}{\alpha(1-\mu)(1+t)^2}\right] dt \geq 0,$$

which completes the proof. ■

Note that by putting $\alpha = 1$ in Theorem 2.1, we obtain the result of Balasubramanian et al [2]. Theorem 2.1 is not at once very useful because we have to check condition (2.4), in order that $V_{\lambda,\alpha}(f) \in S^*(\mu)$. Therefore we will identify some situations where (2.4) holds. We introduce

$$L_{\Lambda_\gamma}(h) = \inf_{z \in U} \int_0^1 \Lambda_\gamma(t) t^{\frac{\alpha}{\gamma}-1}\left[\Re(h(tz)) - \frac{1-(1-\alpha+\alpha\mu)(1+t)}{\alpha(1-\mu)(1+t)^2}\right] dt,$$

and formulate the following result.

Theorem 2.2. *Let $\alpha \geq 1, 1 - \frac{1}{\alpha} \leq \mu \leq 1 - \frac{1}{2\alpha}$. Assume that $\Lambda_\gamma(t)$ is integrable on [0,1] and positive on (0,1). Further suppose that*

$$\frac{\Lambda_\gamma(t) t^{\frac{1}{\gamma}(\alpha-1)}}{(\log \frac{1}{t})^{3-2\alpha(1-\mu)}}$$

is a decreasing function on (0,1). Then for $\frac{1}{2} \leq \gamma \leq 1$ one has $h_{\Lambda_\gamma}(h) \geq 0$, where h is defined above.

Proof. We shall omit some details here because there are many similarities between this proof and the proof of Theorem 1.3 in [2]. So we want to prove the following inequality

$$\int_0^1 \Lambda_\gamma(t)t^{\frac{\alpha}{\gamma}-1}\left[\Re(h(tz)) - \frac{1-(1-\alpha+\alpha\mu)(1+t)}{\alpha(1-\mu)(1+t)^2}\right]dt \geq 0, z \in U \tag{2.6}$$

where $h(z)$ is given by

$$h(z) = \frac{1}{(1-z)^2}\left[1 + \frac{\xi+1-2\alpha(1-\mu)}{2\alpha(1-\mu)}z\right], \quad |\xi| = 1.$$

Using the same argument as in [4] we see that it is sufficient to prove (2.6) for $|z| = 1$. It is easily verified that (2.6) holds if

$$\int_0^1 \Lambda_\gamma(t)t^{\frac{\alpha}{\gamma}-1}\{\Re\frac{2\alpha(1-\mu)+[1-2\alpha(1-\mu)]tz}{(1-tz)^2}$$

$$-\frac{t}{|1-tz|^2} - \frac{2[1-(1-\alpha+\alpha\mu)(1+t)]}{(1+t)^2}\}dt \geq 0,$$

for $|z| = 1, z \neq 0$. Further calculations result into the equivalent condition

$$H(y) = \int_0^1 \Lambda_\gamma(t)t^{\frac{\alpha}{\gamma}-1}\left[tA_1(y,t) - 2(1-\alpha(1-\mu))A_2(y,t)\right]dt \geq 0,$$

where

$$A_1(y,t) = \frac{3-4(1+y)t+2(4y-1)t^2+4(y-1)t^3-t^4}{(1+t^2-2yt)^2(1+t)^2},$$

and

$$A_2(y,t) = \frac{1-t}{(1+t^2-2ty)(1+t)},$$

with $-1 \leq y = Imz < 1$. A series expansion of H is obtained by

$$H(y) = \sum_{k=0}^{\infty} \widetilde{H}_k(1+y)^k, \quad |1+y| < 2,$$

where $\widetilde{H}_k$ is a positive multiple of

$$\widetilde{h}_k = \int_0^1 t^{\frac{\alpha}{\gamma}-1}\Lambda_\gamma(t)[S_k(t) - 2(1-\alpha(1-\mu))U_k(t)]dt,$$

with

$$S_k(t) = \frac{(k+3)t^{k+1}}{(1+t)^{2k+4}}\left(1-2t+\frac{k-1}{k+3}t^2\right) \quad \text{and} \quad U_k(t) = \frac{t^{k+1}}{(1+t)^{2k+4}}(1-t^2).$$

We can see that $A \equiv S_k(t) - 2(1-\alpha(1-\mu))U_k(t)$ has just one zero in (0,1). We denote this zero by t_k; it is well known that $A > 0$ if $0 \leq t < t_k$, while $A < 0$ if $t_k < t < 1$.

Let h_k be of the form:

$$h_k = \int_0^1 t^{\frac{1}{\gamma}-1}\left(\log(\frac{1}{t})\right)^{1+2\mu_1}(S_k(t) - 2\mu_1 U_k(t))dt,$$

where $\mu_1 = 1 - \alpha(1-\mu)$. Define

$$\widetilde{\Lambda}_\gamma(t) = \Lambda_\gamma(t) - \frac{\Lambda_\gamma(t_k)\left(\log(\frac{1}{t})\right)^{1+2\mu_1} t^{-\frac{1}{\gamma}(\alpha-1)}}{\left(\log(\frac{1}{t_k})\right)^{1+2\mu_1}(t_k)^{-\frac{1}{\gamma}(\alpha-1)}}.$$

The assumption that $\frac{\Lambda_\gamma(t)t^{\frac{1}{\gamma}(\alpha-1)}}{(\log\frac{1}{t})^{1+2\mu_1}}$ is decreasing, implies that $\widetilde{\Lambda}_\gamma(t)$ and $S_k(t) - 2\mu_1 U_k(t)$ have the same sign in (0,1). Therefore

$$0 \leq \int_0^1 t^{\frac{\alpha}{\gamma}-1}\widetilde{\Lambda}_\gamma(t)(S_k(t) - 2\mu_1 U_k(t))dt = \widetilde{h}_k - \frac{\Lambda_\gamma(t_k)}{\left(\log(\frac{1}{t_k})\right)^{1+2\mu_1}(t_k)^{-\frac{1}{\gamma}(\alpha-1)}} h_k.$$

If we can prove that $h_k \geq 0$, then it follows that $\widetilde{h}_k \geq 0$ and also $L_{\Lambda_\gamma}(h) \geq 0$ as desired. Along similar lines as in the proof of Theorem 1.3 in [2], one can observe that this is true for $\frac{1}{2} \leq \gamma \leq 1, 1 - \frac{1}{\alpha} \leq \mu \leq 1 - \frac{1}{2\alpha}$, which completes the proof. ■

We remark that the spacial case of Theorem 2.2 where $\alpha = 1$, has been studied by Balasubramanian et al [2].

Before stating our next results we will find an equivalent condition for the function $g(t) = \frac{\Lambda_\gamma(t)t^{\frac{1}{\gamma}(\alpha-1)}}{(\log\frac{1}{t})^{3-2\alpha(1-\mu)}}$ to be decreasing on (0,1), where $\Lambda_\gamma(t) = \int_t^1 \lambda(s)s^{-\frac{\alpha}{\gamma}}ds$.

Taking the logarithmic derivative of $g(t)$ and using the fact that $\Lambda'_\gamma(t) = -\lambda(t)t^{-\frac{\alpha}{\gamma}}$, we have

$$\frac{g'(t)}{g(t)} = \frac{-\lambda(t)t^{-\frac{\alpha}{\gamma}}}{\Lambda_\gamma(t)} + \frac{\alpha-1}{\gamma t} + \frac{3-2\alpha(1-\mu)}{t\log\frac{1}{t}}.$$

Note that $g(t)0$, so $g'(t) \leq 0$ for $t \in (0,1)$ is equivalent with the inequality

$$\psi(t) = \Lambda_\gamma(t) - \frac{\gamma\lambda(t)t^{-\frac{\alpha}{\gamma}+1}\log\frac{1}{t}}{(\alpha-1)\log\frac{1}{t} + \gamma(3-2\alpha(1-\mu))} \leq 0 \quad for \quad t \in (0,1).$$

Clearly $\psi(1) = 0$, and for completing the proof it is sufficient to show that $\psi(t)$ is increasing on (0,1). But a simple calculation shows that

$$\psi'(t) = \lambda(t)\frac{(\log\frac{1}{t})^2(\alpha-1)(1-\gamma) + \gamma^2\mu_1(1-\mu_1) - \mu_1\gamma(\alpha+\gamma-2)\log\frac{1}{t}}{((\alpha-1)\log\frac{1}{t} + \gamma\mu_1)^2}$$
$$-\lambda'(t)\frac{t(\log\frac{1}{t})^2(\alpha-1)\gamma + \gamma^2\mu_1 t\log\frac{1}{t}}{((\alpha-1)\log\frac{1}{t} + \gamma\mu_1)^2},$$

where $\mu_1 = 3 - 2\alpha(1-\mu)$. Therefore $\psi'(t) \geq 0$ for $t \in (0,1)$ is equivalent with the following inequality

$$\lambda(t)\left[(\log\frac{1}{t})^2(\alpha-1)(1-\gamma)+\gamma^2\mu_1(1-\mu_1)-\mu_1\gamma(\alpha+\gamma-2)\log\frac{1}{t}\right] \tag{2.7}$$
$$\geq \lambda'(t)\left[t(\log\frac{1}{t})^2(\alpha-1)\gamma+\gamma^2\mu_1 t\log\frac{1}{t}\right],$$

where $\mu_1 = 3 - 2\alpha(1-\mu)$.

3 Applications

Theorem 3.1. *Suppose that* $1 \leq \alpha < 2, \frac{1}{2} \leq \gamma \leq 1, 1-\frac{1}{\alpha} \leq \mu \leq 1-\frac{1}{2\alpha}$ *and that* $a, p, \mu, \gamma, \alpha$ *are related by any of the following conditions*

(i) $-1 < a \leq \frac{2-\alpha}{\gamma} + \frac{(p-1)(\alpha-1)}{\mu_1\gamma} - 1$ *and* $\mu_1 \leq p \leq \mu_1 + 1$.

(ii) $-1 < a \leq \frac{1}{\gamma} - 1$ *and* $p > 1 + \mu_1$.

If $g(t)$ *is defined by (2.2) and* $\beta = \beta(a,p,\gamma,\mu,\alpha)$ *is given by*

$$\frac{\beta}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)}\int_0^1 t^a(\log\frac{1}{t})^{p-1}g(t)dt \tag{3.1}$$

then for $f \in P_\gamma(\alpha,\beta)$, *the generalized Komatu operator defined by*

$$F(z) = \left\{\frac{(1+a)^p}{\Gamma(p)}\int_0^1 t^a(\log\frac{1}{t})^{p-1}\left(\frac{f(tz)}{t}\right)^\alpha dt\right\}^{\frac{1}{\alpha}} \tag{3.2}$$
$$= \left\{\left(\sum_{n=1}^{\infty}\frac{(1+a)^p}{(n+a)^p}z^{n+\alpha-1}\right) * f^\alpha(z)\right\}^{\frac{1}{\alpha}}$$

belongs to $S^*(\mu)$. *The value of* β *is sharp.*

Proof. Set $\lambda(t) = \frac{(1+a)^p}{\Gamma(p)}t^a(\log\frac{1}{t})^{p-1}$, then it easy to see that $\frac{\lambda'(t)}{\lambda(t)} = \frac{a}{t} - \frac{p-1}{t\log\frac{1}{t}}$. Substituting the value of the above expression in (2.7) we obtain

$$(\log\frac{1}{t})^2(\alpha-1)(1-\gamma-a\gamma)+\gamma\log\frac{1}{t}[(-\alpha-\gamma+2-a\gamma)\mu_1+(p-1)(\alpha-1)]+$$
$$\gamma^2\mu_1(p-\mu_1) \geq 0,$$

which is true for all $t \in (0,1)$ by the hypothesis of Theorem 3.1.

To prove the sharpness, let $f \in P_\gamma(\alpha,\beta)$ be the function for which

$$(1-\gamma)\left(\frac{f(z)}{z}\right)^\alpha + \gamma\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^\alpha = \beta + (1-\beta)\frac{1+z}{1-z}.$$

Using the series expansion we see that

$$\left(\frac{f(z)}{z}\right)^{\alpha} = 1 + \sum_{n=1}^{\infty} \frac{2(1-\beta)}{1+\frac{\gamma}{\alpha}n} z^n.$$

Further, using

$$\int_0^1 t^{a+n-1} (\log \frac{1}{t})^{p-1} dt = \frac{\Gamma(p)}{(a+n)^p}, \tag{3.3}$$

and considering (3.2), we obtain

$$\left(\frac{F(z)}{z}\right)^{\alpha} = 1 + 2(1+a)^p(1-\beta) \sum_{n=1}^{\infty} \frac{1}{(n+a+1)^p(1+\frac{\gamma}{\alpha}n)} z^n, \tag{3.4}$$

and

$$\frac{F'(z)(F(z))^{\alpha-1}}{z^{\alpha-1}} = 1 + 2(1+a)^p(1-\beta) \sum_{n=1}^{\infty} \frac{n+\alpha}{\alpha(n+a+1)^p(1+\frac{\gamma}{\alpha}n)} z^n. \tag{3.5}$$

Expanding $g(t)$ in (2.2) into a power series, we obtain

$$g(t) = 1 + \frac{2}{1-\mu} \sum_{n=1}^{\infty} \frac{(-1)^n(n+\alpha-\alpha\mu)}{(n\gamma+\alpha)} t^k,$$

which, when inserted into (3.1) and using (3.3), leads to

$$\frac{\beta}{1-\beta} = -1 + \frac{2(1+a)^p}{1-\mu} \sum_{n=1}^{\infty} \frac{(-1)^n(n+\alpha-\alpha\mu)}{(a+n+1)^p(n\gamma+\alpha)}.$$

This yields

$$\frac{1}{1-\beta} = \frac{2(1+a)^p}{1-\mu} \sum_{n=1}^{\infty} \frac{(-1)^n(n+\alpha-\alpha\mu)}{(a+n+1)^p(n\gamma+\alpha)}. \tag{3.6}$$

Dividing (3.5) through (3.4) and substituting $z = -1$ in this equation and further using (3.6), it is easily seen that $\Re\left(\frac{(-1)F'(-1)}{F(-1)}\right) - \mu$ is zero, which means that the result is sharp. This completes the proof. ∎

Theorem 3.2. *Let $a, b, c, > 0$, $1 \le \alpha < 2$, $\frac{1}{2} \le \gamma \le 1$, $1 - \frac{1}{\alpha} \le \mu \le 1 - \frac{1}{2\alpha}$ and $g(t)$ be defined by (2.2). Suppose that β is given by*

$$\frac{\beta}{1-\beta} = -\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} \times$$

$$\int_0^1 t^{b-1}(1-t)^{c-a-b} F(c-a, 1-a; c-a-b+1; 1-t) g(t) dt.$$

Then for $f \in P_{\gamma}(\alpha, \beta)$ the function

$$H(z) := H_{a,b,c,\alpha}(f)(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} \times$$

$$\left(\int_0^1 t^{b-1}(1-t)^{c-a-b} F(c-a, 1-a; c-a-b+1; 1-t) \left(\frac{f(tz)}{t} \right)^{\alpha} dt \right)^{\frac{1}{\alpha}}$$

belongs to $S^*(\mu)$ *whenever* $\frac{2-\alpha}{\gamma} - 2 + 2\alpha(1-\mu) > 0$ *and* a, b, c, γ, α *are related by any one of the following conditions,*

(i) $a \in (0,1]$, $0 \le b < \frac{2-\alpha}{\gamma} - 2 + 2\alpha(1-\mu)$ *and* $c - a - b \ge 2 - 2\alpha(1-\mu)$.

(ii) $a + 2 - 2\alpha(1-\mu) < \frac{2-\alpha}{\gamma}$, *with*

$$b + 2 - 2\alpha(1-\mu) \le c - a \le \frac{(2-\alpha)(c-a-b+1)}{\gamma[(c-a-b+1)a + (2 - 2\alpha(1-\mu))(1-a)]}.$$

The value of β *is sharp.*

Proof. Set $\lambda(t) = kt^{b-1}(1-t)^{c-a-b}\phi(1-t), k = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}$, and $\phi(1-t) = F(c-a, 1-a; c-a-b+1; 1-t)$. Then we have

$$H(z) = \left(\int_0^1 \lambda(t) \left(\frac{f(tz)}{t} \right)^{\alpha} dt \right)^{\frac{1}{\alpha}}.$$

According to (2.7), it suffices to verify the inequality

$$\begin{aligned} &\frac{(\log \frac{1}{t})^2(\alpha-1)}{1-t} \left((\frac{1}{\gamma} - b)(1-t) + t(c-a-b) \right) \\ &+\mu_1 \gamma \frac{\log \frac{1}{t}}{1-t} \left(\left[\frac{2-\alpha}{\gamma} - b \right] (1-t) + t(c-a-b) \right) + \mu_1\gamma(1-\mu_1) \\ &\ge -\frac{\phi'(1-t)}{\phi(1-t)} \left(t(\log \frac{1}{t})^2(\alpha-1) + \gamma\mu_1 t \log \frac{1}{t} \right). \end{aligned} \tag{3.7}$$

Case (i): let $0 < a \le 1$ and $0 \le b < \frac{2-\alpha}{\gamma} - 2 + 2\alpha(1-\mu)$ and $c - a - b \ge 2 - 2\alpha(1-\mu)$. The hypotheses imply that each term between the brackets is positive for $t \in (0,1)$, while it is easy to see that each of the Maclaurin coefficients of ϕ is nonnegative, so that $\frac{\phi'}{\phi}$ is nonnegative on [0,1]. Therefore, in view of $\frac{\log \frac{1}{t}}{1-t} \ge 1$, the inequality (3.7) holds for $t \in (0,1)$ if

$$\left(\frac{2-\alpha}{\gamma} - b + (1-\mu_1) \right)(1-t) + (c-a-b+(1-\mu_1))t \ge 0,$$

which clearly holds under the hypotheses of the theorem.

Case (ii): Assume that $a + 2 - 2\alpha(1-\mu) < \frac{2-\alpha}{\gamma}$, and

$$b + 2 - 2\alpha(1-\mu) \le c - a \le \frac{(2-\alpha)(c-a-b+1)}{\gamma[(c-a-b+1)a + (2 - 2\alpha(1-\mu))(1-a)]}.$$

Now, for convenience, we put $\mu_1 = 3 - 2\alpha(1-\mu), A = c-a, B = 1-a$, $C = c-a-b+1$ and $\phi(t) = F(A,B;C;t)$. Then by a simple calculation we obtain

$$\frac{t\phi'(1-t)}{\phi(1-t)} = \frac{AB}{C}\left(\frac{tF(A+1,B+1;C+1;1-t)}{F(A,B;C;1-t)}\right), \quad t\in(0,1). \tag{3.8}$$

Using (3.8) and the following identity, which can obtained by comparing the coefficients on both sides:

$$\begin{aligned} &t(1-t)\frac{AB}{C}F(A+1,B+1;C+1;t) \\ &= -(C-1-At)F(A,B;C;t) + (C-1)F(A,B-1;C-1;t), \end{aligned}$$

(3.7) results into

$$\begin{aligned} &\frac{(\log\frac{1}{t})^2(\alpha-1)}{1-t}\left[\frac{1-t}{\gamma}F(A,B;C;1-t) + (C-1)F(A,B-1;C-1;1-t)\right] \\ &+\mu_1\gamma\frac{\log\frac{1}{t}}{1-t}\left[\frac{2-\alpha}{\gamma}(1-t)F(A,B;C;1-t) + (C-1)F(A,B-1;C-1;1-t)\right] \\ &\geq (\mu_1-1)\mu_1\gamma F(A,B;C;1-t). \end{aligned} \tag{3.9}$$

By making use of the series expansion of the hypergeometric function, we find that (3.9) is equivalent to

$$\begin{aligned} &(c-a-b)\left(\mu_1\gamma\frac{\log\frac{1}{t}}{1-t} + \frac{(\log\frac{1}{t})^2(\alpha-1)}{1-t}\right) + \\ &\frac{(\log\frac{1}{t})^2(\alpha-1)}{1-t}\sum_{n=0}^{\infty}\frac{(A,n)(B,n)}{(C,n)(1,n+1)}\left(\frac{1}{\gamma} + (\frac{1}{\gamma}-a)n + (c-a)(-a)\right)(1-t)^{n+1} + \\ &\mu_1\gamma\frac{\log\frac{1}{t}}{1-t}\sum_{n=0}^{\infty}\frac{(A,n)(B,n)}{(C,n)(1,n+1)}\left(\frac{2-\alpha}{\gamma} + (\frac{2-\alpha}{\gamma}-a)n + (c-a)(-a)\right)(1-t)^{n+1} \\ &\geq (\mu_1-1)\mu_1\gamma F(A,B;C;1-t). \end{aligned} \tag{3.10}$$

The above hypotheses imply that $c-a-b \geq 0, (c-a)(-a) + \frac{1}{\gamma} + (\frac{1}{\gamma}-a) \geq 0$ and $\frac{2-\alpha}{\gamma} + (\frac{2-\alpha}{\gamma}-a)n + (c-a)(-a) \geq 0$ for all n. So the square bracketed terms in the inequality (3.10) are nonnegative. Hence, in view of $\frac{\log\frac{1}{t}}{1-t} \geq 1$ for $t\in(0,1)$, it suffices to show that

$$c-a-b-(\mu_1-1) + \sum_{n=0}^{\infty}\frac{(A,n)(B,n)}{(C,n+1)(1,n+1)}\psi(n)(1-t)^{n+1} \geq 0, t\in(0,1), \tag{3.11}$$

where

$$\begin{aligned} \psi(n) &= (c-a-b+1+n)\left\{\frac{2-\alpha}{\gamma} + (\frac{2-\alpha}{\gamma}-a)n + (c-a)(-a)\right\} \\ &-(\mu_1-1)(c-a+n)(1-a+n). \end{aligned}$$

But, by hypothesis, we see that $\psi(n)$ is an increasing function of $n \geq 0$, and therefore $\psi(n) \geq \psi(0) \geq 0$. Thus the inequality (3.11) holds. Following the same lines as in the proof of the previous theorem, one can see that the result is sharp. This completes the proof.

Note that for $\alpha = 1$ we obtain the result of Balasubramanian et al [2]. ∎

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