On the existence of projective embeddings of multiveblen configurations

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Abstract

We prove that from among simple multiveblen configurations only combinatorial Grassmannians can be embedded into a Desarguesian projective space. The class of regular multiveblen configurations which are projectively embeddable is determined.

Introduction

The class of multiveblen configurations was introduced in [11]. The aim of this note is to determine which multiveblen configurations can be embedded into a Desarguesian projective space (comp. [14] (and [5]) for more information on configurations in projective geometry).

The *Veblen* (or *Veblen-Young*) *configuration* is a well known classical $(6_2\,4_3)$-configuration of projective geometry[1]. Sometimes it is also called *a Pasch Configuration*, though the original Pasch configuration originates in ordered (Euclidean) geometry, while no geometrical order is involved in the definition of the Veblen configuration. A multiveblen configuration (in short: MVC) is a partial Steiner triple system (i.e. a partial linear space with the lines of size 3, cf. [13]), whose construction generalizes the construction of the Desargues configuration and of the 10_3G-configuration of Kantor consisting in completing three Veblen configurations on three concurrent lines by a single new line (see Figure 1). Loosely

[1]Formally, it consists of four pairwise intersecting lines, no three concurrent, together with the corresponding intersection points.

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speaking, *a multiveblen configuration* $\mathfrak{M} = \mathcal{M}_n^p \rhd\!\!\!\!\rhd \mathfrak{H}$ (shorter: $\mathfrak{M} = \mathcal{M}_n \rhd\!\!\!\!\rhd \mathfrak{H}$) can be visualized as a system of $\binom{n}{2}$ Veblen configurations on n concurrent lines of the size 3 (through a point p) completed by another multiveblen configuration $\mathfrak{H}$ (of the form $\mathfrak{H} = \mathcal{M}_{n-2} \rhd\!\!\!\!\rhd \mathfrak{H}'$, or by some "combinatorial Grassmannian" $\mathfrak{H}$), whose lines join "second points of intersection" in the corresponding Veblen configurations. This verbal presentation does not characterize $\mathfrak{M}$ uniquely; the additional parameter $\mathcal{P}$, a graph on n vertices is used to make the definition correct and we write $\mathfrak{M} = \mathcal{M}_n^p \rhd_{\mathcal{P}} \mathfrak{H}$ (cf. 1.1).

The class of multiveblen configurations contains, in particular, structures which generalize the Desargues configuration considered as a perspective of two triangles, and which can be visualized as a perspective of two n-simplices in a projective space. These structures can also be represented in a pure combinatorial way as *combinatorial Grassmannians* $\mathbf{G}_2(n+2)$ (cf. [10]).

In a sense, a multiveblen configuration was invented as a solution of a (rather technical) problem to construct and classify sufficiently regular configurations in which any two lines through a given (fixed) point yield a Veblen configuration. This was the idea of constructing structures defined in [11], of the form $\mathcal{M}_n^p \rhd_{\mathcal{P}} \mathfrak{H}$ with arbitrary $\mathfrak{H}$. A huge variety of the obtained structures[2] forced us to restrict ourselves in the paper to the case when $\mathfrak{H}$ is again a multiveblen configuration, or a combinatorial Grassmannian. In essence, equivalently, we could require that $\mathfrak{H}$ is a MVC, a point, or a line of size 3. In any case the multiveblen configurations seem interesting on their own, due to their simple and well visualizable internal structure, and close connections with the (classical) Veblen configuration. Since the Veblen axiom is a fundamental axiom of projective geometry, multiveblen configurations can be considered as, loosely speaking, locally projective (for a system of pairwise not collinear points). Then the question which of them are "really" projective i.e. which can be realized in a (Desarguesian) projective space seems natural.

Clearly, both combinatorial Grassmannians and the 10_3G configuration can be embedded into a projective space. In our note we prove that these are the only simple MVC (i.e. those $\mathfrak{M}$ where $\mathfrak{H}$ is a combinatorial Grassmannian) which can be projectively embedded. The problem to characterize *all* the projectively embeddable MVC is much more complex because the class of all MVC contains configurations of various quite irregular structure. We distinguish the class of *regular* MVC and prove that in this class, besides combinatorial Grassmannians exactly one new series of projectively embeddable structures appear; configurations in this series generalize the 10_3G configuration.

1 Notation

First, we briefly recall the definitions of the structures considered in the paper. Let X be a nonempty set and k be an integer; we write $\wp_k(X)$ for the family of k-element subsets of X. Two graphs on X are especially important (cf. [16]):

the empty graph $N_X = \langle X, \emptyset \rangle$ and the complete graph $K_X = \langle X, \wp_2(X) \rangle$.

[2] See e.g. [12] for a detailed discussion of the case when the degree n of p is 4 – there are at least 11 distinct isomorphism types of such structures.

Frequently, only the type of a graph will be needed; we write K_n for the type of K_X where $|X| = n$, and similarly N_n for the type of N_X. In what follows we shall also frequently identify a set $\mathcal{P} \subset \wp_2(X)$ with the graph $\langle X, \mathcal{P} \rangle$. If $\mathcal{P} \subset \wp_2(X)$ and $Y \subset X$ we write $\mathcal{P} \curlywedge Y$ for the graph $\mathcal{P} \cap \wp_2(Y)$. Given an ordering $x_1, \ldots, x_n$ of the elements of X we write

L_n for the type of the linear graph $\langle X, \big\{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}\big\}\rangle$.

The structure

$$\mathbf{G}_2(X) := \langle \wp_2(X), \wp_3(X), \subset \rangle \cong \langle \wp_2(X), \big\{\wp_2(Y) \colon Y \in \wp_3(X)\big\}, \in \rangle$$

is referred to as *a combinatorial Grassmannian* (cf. [10], [11], [9])[3]. Then $\mathbf{G}_2(|X|)$ is the type of $\mathbf{G}_2(X)$ – it is a $\big(\binom{n}{2}_{n-2}\binom{n}{3}_3\big)$-configuration ($n = |X|$; generally, a $(\nu_r\, b_\kappa)$-configuration is a configuration with ν points of degree r each and b lines of size κ each), so it is a partial Steiner triple system. In particular, $\mathbf{G}_2(5)$ is the *Desargues configuration* (cf. [7]).

Construction 1.1. Let $\mathfrak{H} = \langle \wp_2(X), \mathcal{L} \rangle$ be a partial Steiner triple system and $\mathcal{P}$ be a non-oriented graph without loops defined on X. We take any two distinct elements $p_1, p_2 \notin X$ and put $p = \{p_1, p_2\}$, $X' = X \cup p$. Consider the following families of blocks:

$$\begin{aligned}
\mathcal{L}_1 &= \Big\{\big\{\{p_1,p_2\},\{p_1,i\},\{p_2,i\}\big\} : i \in X\Big\},\\
\mathcal{L}_2 &= \Big\{\big\{\{i,j\},\{p_1,i\},\{p_2,j\}\big\}, : i,j \in X,\ i \neq j,\ \{i,j\} \notin \mathcal{P}\Big\},\\
\mathcal{L}_3 &= \Big\{\big\{\{i,j\},\{p_1,i\},\{p_1,j\}\big\}, \big\{\{i,j\},\{p_2,i\},\{p_2,j\}\big\} : i,j \in X,\ \{i,j\} \in \mathcal{P}\Big\}.
\end{aligned}$$

The structure $\langle \wp_2(X'), \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \rangle$ will be denoted by $\mathcal{M}^p_X \triangleright_{\mathcal{P}} \mathfrak{H}$ and it will be called *the multiveblen configuration with center p, consistency graph $\mathcal{P}$ defined on X, and axial configuration $\mathfrak{H}$*. A multiveblen configuration is *simple* if it has a combinatorial Grassmannian as its axial configuration.

A particular role is played in the sequel by the structure

$$\mathfrak{B}(X) = \mathcal{M}^p_X \triangleright_{N_X} \mathbf{G}_2(X).$$

We write $\mathfrak{B}(n) := \mathfrak{B}(X)$, where $|X| = n$, for short. ○

It is easy to note that

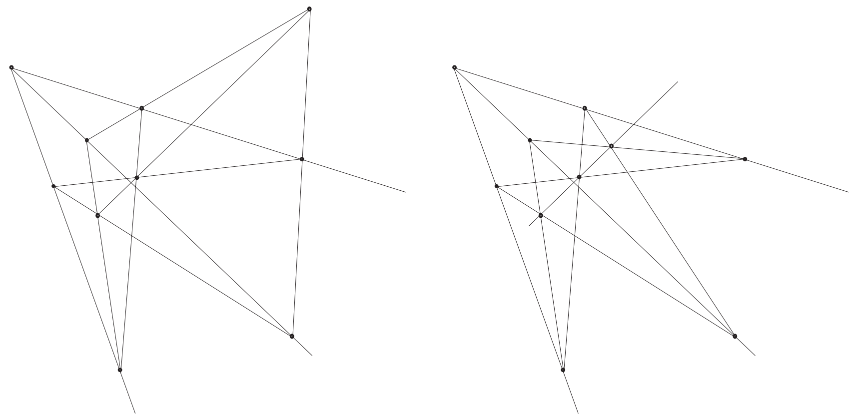
$$\mathcal{M}^p_X \triangleright_{K_X} \mathbf{G}_2(X) = \mathbf{G}_2(X \cup p).$$

The structure $\mathfrak{V}^o := \mathfrak{B}(3)$ is the 10_3G-configuration of Kantor (cf. [6]); in the paper this one will also be called the *Veronese configuration*.[4] The line $\mathbf{G}_2(X)$ of

[3] Classical Grassmann space, as considered in geometry, is an incidence structure defined over the lattice of subspaces of a projective space, cf. [15]. Its points are the k-dimensional subspaces ($k \geq 1$ is a fixed integer) and its lines are the pencils. If $k+1$ is less than the dimension of the space then an equivalent structure is obtained when we adopt the $(k+1)$-dimensional subspaces as the lines (and inclusion as the incidence). Passing to the lattice of subsets of a set X and replacing *dimension* by *cardinality* we define, by analogy, the combinatorial Grassmannian $\mathbf{G}_k(X)$.

[4] This terminology may be justified by the fact that $\mathfrak{V}^o$ is isomorphic with one of the combinatorial Veronese spaces, cf. [9], [8] (which on the other hand, generalize classical projective Veronese spaces, cf. [15], [3]). In general, (see [11, Prop. 6]) the incidence structure $\mathfrak{B}(n)$ is isomorphic to the dual of a suitable combinatorial Veronese space (i.e. isomorphic to $\mathbf{V}^*_n(3)$ dual to $\mathbf{V}_n(3)$ in the notation of [9]).

$\mathfrak{V}^o = \mathfrak{B}(X)$ is referred to as *the axis* of $\mathfrak{V}^o$.



The Desargues Configuration The 10_3G configuration (the Veronese Configuration)

Figure 1: Three Veblen configurations may yield two 10_3 configurations.

Fact 1.2. *If $\mathfrak{H}$ is a $\big(\binom{n}{2}_{n-2}\binom{n}{3}_3\big)$-configuration then $\mathcal{M}^p_X \rhd_{\mathcal{P}} \mathfrak{H}$ is a $\big(\binom{n+2}{2}_n\binom{n+2}{3}_3\big)$-configuration.*

The construction of the structure $\mathcal{M}^p_X \rhd_{\mathcal{P}} \mathfrak{H}$ can be visualized in a more geometrical vein, which is more convenient in the analysis of the obtained configurations. Let us adopt the notation of 1.1. Next, write

$$a_i = \{p_1, i\}, \quad b_i = \{p_2, i\} \quad \text{for } i \in X$$

and

$$c_z = z \quad \text{for } z \in \wp_2(X), \qquad \mathcal{C} = \big\{c_z \colon z \in \wp_2(X)\big\}.$$

Step A The set p is an arbitrary "abstract new point".

Step B Through p we have the lines L_i, and the points a_i, b_i on L_i, for every $i \in X$.

Step C We have a subset $\mathcal{P}$ of $\wp_2(X)$ distinguished, and after that

if $\{i,j\} \in \mathcal{P}$: we draw lines $A_{i,j} = \overline{a_i, a_j}$ and $B_{i,j} = \overline{b_i, b_j}$; the point $c_{\{i,j\}}$ is common for $A_{i,j}$ and $B_{i,j}$,

if $\{i,j\} \in \wp_2(X) \setminus \mathcal{P}$: we draw lines $G_{i,j} = \overline{a_i, b_j}$; the point $c_{\{i,j\}}$ is common for $G_{i,j}$ and $G_{j,i}$,

for every $\{i,j\} \in \wp_2(X)$. It is seen that the point p and the points a_i, b_i ($i \in X$) have degree n, while (up to now) c_z with $z \in \wp_2(X)$ has degree 2. Moreover, the number of the points c_z is $\binom{n}{2}$.
The quadruple of lines $(L_i, L_j, A_{i,j}, B_{i,j})$ ($(L_i, L_j, G_{i,j}, G_{j,i})$ resp.) with any distinct $i, j \in X$ yields a classical Veblen configuration.

Step D Let $\mathfrak{H}$ be any $\left(\binom{n}{2}_{n-2}, \binom{n}{3}_3\right)$-configuration. Finally, we identify the points c_z constructed above with points of $\mathfrak{H}$ (under some bijection γ) and, consequently, we group the points c_z into $\binom{n}{3}$ new lines obtained as coimages of the lines of $\mathfrak{H}$ under γ.

The resulting configuration will be written as $\mathcal{M}^p_X \triangleright^{\gamma}_{\mathcal{P}} \mathfrak{H}$. If the point set of $\mathfrak{H}$ is $\wp_2(X)$ it is natural to put $\gamma\colon c_{\{i,j\}} \longmapsto \{i,j\}$; comparing with 1.1 we see that $\mathcal{M}^p_X \triangleright^{\gamma}_{\mathcal{P}} \mathfrak{H} \cong \mathcal{M}^p_X \triangleright_{\mathcal{P}} \mathfrak{H}$.
The above interpretation justifies the term *multiveblen* used to name structures of the form $\mathcal{M}^p_X \triangleright_{\mathcal{P}} \mathfrak{H}$.

A multiplied multiveblen configuration (more precisely: a multiplied simple multiveblen configuration) is any structure of the form

$$\mathfrak{M} = \mathcal{M}^{p^k}_{X_{k-1}} \triangleright_{\mathcal{P}_{k-1}} \dots (\mathcal{M}^{p^2}_{X_1} \triangleright_{\mathcal{P}_1} (\mathcal{M}^{p^1}_{X_0} \triangleright_{\mathcal{P}_0} \mathbf{G}_2(X_0))), \tag{1}$$

where p^j are two-element sets, the set X_0 and the p^j are pairwise disjoint, $X_j = X_{j-1} \cup p^j$ for $j = 1, \dots, k$, and $\mathcal{P}_j$ is a graph defined on X_j for $j = 0, \dots, k-1$.

In most parts we shall consider a "standard" representation of the structure defined by (1) taking $X_0 := \{1, \dots, m\}$ ($m \geq 2$) and $p^j := \{m + 2j, m + 2j - 1\}$.

The structure $\mathfrak{M}$ of the form (1) is a simple multiveblen configuration if $k = 1$.

Let $\mathcal{P}', \mathcal{P}''$ be two graphs on a set X. We write $\mathcal{P}' \approx \mathcal{P}''$ if and only if there is a sequence $\mu_{x_1}, \dots, \mu_{x_s}$ of maps ($x_1, \dots, x_s \in X$) such that the composition $\mu_{x_1} \dots \mu_{x_s}$ maps $\mathcal{P}'$ onto $\mathcal{P}''$ and every μ_{x_j} switches connections with the vertex x_j: x_j, y are connected in $\mu_{x_j}(\mathcal{P})$ if and only if they are not connected in $\mathcal{P}$, for an arbitrary graph $\mathcal{P}$ and $y \in X$.

Fact 1.3 ((cf. [11, Prop. 9])). *If $\mathfrak{H}$ is any partial Steiner triple system defined on the set $\wp_2(X)$, p is a two-element set disjoint with X, and $\mathcal{P}'$, $\mathcal{P}''$ are graphs defined on X then $\mathcal{P}' \approx \mathcal{P}''$ yields $\mathcal{M}^p_X \triangleright_{\mathcal{P}'} \mathfrak{H} \cong \mathcal{M}^p_X \triangleright_{\mathcal{P}''} \mathfrak{H}$.*

A suitable converse variant of 1.3 is also provable:

Theorem 1.4. *Let $|X| \geq 5$. If there is an isomorphism of $\mathcal{M}^{p^1}_X \triangleright_{\mathcal{P}_1} \mathfrak{H}_1$ onto $\mathcal{M}^{p^2}_X \triangleright_{\mathcal{P}_2} \mathfrak{H}_2$ which maps p^1 onto p^2 then $\mathfrak{H}_1 \cong \mathfrak{H}_2$ and then there is a graph $\mathcal{P}_3$ such that $\mathcal{P}_1 \approx \mathcal{P}_3 \cong \mathcal{P}_2$. If $\mathcal{M}^{p'}_X \triangleright_{\mathcal{P}_1} \mathbf{G}_2(X) \cong \mathcal{M}^{p''}_X \triangleright_{\mathcal{P}_2} \mathbf{G}_2(X)$ then there is a graph $\mathcal{P}_3$ such that $\mathcal{P}_1 \approx \mathcal{P}_3 \cong \mathcal{P}_2$.*

A detailed classification of the simple multiveblen configurations $\mathcal{M}^p_X \triangleright_{\mathcal{P}} \mathbf{G}_2(X)$ with $|X| \leq 5$ is presented in [11]. One result of that investigations will also be used here:

Fact 1.5. *Let $\mathcal{P}$ be a graph on a set Y and $X \in \wp_4(Y)$. Then $\mathcal{P} \curlywedge X \approx N_4$, $\mathcal{P} \curlywedge X \approx K_4$, or $\mathcal{P} \curlywedge X \approx L_4$.*

2 Projective embeddings

In what follows by a *projective embedding* of a configuration $\mathfrak{K} = \langle S, \mathcal{L} \rangle$ or an *embedding of $\mathfrak{K}$ into a projective space $\mathfrak{P}$* we mean an injective map which associates with the elements of S points of $\mathfrak{P}$ and with the elements of $\mathcal{L}$ lines of $\mathfrak{P}$ and which preserves (in both directions) the incidence (comp. [4]). As a rule, in the sequel we consider only embeddings into Desarguesian spaces.

Clearly, $\mathbf{G}_2(n) \cong \mathcal{M}_{n-2} \triangleright_{K_{n-2}} \mathbf{G}_2(n-2)$ has a standard projective embedding into $PG(n,2)$ (much interesting information on projective embeddings of the structures $\mathbf{G}_2(n)$ can also be found in [2]).

Now, we continue some remarks concerning projective embeddings of the structure $\mathfrak{V}^o = \mathfrak{B}(3)$ given in [9].

Fact 2.1 ([9, Prop. 5.3])**.** *If $\mathfrak{V}^o$ is embedded into a projective space $\mathfrak{P}$ then $\mathfrak{V}^o$ lies on a plane of $\mathfrak{P}$.*

Lemma 2.2. *Let us assume that the configuration $\mathfrak{V}^o$ is embedded into a projective space. Then the lines $\overline{a_i, a_j}$ and $\overline{b_i, b_j}$ meet on the axis of $\mathfrak{V}^o$ for every $1 \le i < j \le 3$.*

Proof. From [9, Prop. 5.3], $\mathfrak{V}^o$ lies on a projective plane Π and one of the following holds:

a) There are lines A, B of Π such that a_1, a_2, a_3 lie on A and b_1, b_2, b_3 lie on B.
b) There is a conic $\mathcal{S}$ in Π such that the a_i and the b_j lie on $\mathcal{S}$.

In the corresponding cases we prove our claim as follows:
a) Let $\{i,j,l\} = \{1,2,3\}$. From the Desargues theorem applied to the triangles (a_i, b_l, a_j) and (b_i, a_l, b_j) we infer that the lines $\overline{a_i, a_j}$, $\overline{b_i, b_j}$ meet on the line $\overline{c_{\{i,l\}}, c_{\{l,j\}}}$, which is the axis of $\mathfrak{V}^o$.
b) The point p is the pole of the axis L of $\mathfrak{V}^o$ conjugated under $\mathcal{S}$ i.e. it is the center of the harmonic homology f with axis L, which leaves $\mathcal{S}$ invariant. Since f interchanges the lines $\overline{a_i, a_j}$ and $\overline{b_i, b_j}$, these lines meet on the axis of f, as required. ∎

Proposition 2.3. *Let $\mathcal{P}$ be a graph on n-element set X, let $Y \in \wp_4(X)$ such that $\mathcal{P} \curlywedge Y \approx L_4$ or $\mathcal{P} \curlywedge Y \approx N_4$. Then there is no projective embedding of the structure $\mathcal{M}^p_X \triangleright_{\mathcal{P}} \mathbf{G}_2(X)$ (in short: of $\mathcal{M}_n \triangleright_{\mathcal{P}} \mathbf{G}_2(n)$).*[5]

Proof. It suffices to prove that there is no projective embedding of the structure $\mathfrak{B} := \mathcal{M}^p_{\{1,2,3,4\}} \triangleright_{\mathcal{P}'} \mathbf{G}_2(\{1,2,3,4\})$, where $\mathcal{P}' = L_4 = \{\{1,2\},\{2,3\},\{3,4\}\}$ or $\mathcal{P}' = N_{\{1,2,3,4\}}$. It is seen that $\mathfrak{B}$ contains two $\mathfrak{V}^o$-configurations $\mathcal{V}_1, \mathcal{V}_2$ with the common center p spanned by the lines L_2, L_3, L_1 and L_2, L_3, L_4 respectively.

Suppose that $\mathfrak{B}$ is embedded into a projective space. From 2.1, $\mathcal{V}_1, \mathcal{V}_2$ lie in corresponding planes Π_1, Π_2 which now have two distinct lines L_2, L_3 in common; thus $\Pi_1 = \Pi_2 =: \Pi$. Let M_i be the axis of $\mathcal{V}_i$. From the definition, M_i passes through $c_{\{2,3\}}$ and from 2.2 we get that M_i passes through the common point of the lines $\overline{a_2, a_3}$, $\overline{b_2, b_3}$. Consequently, $M_1 = M_2$, which does not hold in $\mathfrak{B}$. ∎

[5] Note, that in the case of $\mathcal{P} = N_X$ Proposition 2.3 is also a direct consequence of [9, Theorem 5.10].

The following technical lemma will be useful in the sequel.

Lemma 2.4. *Let $\mathcal{P}$ be the graph on the n-element set X such that $\mathcal{P} \curlywedge A \approx K_3$ for every $A \in \wp_3(X)$. Then $\mathcal{P} \approx K_n$ and $\mathcal{M}_n \rhd_{\mathcal{P}} \mathbf{G}_2(n) \cong \mathbf{G}_2(n+2)$.*

Proof. Suppose that $\mathcal{P} = \emptyset$; then $\mathcal{P} \curlywedge A \approx N_3$ which contradicts assumptions. Thus there is an edge (say $e = \{1,2\}$) of $\mathcal{P}$. For arbitrary $i \in X \setminus e$, since $\mathcal{P} \curlywedge (e \cup \{i\}) \approx K_3$, either $\{1,i\},\{2,i\} \in \mathcal{P}$ or $\{1,i\},\{2,i\} \notin \mathcal{P}$. Let us set

$$X^+ := \big\{i\colon \{1,i\} \in \mathcal{P}\big\} \text{ and } X^+ := \big\{i\colon \{1,i\} \notin \mathcal{P}\big\}.$$

Observing triples $\{i,j,1\}$ with $i,j \in X \setminus e$ we get

$$\begin{aligned} i,j \in X^+ \text{ or } i,j \in X^- \quad &\Longrightarrow \quad \{i,j\} \in \mathcal{P} \\ i \in X^+,\ j \in X^- \quad &\Longrightarrow \quad \{i,j\} \notin \mathcal{P}. \end{aligned}$$

The composition of all the maps μ_i with $i \in X^-$ transforms $\mathcal{P}$ onto K_n. ∎

Now we are in a position to prove a first important result:

Theorem 2.5. *Let $\mathcal{P}$ be an arbitrary graph on n ($n \geq 4$) vertices such that $\mathfrak{B} := \mathcal{M}_n \rhd_{\mathcal{P}} \mathbf{G}_2(n) \ncong \mathbf{G}_2(n+2)$. Then $\mathfrak{B}$ cannot be embedded into a projective space.*

Proof. Let $X = \{1,\dots,n\}$ be the set of the vertices of $\mathcal{P}$. From the assumption and 2.4, there is $A \in \wp_3(X)$ such that $\mathcal{P}_0 := \mathcal{P} \curlywedge A \not\approx K_3$ and thus $\mathcal{P}_0 \approx N_3$. Without loss of generality we can assume that $A = \{1,2,3\}$ and

$$\mathcal{P}_0 = \big\{\{1,2\},\{1,3\}\big\} \text{ or } \mathcal{P}_0 = \emptyset.$$

Applying μ_1, if necessary, we assume that $\mathcal{P}_0 = \big\{\{1,2\},\{1,3\}\big\}$.

Observe $A' = A \cup \{4\}$ and $\mathcal{P}' = \mathcal{P} \curlywedge A'$. Note that a graph equivalent to K_4 defined on A' is one of the following: K_4, a triangle C_3 embedded into A', or consisting of two disjoint edges. Restriction of no one of them is the path $\mathcal{P}_0$ and thus $\mathcal{P}' \approx N_4$ or $\mathcal{P}' \approx L_4$. In any case from 2.3 and 2.3 we get that the substructure $\mathcal{M}_4 \rhd_{\mathcal{P}'} \mathbf{G}_2(A')$ of $\mathfrak{B}$ cannot be projectively embedded. ∎

The result of 2.5 can also be read as follows:

Corollary 2.6. *Let $\mathfrak{B}$ be a simple multiveblen configuration with point degree at least 4. Then $\mathfrak{B}$ can be embedded into a projective space if and only if $\mathfrak{B}$ is a generalized Desargues configuration (a combinatorial Grassmannian).*

A direct analogue of 2.6 for multiplied multiveblen configurations does not hold.

Proposition 2.7. *The structure*

$$\mathfrak{M} = \mathcal{M}^{\{5,6\}}_{\{1,2,3,4\}} \rhd_{\mathcal{P}} \mathbf{\mathfrak{B}}(2), \text{ where } \mathbf{\mathfrak{B}}(2) = \mathcal{M}^{\{3,4\}}_{\{1,2\}} \rhd_{N_{\{1,2\}}} \mathbf{G}_2(\{1,2\}).$$

can be embedded into a Desarguesian projective space if and only if $\mathcal{P}$ is equivalent to the linear graph $\big\{\{1,2\},\{1,3\},\{2,4\}\big\}$. If $\mathfrak{M}$ is embedded into a projective space then $\overline{a_3,a_4}$ and $\overline{b_3,b_4}$ pass through $c_{\{1,2\}}$ as well.

Proof. From the definition, the following triples

$$V_1 := \{\{3,4\},\{1,3\},\{1,4\}\},\ V_2 := \{\{3,4\},\{2,3\},\{2,4\}\},$$
$$V_3 := \{\{1,2\},\{1,3\},\{2,4\}\},\ V_4 := \{\{1,2\},\{1,4\},\{2,3\}\}.$$

are the lines of the Veblen configuration $\mathfrak{B}(2)$.

Let $\mathcal{D}_i$ be the restriction of $\mathfrak{M}$ spanned by the lines L_i, L_3, L_4 for $i = 1,2$. Then $\mathcal{D}_i$ is either a Desargues or a Veronese $\mathfrak{V}^o$ configuration with the axis V_i.

Let us assume that $\mathfrak{M}$ is embedded into a projective Desarguesian space $\mathfrak{P}$. For every two distinct i_1, i_2 in $\{1,2,3,4\}$ we have in $\mathfrak{M}$ a Veblen configuration inscribed into the lines L_{i_1}, L_{i_2} and the point $c_{\{i_1,i_2\}}$ is the point of intersection of the corresponding lines of $\mathfrak{M}$. Let us denote by $c^*_{\{i_1,i_2\}}$ the intersection point (considered in $\mathfrak{P}$) of the lines

- $\overline{a_{i_1}, b_{i_2}}, \overline{b_{i_1}, a_{i_2}}$ when $\{i_1, i_2\} \in \mathcal{P}$ and of
- $\overline{a_{i_1}, a_{i_2}}, \overline{b_{i_1}, b_{i_2}}$ when $\{i_1, i_2\} \notin \mathcal{P}$.

The point $c_{\{1,2\}}$ is determined by the structure of $\mathfrak{B}(2)$ as the intersection point of the lines $\overline{c_{\{1,3\}}, c_{\{2,4\}}} = V_3$ and $\overline{c_{\{1,4\}}, c_{\{2,3\}}} = V_4$. In any case two possibilities arise:

(a) $c_{\{1,2\}} \mathrel{I} \overline{a_1, a_2}, \overline{b_1, b_2}$ (i.e. $\{1,2\} \in \mathcal{P}$), or

(b) $c_{\{1,2\}} \mathrel{I} \overline{a_1, b_2}, \overline{b_1, a_2}$ (i.e. $\{1,2\} \notin \mathcal{P}$).

Step 1: Assume, first, that $\mathcal{D}_1$ and $\mathcal{D}_2$ both are Desargues configurations. Without loss of generality we can assume that the given Desargues configurations represent a perspective of the triangles (a_1, b_3, a_4), (b_1, a_3, b_4) – in $\mathcal{D}_1$ and (a_2, b_4, a_3), (b_2, a_4, b_3) – in $\mathcal{D}_2$. Consequently, in this case

$\{1,4\}, \{2,3\}$ are in $\mathcal{P}$, $\quad \{1,3\}, \{3,4\}, \{2,4\}$ are not in $\mathcal{P}$.

Let us analyze the two possibilities (a) and (b).

Ad (a): Applying several times the Desargues axiom we get $c^*_{\{3,4\}}, c_{\{1,2\}} \mathrel{I} V_3, V_4$ and thus the equality $c^*_{\{3,4\}} = c_{\{1,2\}}$ must hold.

Ad (b): From the Desargues axiom applied to the triangles a_1, b_2, b_3 and b_1, a_2, a_3 we obtain $\mathbf{L}(c_{\{1,2\}}, c_{\{1,3\}}, c_{\{2,3\}})$, which gives $c_{\{2,3\}} \mathrel{I} V_3$. Thus $c_{\{1,2\}}, c_{\{2,3\}} \mathrel{I} V_3, V_4$ yields, contradictory, $V_3 = V_4$.

Step 2: Next, assume that $\mathcal{D}_1$ is a Desargues configuration and $\mathcal{D}_2$ is a Veronese configuration. Without loss of generality we can label the points on the lines L_i in such a way that $\mathcal{D}_1$ represents the perspective of the triangles (a_1, a_3, b_4) and (b_1, b_3, a_4), and $\mathcal{D}_2$ contains the closed hexagon $(a_2, b_4, a_3, b_2, a_4, b_3)$. In this case

$\{1,3\}$ is in $\mathcal{P}$, $\quad \{2,3\}, \{3,4\}, \{2,4\}, \{1,4\}$ are not in $\mathcal{P}$.

Let us analyze the two possibilities (a) and (b).

Ad (a): Note that the points a_1, a_2, b_4 are not collinear (otherwise an extra incidence $b_4 \mathrel{I} \overline{a_1, a_2}$ holds in $\mathfrak{M}$) and, analogously, b_1, b_2, a_4 are not collinear. From the Desargues axiom we have $\mathbf{L}(c_{\{1,2\}}, c_{\{2,4\}}, c_{\{1,4\}})$, so $c_{\{2,4\}} \mathrel{I} V_4$ which gives $V_3 = V_4$.

Ad (b): Since $b_2 \not\mathrel{I} \overline{a_1, a_3}$ the points a_1, a_3, b_2 are not collinear. From the Desargues axiom we infer that $\mathbf{L}(c_{\{1,3\}}, c_{\{2,3\}}, c_{\{1,2\}})$; then $c_{\{2,3\}} \mathrel{I} V_3$ and thus $V_3 = V_4$.

Step 3: Finally, let us assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ both are Veronese configurations. From 2.2 we get that $c_{\{3,4\}}, c^*_{\{3,4\}} \mid V_1, V_2$, which yields a contradiction.

Therefore, only in the case analyzed in Step 1 we can expect that $\mathfrak{M}$ can be embedded into a projective space. On the other hand, let $\mathfrak{P}$ be the projective 3-space over a field with characteristic $\neq 2$ and let β be a scalar with $\beta \neq 1, -1$. It is a matter of a simple computation that the following map embeds $\mathfrak{M}$ into $\mathfrak{P}$:

$$
\begin{gathered}
p \longmapsto [1,0,0,0], \\
a_1 \longmapsto [1,1,-1,1], a_2 \longmapsto [1,1,1,1], a_3 \longmapsto [1,1,-1,-1], a_4 \longmapsto [1,1,1,-1], \\
b_1 \longmapsto [1,\beta,-\beta,\beta], b_2 \longmapsto [1,\beta,\beta,\beta], b_3 \longmapsto [1,\beta,-\beta,-\beta], b_4 \longmapsto [1,\beta,\beta,-\beta], \\
c_{\{1,2\}} \longmapsto [0,0,1,0], c_{\{2,3\}} \longmapsto [0,0,1,1], c_{\{1,4\}} \longmapsto [0,0,1,-1], \\
c_{\{1,3\}} \longmapsto [1+\beta, 2\beta, -2\beta, 0], \\
c_{\{2,4\}} \longmapsto [1+\beta, 2\beta, 2\beta, 0], c_{\{3,4\}} \longmapsto [1+\beta, 2\beta, 0, -2\beta]. \quad (2)
\end{gathered}
$$

provided that $\mathcal{P}$ consists of the edges $\{1,2\}$, $\{1,4\}$, and $\{2,3\}$. ∎

Let $\mathfrak{R}_4$ be the projectively embeddable $(15_4\,20_3)$-multiveblen configuration constructed in the proof of 2.7. It is evident that $\mathfrak{R}_4$ is not isomorphic to $\mathbf{G}_2(6)$, the second projectively embeddable $(15_4\,20_3)$-multiveblen configuration. A picture of $\mathfrak{R}_4$ is presented in Figure 2.

Generally, investigations on general (iterated) multiveblen configurations are much more complex. The main reason is that a representation of a multiveblen configuration $\mathfrak{M}$ in the form (1) is not unique in the sense that $\mathfrak{M}$ does not determine k, nor $|X_0|$, nor the $\mathcal{P}_i$. Let us point out three simple examples. Let p, q, r be pairwise disjoint two-element sets.

$\mathcal{M}^r_{p\cup q}\triangleright_{K_{p\cup q}}(\mathcal{M}^q_p\triangleright_{K_p}\mathbf{G}_2(p)) \cong \mathcal{M}^r_{p\cup q}\triangleright_{K_{p\cup q}}\mathbf{G}_2(p\cup q)$

– k in (1) is not determined by the configuration $\mathfrak{M}$.

$\mathcal{M}^r_{p\cup q}\triangleright_{K_{p\cup q}}\mathfrak{B}(p) \cong \mathcal{M}^r_{p\cup q}\triangleright_{K_{p\cup q}}(\mathcal{M}^q_p\triangleright_{N_p}\mathbf{G}_2(p)) \cong \mathcal{M}^r_{p\cup q}\triangleright_{L_4}\mathbf{G}_2(p\cup q)$

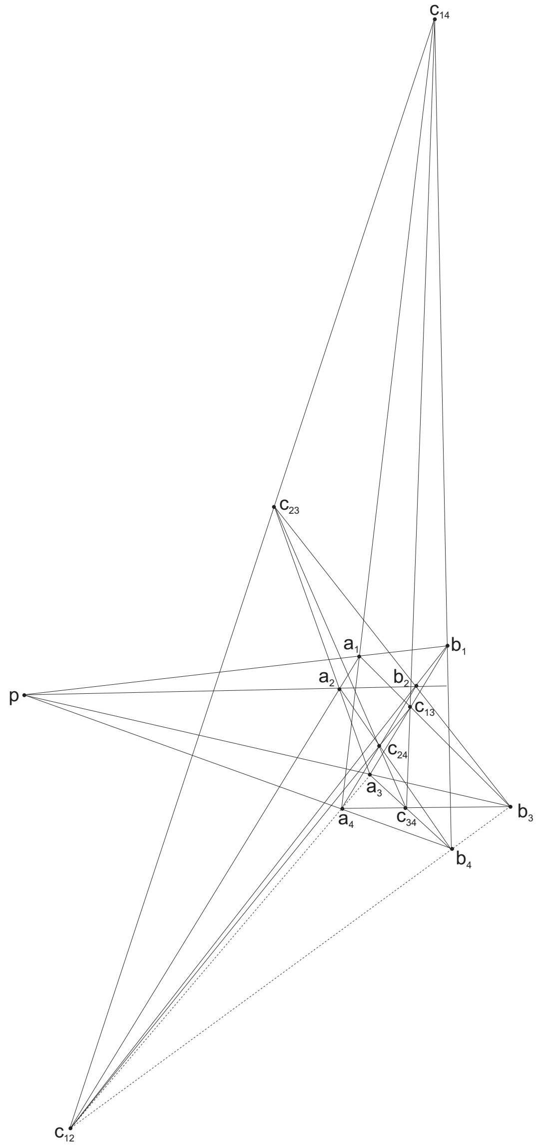
(cf. [11, Prop. 18]) – a multiveblen configuration $\mathcal{M}^r_X\triangleright_{\mathcal{P}}\mathfrak{H}$ defined in 1.1 does not determine its consistency graph $\mathcal{P}$ and its axial configuration $\mathfrak{H}$.

$$\mathcal{M}^r_{p\cup q}\triangleright_{K_{p\cup q}}(\mathcal{M}^q_p\triangleright_{N_p}\mathbf{G}_2(p)) \cong \mathcal{M}^r_{p\cup q}\triangleright_{K_{p\cup q}}\mathfrak{B}(p) \not\cong$$
$$\not\cong \mathbf{G}_2(p\cup q\cup r) \cong \mathcal{M}^r_{p\cup q}\triangleright_{K_{p\cup q}}(\mathcal{M}^q_p\triangleright_{K_p}\mathbf{G}_2(p))$$

even though $N_p \approx K_p$. This yields, in particular, that even if $\mathcal{P}'_0 \approx \mathcal{P}''_0$ the two structures $\mathcal{M}^r_{X_0\cup q}\triangleright_{\mathcal{P}_1}\mathcal{M}^q_{X_0}\triangleright_{\mathcal{P}'_0}\mathbf{G}_2(X_0)$ and $\mathcal{M}^r_{X_0\cup q}\triangleright_{\mathcal{P}_1}\mathcal{M}^q_{X_0}\triangleright_{\mathcal{P}''_0}\mathbf{G}_2(X_0)$ may stay nonisomorphic (comp. 1.3).

Let us say that a multiveblen configuration $\mathfrak{M}$ is *regular* if and only if it can be represented in the form (1), where $\mathcal{P}_{i-1} = \mathcal{P}_i \curlywedge X_{i-1}$ for $i = 1, \ldots, k-1$. To determine possible projective embeddings of regular multiveblen configurations we use intensively the following lemma, which follows from 2.7 and 2.3.

Lemma 2.8. *Let $\mathfrak{M} = \mathcal{M}^q_{X\cup p}\triangleright_{\mathcal{P}_1}(\mathcal{M}^p_X\triangleright_{\mathcal{P}_0}\mathbf{G}_2(X))$, where $|X| \geq 2$, p, q are two-element sets such that X, p, q are pairwise disjoint, $\mathcal{P}_1$ is a graph on $X\cup p$, and $\mathcal{P}_0 = \mathcal{P}_1 \curlywedge X$. Assume that $\mathfrak{M}$ has a projective embedding. Then the following holds:*



We see that while $\mathbf{G}_2(6)$ represents a perspective of two tetrahedrons (cf. [10]), the configuration $\mathfrak{R}_4$ also represents a perspective, a perspective of some kind of two 4-tuples (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4). A perspective of the same type can be seen with the point $c_{\{3,4\}}$ as the perspective center.
Note also that the Desargues axiom applied to this configuration forces the line which joins intersection points of the lines in pairs $(\overline{a_1, a_3}, \overline{a_2, a_4})$ and $(\overline{b_1, b_3}, \overline{b_2, b_4})$ to pass through p.

Figure 2: The projectively embeddable $\left(15_4\, 20_3\right)$-multiveblen configuration $\mathfrak{R}_4$, constructed in the proof of 2.7.

(i) $|\wp_2(X) \setminus \mathcal{P}_0| \leq 1$ *(i.e. either $\mathcal{P}_0$ is the complete graph K_X or it is K_X with exactly one edge deleted).*

(ii) *If* $|\wp_2(X) \setminus \mathcal{P}_0| = 1$ *then* $|X| \leq 3$.

(iii) *One of the following three conditions holds:*

1. *$\mathcal{P}_1$ contains p and every pair $\{s, t\}$ with $s \in p$ and $t \in X$,*
2. *$\mathcal{P}_1$ contains p and $\{s, t\} \notin \mathcal{P}_1$ for every $s \in p$, $t \in X$,*
3. *$p \notin \mathcal{P}_1$, $\{i_2, t\} \notin \mathcal{P}_1$ and $\{i_1, t\} \in \mathcal{P}_1$ for every $t \in X$, where $p = \{i_1, i_2\}$.*

In every one of the above three cases either $\mathcal{P}_1 \approx K_{X \cup p}$ or $\mathcal{P}_1$ is equivalent to $K_{X \cup p}$ with one edge (taken from $\wp_2(X)$) deleted.

Keeping in mind the equality $\mathbf{G}_2(X \cup p) \cong \mathcal{M}^{p}_{X} \triangleright_{K_X} \mathbf{G}_2(X)$ we see that every combinatorial Grassmannian $\mathbf{G}_2(X)$ can be presented in the form (1), where either $|X_0| = 2$ and then the series (1) begins with a single point $\mathbf{G}_2(X_0) = \mathbf{G}_2(2)$, or $|X_0| = 3$ and then (1) begins with a single line $\mathbf{G}_2(X_0) = \mathbf{G}_2(3)$.

Let $n \geq 2$ be an integer. We define the regular multiveblen configuration $\mathfrak{R}_n$ as follows. If $n = 2k$ for some integer k then we take $m = 2$, if $n = 2k+1$ we set $m = 3$; in both cases we obtain $n = m + 2(k-1)$. Let $q = \{1,2\}$, $X = \{1,\dots,m\}$, $p^j = \{m+2j-1, m+2j\}$ for $j = 1,\dots,k$; in particular, $p^k = \{m+2k-1, m+2k\}$. Let $\mathcal{P} = \wp_2(X \cup p^1 \cup \dots \cup p^{k-1}) \setminus \{q\}$. If $m = 2$ then $\mathcal{P}_0 := \mathcal{P} \curlywedge X = N_X$; if $m = 3$ then $\mathcal{P}_0 \approx N_X$. Let us put $\mathcal{P}_j = \mathcal{P} \curlywedge (X \cup p^1 \cup \dots \cup p^j)$ for $0 < j < k$. We set

$$\mathfrak{R}_n := \mathcal{M}^{p^k}_{X\cup p^1 \cup \dots \cup p^{k-1}} \triangleright_{\mathcal{P}_{k-1}} \Big(\dots \big(\mathcal{M}^{p^2}_{X \cup p^1} \triangleright_{\mathcal{P}_1} (\mathcal{M}^{p^1}_X \triangleright_{\mathcal{P}_0} \mathbf{G}_2(X)) \dots \Big).$$

In particular, $\mathfrak{R}_2$ is simply $\mathfrak{B}(q)$, $\mathfrak{R}_3 \cong \mathfrak{B}(3)$ is the Veronese configuration $\mathfrak{V}^o$, and (up to an isomorphism) $\mathfrak{R}_4$ is the configuration on Figure 2, defined in the proof of 2.7. In general, $\mathfrak{R}_n$ is a $\big(\binom{n+2}{2}_n \binom{n+2}{3}_3\big)$-configuration. Intuitively, we can write $\mathfrak{R}_{n+2} = \mathcal{M}_n \triangleright_{\mathcal{P}_{n-1}} \mathfrak{R}_n$.

Construction 2.9. Let $X = \{1,2,3\}$, $r = \{1,2\}$, $p^1 = p = \{4,5\}$, $p^2 = q = \{6,7\}$, and let $\mathcal{P}_1 = \wp_2(X \cup p) \setminus r$ be a graph on $X \cup p$; let $\mathcal{P}_0 = \mathcal{P}_1 \curlywedge X$. Evidently $\mathcal{P}_0 \approx N_X$ and thus $\mathfrak{M}_1 := \mathcal{M}^p_X \triangleright_{\mathcal{P}_0} \mathbf{G}_2(X) \cong \mathfrak{V}^o$ is a Veronese configuration. By definition, $\mathfrak{R}_5 = \mathcal{M}^q_{X\cup p} \triangleright_{\mathcal{P}_1} \mathfrak{M}_1$. Next, let $\mathfrak{P}$ be the projective 3-space over a field with characteristic unequal to two and let β, γ be two scalars such that $\gamma, \beta \neq 1, -1, 0$ and $\gamma \neq 2, -2$. Let us consider the following map F defined on the points of $\mathfrak{R}_5$:

$$\begin{array}{l}
q \longmapsto [1,0,0,0], \quad q' = p = c_{\{4,5\}} \longmapsto [1+\beta, 2\beta, 0, -2\beta], \\
a_1 \longmapsto [1,1,-1,1], \ \ b_1 \longmapsto [1,\beta,-\beta,\beta], \ \ a_2 \longmapsto [1,\beta,\beta,\beta], \ \ b_2 \longmapsto [1,1,1,1], \\
b_4 \longmapsto [1,1,-1,-1], a_4 \longmapsto [1,\beta,-\beta,-\beta], b_5 \longmapsto [1,\beta,\beta,-\beta], a_5 \longmapsto [1,1,1,-1], \\
c'_{\{1,2\}} := c_{\{1,2\}} \longmapsto [0,0,1,0], \\
a'_2 := c_{\{2,4\}} \longmapsto [0,0,1,1], \ \ b'_1 := c_{\{1,5\}} \longmapsto [0,0,1,-1], \\
a'_1 := c_{\{1,4\}} \longmapsto [1+\beta, 2\beta, -2\beta, 0], \quad b'_2 := c_{\{2,5\}} \longmapsto [1+\beta, 2\beta, 2\beta, 0], \\
a_3 \longmapsto [1, \gamma u, u, \gamma u], \quad b_3 \longmapsto [1, \gamma v, v, \gamma v], \\
\text{ where } u = \frac{2\beta}{\gamma(1+\beta)+\beta-1}, v = \frac{2\beta}{\gamma(1+\beta)-\beta+1}; \\
\text{ set } \mu := \beta u - v, \ \lambda := \beta v - u, \ \beta' := \beta - 1, \ \omega := \beta(u-v) \text{ (then } \beta(v-u) = -\omega) \\
c'_{\{1,3\}} := c_{\{1,3\}} \longmapsto [\mu, \beta'\gamma uv + \omega, \beta' uv - \omega, \beta'\gamma uv + \omega], \\
c'_{\{2,3\}} := c_{\{2,3\}} \longmapsto [\lambda, \beta'\gamma uv - \omega, \beta' uv - \omega, \beta'\gamma uv - \omega], \\
a'_3 := c_{\{3,4\}} \longmapsto [\lambda, \beta'\gamma uv - \omega, \beta' uv + \omega, \beta'\gamma uv + \omega], \\
b'_3 := c_{\{3,5\}} \longmapsto [\mu, \beta'\gamma uv + \omega, \beta' uv + \omega, \beta'\gamma uv - \omega].
\end{array} \tag{3}$$

A direct though tedious computation shows that the map F embeds $\mathfrak{R}_5$ into $\mathfrak{P}$. Thus our construction proves that *the* $(21_5\, 35_3)$*-multiveblen configuration* $\mathfrak{R}_5$ *can be embedded into a projective space*.

The above embedding can be further extended to an embedding of $\mathfrak{R}_7$ (its soundness was verified with the help of Maple V, $\rho \neq 1, -1, 2, -2, \frac{u}{v}, \frac{v}{u}, \beta, \ \beta\rho \neq 1$, and similar); we put:

$$\begin{array}{ll}
F(a_7) = [1, w_1, w_2, w_3], & F(b_7) = [1, \rho w_1, \rho w_2, \rho w_3], \\
F(b_6) = [1, w_1, -w_2, w_3], & F(a_6) = [1, \rho w_1, -\rho w_2, \rho w_3].
\end{array}$$

After that the coordinates of the points $F(c_{\{i,j\}})$ with $i \in \{6,7\}$ can be directly computed.

We see that the above procedure can be continued by adding suitable pairs $a_8, b_8,\ a_9, b_9,\ \ldots\ a_{2k}, b_{2k},\ a_{2k+1}, b_{2k+1}$ on lines through q, which should yield an embedding of $\mathfrak{R}_{2k+1}$ into $\mathfrak{P}$. $\bigcirc$

Construction 2.10. Now, we shall extend the embedding of $\mathfrak{R}_4$ given in the proof of 2.7 to a projective embedding of $\mathfrak{R}_6$. First, we note that the substructure of $\mathfrak{R}_5$ spanned by the points on the lines through q and a_1, a_2, a_4, a_5 is exactly $\mathfrak{R}_4$; its embedding (2) given in the proof of 2.7 differs from that of 2.9 in the ordering of some symbols in pairs a_i, b_i.

Let F be defined as in 2.9. Let us write $\sigma i = i$ for for $i \leq 2$ and $\sigma i = i+1$ for $2 < i \leq 6$ and let us label the points on the lines of $\mathfrak{R}_6$ through $p^3 = F(q)$ by a_i, b_i in a standard way. It is a matter of simple (computer-aided) computation that the map G defined by $G(a_i) = F(a_{\sigma i})$, $G(b_i) = F(b_{\sigma i})$, $G(c_{\{i,j\}}) = F(c_{\{\sigma i, \sigma j\}})$ is an embedding of $\mathfrak{R}_6$ into $\mathfrak{P}$.

We see that this embedding can be further extended to an embedding of $\mathfrak{R}_{2k}$ to $\mathfrak{P}$. $\bigcirc$

Finally we obtain our main result

Theorem 2.11. *Let $\mathfrak{M}$ be a regular multiveblen configuration. If $\mathfrak{M}$ can be embedded into a projective space then either $\mathfrak{M} \cong \mathbf{G}_2(n)$ or $\mathfrak{M} \cong \mathfrak{R}_n$ for some integer n.*

Proof. Let $\mathfrak{M}$ be defined by (1); we define inductively

$$\mathfrak{M}_0 := \mathbf{G}_2(X_0), \quad \mathfrak{M}_j := \mathcal{M}^{p^j}_{X_{j-1}} \triangleright_{\mathcal{P}_{j-1}} \mathfrak{M}_{j-1},$$

and then $\mathfrak{M} = \mathfrak{M}_k$. Recall that the structure $\mathfrak{M}_j$ is defined on $\wp_2(X_j)$. Let $m = |X_0|$ and let $n \geq 4$ be the degree of a point in $\mathfrak{M}$. From the construction, $|X_k| = m + 2k$; we have $n = (m+2k) - 2$ and thus $|X_k| = n + 2$. From the assumptions, the graph $\mathcal{P}_{k-1}$ determines all the graphs $\mathcal{P}_j$ with $j < k-1$.

Clearly, if $\mathfrak{M}_k$ can be projectively embedded, its subspace $\mathfrak{M}_{k-1}$ can be projectively embedded as well; continuing we obtain that $\mathfrak{M}_j$ can be projectively embedded for every $j = 0, \ldots, k-1$.

If $k = 1$ then for $m < 4$ our claim is evident: $\mathfrak{M}$ is either the Veblen configuration $\mathbf{G}_2(4)$ ($m = 2$), or the Desargues configuration $\mathbf{G}_2(5) = \mathcal{M}^{p^1}_{X_0} \triangleright_{K_{X_0}} \mathbf{G}_2(X_0) = \mathcal{M}_3 \triangleright_{K_3} \mathbf{G}_2(3)$, or the Veronese configuration $\mathfrak{V}^o = \mathcal{M}_3 \triangleright_{N_3} \mathbf{G}_2(3)$. If $m \geq 4$ from 2.5 we get that $\mathfrak{M}$ admits a projective embedding if and only if $\mathfrak{M} \cong \mathbf{G}_2(X_0 \cup p)$ for some two-element set p.

If $k > 1$ we apply consecutively 2.8 to determine possible $\mathcal{P}_j$ for $j = 0, \ldots, k-2$. Assume that there are distinct $i_0, j_0 \in X_0$ with $X' := \{i_0, j_0\} \notin \mathcal{P}_0$. In view of 2.8 we have $m = |X_0| \leq 3$; from 2.7, 2.8, and 2.9 we infer that either $\mathfrak{M}_2$ is $\mathfrak{R}_4$ ($m = 2$, $\mathfrak{M}_1 \cong \mathfrak{B}(2)$) or it is $\mathfrak{R}_5$ ($m = 3$, $\mathfrak{M}_1 \cong \mathfrak{V}^o$). In both cases $X' \notin \mathcal{P}_l$ for $l > 0$ but $|X_l| > 3$ so, in view of 2.8, the graph $\mathcal{P}_l$ has the form $\wp_2(X_l) \setminus X'$ and $\mathfrak{M}_l = \mathfrak{R}_{m+2l}$. (In accordance with 2.9 and 2.10 structures $\mathfrak{M}_l$ with $l > 2$ can be projectively embedded and after all $\mathfrak{M}$ can be projectively embedded.)

Finally, suppose that there is a pair $s := \{i,j\}$ of distinct elements of X_l such that $s \notin \mathcal{P}_l$ for some $l > 0$ but $\mathcal{P}_{l'} = K_{X_{l'}}$ for all $l' < l$. This yields, in particular, that

$\mathfrak{M}_{l'+1} = \mathbf{G}_2(X_{l'})$ for every $l' < l$ and thus $\mathfrak{M}_l = \mathbf{G}_2(X_{l-1})$. If $l = k-1$ from 2.5 we obtain that necessarily $\mathfrak{M}_k \cong \mathbf{G}_2(X_l)$. Assume that $l < k-1$ and have a look at $\mathfrak{M}_{l+1}$. Applying 2.8 we obtain $|X_l| \leq 3$, which is impossible and thus $\mathcal{P}_l = K_{X_l}$. ∎

A Starting from another representation of the Veblen configuration

The construction of a multiveblen configuration as defined in (1) can also begin with the representation of the Veblen configuration $\mathfrak{H}$ in the form $\mathfrak{H} = \mathbf{G}_2^*(4) = \langle \wp_2(X), \wp_1(X), \supset \rangle$ (where $|X| = 4$, cf. [10]). To complete our results we prove

Proposition A.1. *Neither* $\mathcal{M}_4 \triangleright_{K_4} \mathbf{G}_2^*(4)$ *nor* $\mathcal{M}_4 \triangleright_{L_4} \mathbf{G}_2^*(4)$ *can be embedded into a Desarguesian projective space.*

Proof. Let $X = \{1,2,3,4\}$ and $\mathcal{P}$ be a graph defined on X such that $\mathcal{P} \approx K_4$ or $\mathcal{P} \approx L_4$. Say, $\mathcal{P} = K_X$ or $\mathcal{P} = \{\{1,2\},\{2,3\},\{3,4\}\}$. Consider $\mathfrak{M} = \mathcal{M}_X \triangleright_{\mathcal{P}} \mathbf{G}_2^*(X)$.

Suppose that $\mathfrak{M}$ is embedded into a Desarguesian projective space $\mathfrak{P}$. In any case $\mathcal{P} \curlywedge Y \approx K_Y$, where $Y = \{1,2,4\}$. Consequently, $\{1,2\}$, $\{1,4\}$ and $\{2,4\}$ are collinear in $\mathfrak{P}$. From the definition of $\mathbf{G}_2^*(X)$ the points $\{1,2\}$, $\{1,4\}$, $\{1,3\}$ are collinear as well and thus all the points of $\mathbf{G}_2^*(X)$ should lie on one line of $\mathfrak{P}$, which is impossible. ∎

Proposition A.2. *The structure* $\mathcal{M}_4 \triangleright_{N_4} \mathbf{G}_2^*(4)$ *can be embedded into a Desarguesian projective space.*[6]

Moreover, when $\mathcal{M}_4 \triangleright_{N_4} \mathbf{G}_2^*(4)$ *is embedded into a Desarguesian projective space* $\mathfrak{P}$ *then the characteristic of the coordinate field of* $\mathfrak{P}$ *is* 2, *that is* $\mathfrak{P}$ *satisfies the projective Fano axiom.*

Proof. Write $\mathfrak{R}_4^* := \mathcal{M}_X^p \triangleright_{N_X} \mathbf{G}_2^*(X)$, where $X = \{1,2,3,4\}$. Let us start with an analysis of possible embeddings of $\mathfrak{R}_4^*$ into a projective space $\mathfrak{P}$. As usual we write $c^*_{\{i,j\}}$ for the common point of $\overline{a_i, a_j}$, $\overline{b_i, b_j}$, which exists in $\mathfrak{P}$. Observing the triangles a_1, a_2, b_3 and b_1, b_2, a_3 of $\mathfrak{P}$ with the perspective center p we obtain that the points $c_{\{1,3\}}$, $c_{\{2,3\}}$ and $c^*_{\{1,2\}}$ are collinear; similarly, the points $c_{\{1,4\}}$, $c_{\{2,4\}}$, $c^*_{\{1,2\}}$ are collinear, which gives $c^*_{\{1,2\}} = c_{\{3,4\}}$. With the same technique we obtain

$$c_{\{i,j\}} = c^*_{\{i',j'\}} \text{ whenever } \{i,j,i',j'\} = X. \tag{4}$$

Therefore, the given points yield an embedding of $\mathcal{M}_X^p \triangleright_{K_X} \mathbf{G}_2(X)$ as well – it suffices to note that with $\varkappa(u) = X \setminus u$ for $u \in \wp(X)$ we obtain: $c_{u_1}, c_{u_2}, c_{u_3}$ is a line of $\mathbf{G}_2^*(X)$ if and only if $c^*_{u_1}, c^*_{u_2}, c^*_{u_3}$ is a line of $\mathbf{G}_2(X)$. And conversely, every projective embedding of $\mathcal{M}_X^p \triangleright_{K_X} \mathbf{G}_2(X)$ which satisfies (4) yields a projective embedding of $\mathfrak{R}_4^*$.

[6] From a more general perspective, the existence of the required embedding is a consequence of some results on linear completions of multiveblen configurations: $\mathcal{M}_4 \triangleright_{N_4} \mathbf{G}_2^*(4)$ and $\mathcal{M}_4 \triangleright_{K_4} \mathbf{G}_2(4) = \mathbf{G}_2(6)$ have the common linear completion: the projective Fano 3-space, cf. [12].

Next, since $c_{\{3,4\}} \mathrel{I} \overline{a_1, a_2}, \overline{a_3, b_4}$, $c_{\{2,4\}} \mathrel{I} \overline{a_2, b_4}, \overline{a_1, a_3}$, and $c_{\{1,4\}} \mathrel{I} \overline{a_1, b_4}, \overline{a_2, a_3}$ the points a_1, a_2, a_3, b_4 yield in $\mathfrak{P}$ a quadrangle with the diagonal points $c_{\{1,4\}}$, $c_{\{2,4\}}$, $c_{\{3,4\}}$, which are collinear and thus $\mathfrak{P}$ contains a closed Fano configuration.

Finally, to construct an embedding of $\mathfrak{R}^*_4$ into a Desarguesian projective space it suffices to take a closed Fano configuration in a projective space $\mathfrak{P}$ (thus coordinatized by a field with characteristic 2; to ensure that the procedure works one can assume that $\dim(\mathfrak{P}) > 2$), a point p not on a plane that contains this configuration, and an image of this Fano configuration under suitable homology. ∎

However, no series of projectively embeddable iterated multiveblen configurations may start from $\mathbf{G}^*_2(4)$.

Proposition A.3. *Let* $X_0 = \{1,2,3,4\}$, $p^1 = \{5,6\}$, $X_1 = X_0 \cup p^1$, *and* $p^2 = \{7,8\}$. *Assume that* $\mathcal{P}_0 \approx N_{X_0}$. *The structure* $\mathcal{M}^{p^2}_{X_1} \rhd_{\mathcal{P}_1} \mathfrak{R}^*_4 = \mathcal{M}^{p^2}_{X_1} \rhd_{\mathcal{P}_1} \big(\mathcal{M}^{p^1}_{X_0} \rhd_{\mathcal{P}_0} \mathbf{G}^*_2(X_0)\big)$ *cannot be embedded into a Desarguesian projective space for any graph* $\mathcal{P}_1$ *defined on* X_1.

Proof. Write $\mathfrak{M} = \mathcal{M}^{p^2}_{X_1} \rhd_{\mathcal{P}_1} \big(\mathcal{M}^{p^1}_{X_0} \rhd_{\mathcal{P}_0} \mathbf{G}^*_2(X_0)\big)$. Suppose that $\mathfrak{M}$ is embedded into a Desarguesian projective space $\mathfrak{P}$, then from A.2 $\mathfrak{P}$ satisfies the Fano axiom, because $\mathfrak{M}$ contains $\mathfrak{R}^*_4$ embedded into $\mathfrak{P}$. Observe the quadrangle $c_{\{1,5\}}$, $c_{\{1,6\}}$, $c_{\{2,5\}}$, $c_{\{2,6\}}$. Its diagonal points in $\mathfrak{P}$ are the following: $p^1 = c_{\{5,6\}}$, $c_{\{1,2\}}$, and $c^*_{\{1,2\}} = c_{\{3,4\}}$; thus $\mathbf{L}(p^1, c_{\{1,2\}}, c_{\{3,4\}})$. Analogously we obtain $\mathbf{L}(p^1, c_{\{1,3\}}, c_{\{2,4\}})$, and therefore the point p^1 is the common point of the lines $\overline{c_{\{1,2\}}, c_{\{3,4\}}}$ and $\overline{c_{\{1,3\}}, c_{\{2,4\}}}$. Next, let us observe the substructure $\mathfrak{N}$ of $\mathfrak{M}$ spanned by the lines that pass through p^2 and have numbers in X_0; it is embedded into the same projective space $\mathfrak{P}$ and it is seen that $\mathfrak{N} = \mathcal{M}^{p^2}_{X_0} \rhd_{\mathcal{P}_1 \curlywedge X_0} \mathbf{G}^*_2(X_0)$. This yields that (in particular) $\mathcal{P}_1 \curlywedge X_0 \approx N_{X_0}$; what is more important, analogous reasoning applied to the quadrangles a_1, a_2, b_1, b_2 and a_1, a_3, b_1, b_3 gives that p^2 is the common point of the lines $\overline{c_{\{1,2\}}, c_{\{3,4\}}}$ and $\overline{c_{\{1,3\}}, c_{\{2,4\}}}$. This finally gives $p^1 = p^2$, which is impossible. ∎

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