On subordinations for certain analytic functions associated with Fox-Wright psi function

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Abstract

The aim of the present paper is to investigate several interesting properties of a linear operator $L^{p}_{q,s}(\alpha_i)$ associated with the Fox-Wright psi function.

1 Introduction

Let A denote the class of functions that are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$ and consisting of the functions f of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in N = \{1, 2, 3, ...\}), \tag{1.1}$$

where f is analytic and p-valent in U.
Given two functions $f(z)$ and $g(z)$ which are analytic in U, then we say that the function $g(z)$ is subordinate to $f(z)$, if there exists an analytic function $\omega(z)$ in U such that $|\omega(z)| < 1$ for $(z \in U)$ and $g(z) = f(\omega(z))$. This relation is denoted by $g(z) \prec f(z)$. In case $f(z)$ is univalent in U, we have that the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$.
For analytic functions given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

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let $f * g$ denote the Hadamard product or convolution of f and g, defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.2}$$

Next for real parameters A and B such that $-1 \leq B < A \leq 1$, we define the function

$$h(A, B; z) = \frac{1 + Az}{1 + Bz} \quad (z \in U). \tag{1.3}$$

It is obvious that $h(A, B; z)$ for $-1 \leq B \leq 1$ is the conformal map of the unit disk U onto the disk symmetrical with respect to the real axis having the center $\frac{1-AB}{1-B^2}$ and the radius $\frac{A-B}{1-B^2}$ for $B \neq \mp 1$. Furthermore the boundary circle intersects the real axis at the points $\frac{1-A}{1-B}$ and $\frac{1+A}{1+B}$.
The Fox-Wright psi function is defined by [4, p. 50]

$$\cdot_q\psi_s \begin{bmatrix} (\alpha_i, A_i)_{1,q} & \\ (\beta_i, B_i)_{1,s} & z \end{bmatrix} = \cdot_q\psi_s \begin{bmatrix} (\alpha_1, A_1), & . & . & . & , (\alpha_q, A_q); & \\ (\beta_1, B_1), & . & . & . & , (\beta_s, B_s); & z \end{bmatrix} \tag{1.4}$$

$$= \sum_{n=0}^{\infty} \left(\prod_{i=1}^{q} \Gamma(\alpha_i + A_i n) \right) \left(\prod_{i=1}^{s} \Gamma(\beta_i + B_i n) \right)^{-1} \frac{z^n}{n!},$$

where $\alpha_i \in C (i = 1, ..., q)$, $\beta_i \in C (i = 1, ..., s)$ and the coefficients $A_i \in R_+$ $(i = 1, ..., q)$ and $B_i \in R_+ (i = 1, ..., s)$ such that

$$1 + \sum_{i=1}^{s} B_i - \sum_{i=1}^{q} A_i \geq 0, \quad (q, s \in N_0 = N \cup \{0\}).$$

The normalized Fox-Wright psi function $\cdot_q\psi_s^*(z)$ in series form is represented as

$$\cdot_q\psi_s^* \begin{bmatrix} (\alpha_i, A_i)_{1,q} & \\ (\beta_i, B_i)_{1,s} & z \end{bmatrix} = \frac{\prod_{i=1}^{s} \Gamma(\beta_i)}{\prod_{i=1}^{q} \Gamma(\alpha_i)} \cdot_q\psi_s \begin{bmatrix} (\alpha_1, A_1), & . & . & . & , (\alpha_q, A_q); & \\ (\beta_1, B_1), & . & . & . & , (\beta_s, B_s); & z \end{bmatrix}. \tag{1.5}$$

The $\cdot_q\psi_s(z)$ is a special case of Fox's H-function $H_{k,l}^{m,n}(z)$ (see e.g.[4, p. 50])and $\cdot_q\psi_s^*(z)$ is a generalization of the familiar generalized hypergeometric function $\cdot_q F_s(z)$,

$$\cdot_q F_s \begin{bmatrix} (\alpha_i)_{1,q} & \\ (\beta_i)_{1,s} & z \end{bmatrix} = \cdot_q F_s \begin{bmatrix} (\alpha_1), & . & . & . & , (\alpha_q); & \\ (\beta_1), & . & . & . & , (\beta_s); & z \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n, ..., (\alpha_q)_n}{(\beta_1)_n, ..., (\beta_s)_n} \frac{z^n}{n!},$$

where $(\alpha)_n$ is the Pochhammer symbol,defined in terms of the gamma function Γ by

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}$$

Corresponding to a function $\pounds_p(\alpha_1,...,\alpha_q;A_1,...,A_q;\beta_1,...,\beta_s;B_1,...,B_s;z)$ defined by

$$\pounds_p(\alpha_1,...,\alpha_q;A_1,...,A_q;\beta_1,...,\beta_s;B_1,...,B_s;z) = z^p \cdot {}_q\psi_s^*(z).$$

We consider a linear operator

$$L_{q,s}^p(\alpha_1,...,\alpha_q;A_1,...,A_q;\beta_1,...,\beta_s;B_1,...,B_s) : A(p) \to A(p)$$

defined by the convolution

$$\begin{aligned} &L_{q,s}^p(\alpha_1,...,\alpha_q;A_1,...,A_q;\beta_1,...,\beta_s;B_1,...,B_s)f(z) \\ &= \pounds_p(\alpha_1,...,\alpha_q;A_1,...,A_q;\beta_1,...,\beta_s;B_1,...,B_s;z) * f(z). \end{aligned}$$

For brevity, we write

$$L_{q,s}^p(\alpha_i) = L_{q,s}^p(\alpha_1,...,\alpha_q;A_1,...,A_q;\beta_1,...,\beta_s;B_1,...,B_s) \quad (i = 1,...,q).$$

Thus, after some calculations,we get

$$z(A_iL_{q,s}^p(\alpha_i)f(z))' = \alpha_iL_{q,s}^p(\alpha_i+1)f(z) - (\alpha_i - A_ip)L_{q,s}^p(\alpha_i)f(z) \quad (i = 1,...,q). \tag{1.6}$$

Special cases of the operator $L_{q,s}^p(\alpha_i)$ $(i = 1,...,q)$ includes Dziok-Srivastava linear operator (cf. [5, 6, 3]), Hohlov linear operator [7], the Carlson-Shaffer linear operator [2], the Ruscheweyh derivative operator [13], the generalized Bernardi-Libera-Livingston linear integral operator (cf. [1, 9, 10]), and the Srivastava-Owa fractional derivative operators (cf. [11, 12]).
Our aim in the present paper is to derive several interesting properties and characteristics of the linear operator $L_{q,s}^p(\alpha_i)$ $(i = 1,...,q)$ by the application of the differential subordination method.

2 Main Results

We begin by recalling the following Lemmas which will be required in our investigation.

Lemma 2.1. *(see[14]). Let $h(z)$ be analytic and convex univalent in U, $h(0) = 1$ and let $g(z) = 1 + b_1z + b_2z^2 + ...$ be analytic in U. If*

$$g(z) + \frac{zg'(z)}{c} \prec h(z) \tag{2.1}$$

then for $Re(c) \geq 0$

$$g(z) \prec \frac{c}{z^c}\int_0^z t^{c-1}h(t)dt. \tag{2.2}$$

Lemma 2.2. *(see[8]). The function $(1-z)^\gamma \equiv e^{\gamma log(1-z)}$, $\gamma \neq 0$ is univalent in U iff γ is either in closed disk $|\gamma - 1| \leq 1$ or in the closed disk $|\gamma + 1| \leq 1$.*

Lemma 2.3. *(see[15]). Let $q(z)$ be univalent in U and let $\theta(\omega)$ and $\phi(\omega)$ be analytic in a domain D containing $q(U)$ with $\phi(\omega) \neq 0$ when $\omega \in q(U)$. Set $Q(z) = zq^{'}(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*
(i) $Q(z)$ is starlike(univalent) in U;
(ii)$Re\left(\frac{zh^{'}(z)}{Q(z)}\right) = Re\left(\frac{\theta^{'}(q(z))}{\phi(q(z))} + \frac{zQ^{'}(z)}{Q(z)}\right) > 0 \quad (z \in U)$
if $p(z)$ is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$, and

$$\theta(p(z)) + zp^{'}(z)\phi(p(z)) \prec \theta(q(z)) + zq^{'}(z)\phi(q(z)) = h(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Theorem 2.4. *Let $\alpha_i > 0, A_i > 0 \quad (i = 1, ..., q)$, $\lambda > 0$ and $-1 \leq B < A \leq 1$. If $f(z) \in A(p)$ satisfies*

$$(1-\lambda)\frac{L^p_{q,s}(\alpha_i)f(z)}{z^p} + \lambda\frac{L^p_{q,s}(\alpha_i+1)f(z)}{z^p} \prec h(A, B; z), \tag{2.3}$$

then

$$Re\left(\left(\frac{L^p_{q,s}(\alpha_i)f(z)}{z^p}\right)^{\frac{1}{m}}\right) > \left(\frac{\alpha_i}{\lambda A_i}\int_0^1 u^{\frac{\alpha_i}{\lambda A_i}-1}\left(\frac{1-Au}{1-Bu}\right)du\right)^{\frac{1}{m}} \quad (m \geq 1). \tag{2.4}$$

The result is sharp.

Proof. Let

$$g(z) = \frac{L^p_{q,s}(\alpha_i)f(z)}{z^p} \tag{2.5}$$

for$f(z) \in A(p)$. Then the function $g(z) = 1 + b_1z + ...$ is analytic in U. By making use of (1.6) and (2.5), we obtain

$$\frac{L^p_{q,s}(\alpha_i+1)f(z)}{z^p} = g(z) + \frac{A_izg^{'}(z)}{\alpha_i}. \tag{2.6}$$

From (2.3), (2.5), and (2.6), we get

$$g(z) + \lambda\frac{A_izg^{'}(z)}{\alpha_i} \prec h(A, B; z). \tag{2.7}$$

Now an application of Lemma 2.1 leads to

$$g(z) \prec \frac{\alpha_i}{\lambda A_i}z^{\frac{-\alpha_i}{\lambda A_i}}\int_0^1 t^{\frac{\alpha_i}{\lambda A_i}-1}\left(\frac{1+At}{1+Bt}\right)dt \tag{2.8}$$

or

$$\frac{L^p_{q,s}(\alpha_i)f(z)}{z^p} = \frac{\alpha_i}{\lambda A_i}\int_0^1 u^{\frac{\alpha_i}{\lambda A_i}-1}\left(\frac{1+Au\omega(z)}{1+Bu\omega(z)}\right)du, \tag{2.9}$$

where $\omega(z)$ is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1 \quad (z \in U)$.
In view of $-1 \leq B < A \leq 1$, $\alpha_i > 0$ and $A_i > 0$, it follows from (2.9) that

$$Re\left(\frac{L^p_{q,s}(\alpha_i)f(z)}{z^p}\right) > \frac{\alpha_i}{\lambda A_i}\int_0^1 u^{\frac{\alpha_i}{\lambda A_i}-1}\left(\frac{1-Au}{1-Bu}\right)du \quad (z \in U). \tag{2.10}$$

Therefore ,with the aid of elementary inequality $Re(\omega^{\frac{1}{m}}) \geq (Re\omega)^{\frac{1}{m}}$ for $Re\omega > 0$ and $m \geq 1$, the inequality (2.4) follows directly from (2.10).
To show the sharpness of (2.4), we take $f(z) \in A(p)$ defined by

$$\frac{L^p_{q,s}(\alpha_i)f(z)}{z^p} = \frac{\alpha_i}{\lambda A_i}\int_0^1 u^{\frac{\alpha_i}{\lambda A_i}-1}\left(\frac{1+Auz}{1+Buz}\right)du.$$

For this function,we find that

$$(1-\lambda)\frac{L^p_{q,s}(\alpha_i)f(z)}{z^p} + \lambda\frac{L^p_{q,s}(\alpha_i+1)f(z)}{z^p} = \frac{1+Az}{1+Bz}$$

and

$$\frac{L^p_{q,s}(\alpha_i)f(z)}{z^p} \to \frac{\alpha_i}{\lambda A_i}\int_0^1 u^{\frac{\alpha_i}{\lambda A_i}-1}\left(\frac{1-Au}{1-Bu}\right)du \quad as \quad z \to -1$$

Hence the proof of the Theorem is complete.

Theorem 2.5. *Let $\alpha_i > 0, A_i > 0 \quad (i = 1, ..., q)$, and $0 \leq \rho < 1$. Let γ be a complex number with $\gamma \neq 0$ and satisfy either $|\frac{2\gamma(1-\rho)\alpha_i}{A_i} - 1| \leq 1$ or $|\frac{2\gamma(1-\rho)\alpha_i}{A_i} + 1| \leq 1$ $(i = 1, ..., q)$. If $f(z) \in A(p)$ satisfies the condition*

$$Re\left(\frac{L^p_{q,s}(\alpha_i+1)f(z)}{L^p_{q,s}(\alpha_i)f(z)}\right) > \rho \quad (z \in U; i = 1, ..., q) \tag{2.11}$$

then

$$\left(\frac{L^p_{q,s}(\alpha_i)f(z)}{z^p}\right)^{\gamma} \prec \frac{1}{(1-z)^{\frac{2\gamma(1-\rho)\alpha_i}{A_i}}} = q(z) \quad (z \in U; i = 1, ..., q), \tag{2.12}$$

where $q(z)$ is the best dominant.

Proof. Let

$$p(z) = \left(\frac{L^p_{q,s}(\alpha_i)f(z)}{z^p}\right)^{\gamma} \quad (z \in U; i = 1, ..., q). \tag{2.13}$$

Then by making use of (1.6),(2.11) and (2.13), we have

$$1 + \frac{zA_i p'(z)}{\gamma\alpha_i p(z)} \prec \frac{1+(1-2\rho)z}{1-z} \quad (z \in U). \tag{2.14}$$

If we take

$$q(z) = \frac{1}{(1-z)^{\frac{2\gamma(1-\rho)\alpha_i}{A_i}}}, \quad \theta(\omega) = 1, \quad \phi(\omega) = \frac{A_i}{\gamma\alpha_i\omega},$$

then $q(z)$ is univalent by the condition of the Theorem 2.5 and Lemma 2.2. Further, it is easy to solve that $q(z),\ \theta(\omega)\ and\ \phi(\omega)$ satisfy the condition of Lemma 2.3. Since

$$Q(z) = zq^{'}(z)\phi(q(z)) = \frac{2(1-\rho)z}{1-z}$$

is univalent starlike in U and

$$h(z) = \theta(q(z)) + Q(z) = \frac{1+(1-2\rho)z}{1-z}.$$

It may be readily checked that the condition (i) and (ii) of Lemma 2.3 are satisfied. Thus, the result follows from (2.14) immediately.The proof is complete.

Corollary 2.6. *Let $\alpha_i > 0, A_i > 0\ \ (i = 1, ..., q)$, and $0 \le \rho < 1$. Let γ be a real number with $\gamma \ge 1$. If $f(z) \in A(p)$ satisfies the condition (2.11), then*

$$Re\left(\frac{L^{p}_{q,s}(\alpha_i)f(z)}{z^p}\right)^{\frac{A_i}{2\gamma(1-\rho)\alpha_i}} > 2^{\frac{-1}{\gamma}}\quad (z \in U; i = 1, ..., q).$$

The bound $2^{\frac{-1}{\gamma}}$ is the best possible.

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