Multiple Solutions for Noncoercive Problems with the p-Laplacian

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Abstract

We consider a nonlinear elliptic equation driven by the p-Laplacian and with a Carathéodory right hand side nonlinearity which exhibits an asymmetric asymptotic behaviour at $+\infty$ and at $-\infty$. These hypotheses imply that the Euler functional of the problem is noncoercive (indefinite). Using critical point theory, we prove the existence of at least two nontrivial smooth solutions. Also in the last section for the asymmetric functionals considered here, we compute the critical groups at infinity.

1 Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z. We consider the following nonlinear elliptic problem

$$\begin{cases} -\mathrm{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) = f(z,x(z)) & \text{for a.a. } z \in Z, \\ x|_{\partial Z} = 0, \end{cases} \tag{1.1}$$

with $p > 1$. Recently there have been multiplicity results concerning problem (1.1), without any symmetry condition on the right hand side nonlinearity $f(z,\cdot)$. We refer to the works of Dancer-Perera [4], Jiu-Su [8], Liu [11], Liu-Liu [10] and Zhang-Chen-Li [15]. In the papers of Jiu-Su [8], Liu [11], Liu-Liu [10], the Euler

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functional is coercive and in the works of Dancer-Perera [4] and Zhang-Chen-Li [15], the asymptotic limits at zero and at infinity exist. Here we examine what happens if the Euler functional is noncoercive and the nonlinearity $f(z,\cdot)$ exhibits as asymmetric behaviour at $+\infty$ and $-\infty$. We should mention that an asymmetric behaviour is also present in the works of Dancer-Perera [4] and Zhang-Chen-Li [15], but their asymptotic limits exist and are related to the Fučik spectrum of the negative Dirichlet p-Laplacian.

2 Mathematical Background

First let us recall some basic facts about the spectrum of the negative p-Laplacian with Dirichlet boundary condition. Let $m \in L^\infty(Z)_+$, $m \neq 0$ and consider the following nonlinearity weighted (with weight m) eigenvalue problem

$$\begin{cases} -\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) = \widehat{\lambda} m(z)|x(z)|^{p-2}x(z) \\ \qquad\qquad\qquad\qquad \text{for a.a. } z \in Z, \\ x|_{\partial Z} = 0, \end{cases} \tag{2.1}$$

with $p > 1$ and $\widehat{\lambda} \in \mathbb{R}$.

Problem (2.1) has a smallest eigenvalue denoted by $\widehat{\lambda}_1(m)$, which is positive, isolated, simple and admits a variational characterization

$$\widehat{\lambda}_1(m) = \inf\left\{ \frac{\|Dx\|_p^p}{\int\limits_Z m|x|^p\,dz} : x \in W_0^{1,p}(Z),\ x \neq 0 \right\}. \tag{2.2}$$

In (2.2) the infimum is realized at an eigenfunction u_1 and by nonlinear regularity theory, we have $u_1 \in C_0^1(\overline{Z})$ (see Lieberman [9]). Moreover, $u_1 \geqslant 0$ and in fact by virtue of the nonlinear strong maximum principle of Vázquez [14], we have that $u_1(z) > 0$ for all $z \in Z$. If $m \equiv 1$, then we write $\widehat{\lambda}_1(1) = \lambda_1$. If $u \in W_0^{1,p}(Z)$ is an eigenfunction corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(m)$, then $u \in C_0^1(\overline{Z})$ and changes sign. Finally, it is clear from (2.2) that

$$\left.\begin{array}{l} m, \overline{m} \in L^\infty(Z)_+,\ \ m \neq \overline{m} \\ m(z) \leqslant \overline{m}(z) \text{ for a.a. } z \in Z \end{array}\right\} \Longrightarrow \widehat{\lambda}_1(\overline{m}) < \widehat{\lambda}_1(m). \tag{2.3}$$

In our analysis of problem (1.1), we will also use Morse theory, in particular critical groups in order to produce new critical points. So let X be a Banach space and let $\varphi \in C^1(X)$ satisfy the PS-condition. We use the following notation

$$\begin{array}{rcll} K^\varphi &=& \left\{x \in X :\ \varphi'(x) = 0\right\} & \text{(critical points of } \varphi), \\ K_c^\varphi &=& \left\{x \in K^\varphi :\ \varphi(x) = c\right\} & \text{(with } c \in \mathbb{R}), \\ \varphi^c &=& \left\{x \in X :\ \varphi(x) \leqslant c\right\} & \text{(with } c \in \mathbb{R}). \end{array}$$

Let $x_0 \in X$ be an isolated critical point of φ with $\varphi(x_0) = c_0$ and let U be a neighbourhood of x_0 containing no other critical point. The critical groups (over $\mathbb{Z}$) of φ at x_0, are defined by

$$C_k(\varphi, x_0) = H_k\big(\varphi^{c_0} \cap U, (\varphi^{c_0} \cap U) \setminus \{x_0\}\big) \quad \forall k \geqslant 0,$$

where $H_k(\cdot,\cdot)$ as the k-th singular relative homology group with integer coefficients.

Suppose that $-\infty < \inf \varphi(K^\varphi)$ are choose $c < \inf \varphi(K^\varphi)$. The critical groups of φ at infinity, are defined by

$$C_k(\varphi,\infty) \;=\; H_k(X,\varphi^c) \quad \forall k \geqslant 0$$

(see Bartsch-Li [1]). If K^φ is finite, then the Morse-type numbers of φ, are defined by

$$M_k \;=\; \sum_{x\in K^\varphi} \operatorname{rank} C_k(\varphi,x)$$

and the Betti-type numbers for φ, are defined by

$$\beta_k \;=\; \operatorname{rank} C_k(\varphi,\infty).$$

By Morse theory (see Bartsch-Li [1], Chang [2, Theorem 6.1, p. 55] and Mawhin-Willem [12]), we have

$$\sum_{k=0}^{m}(-1)^{m-k}M_k \;\geqslant\; \sum_{k=0}^{m}(-1)^{m-k}\beta_k \quad \mathrm{m} \geqslant 0 \tag{2.4}$$

and

$$\sum_{k\geqslant 0}(-1)^{k}M_k \;=\; \sum_{k\geqslant 0}(-1)^{k}\beta_k. \tag{2.5}$$

From the inequality (2.4), we have $\beta_k \leqslant M_k$ for all $k \geqslant 0$. Therefore, if $\beta_k \neq 0$ for some $k \geqslant 0$, then φ must have a critical point $x \in X$ and the critical group $C_k(\varphi,x)$ is nontrivial. The equality (2.5) is known as the "Poincaré-Hopf formula". If $x,y \in X$ are critical points of φ and for some integer $k \geqslant 0$, we have $C_k(\varphi,x) \neq C_k(\varphi,y)$, then obviously $x \neq y$. Finally if $K^\varphi = \{x\}$, then $C_k(\varphi,\infty) = C_k(\varphi,x)$ for all $k \geqslant 0$.

Finally let us recall the notion of the Clarke subdifferential, which will be needed in the statement of our hypotheses. Let X be a Banach space and let $\varphi\colon X \longrightarrow \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of φ at $x \in X$ in the direction $h \in X$ is defined by

$$\varphi^0(x;h) \;=\; \limsup_{\substack{x'\to x\\ \lambda\searrow 0}} \frac{\varphi(x'+\lambda h)-\varphi(x')}{\lambda}.$$

The function $h \longmapsto \varphi^0(x;h)$ is sublinear, continuous and so it is the support function of a nonempty, w^*-compact and convex set $\partial\varphi(x)$, defined by

$$\partial\varphi(x) \;=\; \big\{x^* \in X^* :\; \langle x^*,h\rangle_X \leqslant \varphi^0(x;h) \text{ for all } h \in X\big\}.$$

The multifunction $x \longmapsto \partial\varphi(x)$ is known as the generalized (or Clarke) subdifferential of φ. If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$. The generalized subdifferential has a rich calculus, which generalizes the subdifferential calculus of continuous, convex functions. For details we refer to Clarke [3].

3 Multiple Solutions

The hypotheses on the nonlinearity f are the following:

$\underline{H(f)}$ $f\colon Z \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function, such that

(i) for all $\zeta \in \mathbb{R}$, the function $z \longmapsto f(z,\zeta)$ is measurable;

(ii) for almost all $z \in Z$, the function $\zeta \longmapsto f(z,\zeta)$ is locally Lipschitz and $f(z,0) = 0$;

(iii) there exist $a \in L^\infty(Z)_+$ and $c > 0$, such that

$$|u| \ \leqslant\ a(z) + c|\zeta|^{p-2}$$

for almost all $z \in Z$, all $\zeta \in \mathbb{R}$ and all $u \in \partial f(z,\zeta)$;

(iv) there exist $\vartheta, \beta, \widehat{\beta} \in L^\infty(Z)_+$, $\vartheta(z) \leqslant \lambda_1 \leqslant \beta(z)$ for almost all $z \in Z$, with strict inequalities on sets (not necessary the same) of positive measure and

$$\limsup_{\zeta \to +\infty} \frac{f(z,\zeta)}{\zeta^{p-1}} \ \leqslant\ \vartheta(z)$$

uniformly for almost all $z \in Z$ and

$$\beta(z) \ \leqslant\ \liminf_{\zeta \to -\infty} \frac{f(z,\zeta)}{|\zeta|^{p-2}\zeta} \ \leqslant\ \limsup_{\zeta \to -\infty} \frac{f(z,\zeta)}{|\zeta|^{p-2}\zeta} \ \leqslant\ \widehat{\beta}(z)$$

uniformly for almost all $z \in Z$;

(v) there exist $\delta > 0$ and $0 < \mu < p$, such that

$$\begin{aligned}
&f(z,\zeta)\zeta \ >\ 0 \quad \text{for a.a. } z \in Z \text{ and all } 0 < |\zeta| \leqslant \delta,\\
&f(z,\zeta) \ \geqslant\ 0 \quad \text{for a.a. } z \in Z \text{ and all } \zeta \geqslant 0,\\
&\mu F(z,\zeta) - f(z,\zeta)\zeta \ \geqslant\ 0 \quad \text{for a.a. } z \in Z \text{ and all } |\zeta| \leqslant \delta,
\end{aligned}$$

with $F(z,\zeta) = \int\limits_0^\zeta f(z,r)\,dr$.

Let $\tau_+\colon \mathbb{R} \longrightarrow \mathbb{R}_+$ be the truncation map

$$\tau_+(\zeta) \ =\ \begin{cases} 0 & \text{if} \quad \zeta \leqslant 0,\\ \zeta & \text{if} \quad \zeta > 0. \end{cases}$$

Then we introduce

$$f_+(z,\zeta) \ =\ f(z,\tau_+(\zeta)) \quad \forall (z,\zeta) \in Z \times \mathbb{R}.$$

Clearly f_+ is still a Carathéodory function and

$$f_+(z,\zeta) \ =\ 0 \quad \text{for a.a. } z \in Z \text{ and all } \zeta \leqslant 0.$$

We set

$$F_+(z,\zeta) \;=\; \int_0^{\zeta} f_+(z,r)\,dr.$$

The next lemma is an easy consequence of the strict positivity of $u_1 \in C_0^1(\overline{Z})$ and of the hypothesis on $\vartheta \in L^\infty(Z)_+$. So we omit the proof.

Lemma 3.1. *If $\vartheta \in L^\infty(Z)_+$, $\vartheta(z) \leqslant \lambda_1$ for almost all $z \in Z$ and the inequality is strict on a set of positive measure,*
then there exists $\xi_0 > 0$, such that

$$\|Dx\|_p^p - \int_Z \vartheta |x|^p\,dz \;\geqslant\; \xi_0 \|Dx\|_p^p \quad \forall x \in W_0^{1,p}(Z).$$

Let $\varphi\colon W_0^{1,p}(Z) \longrightarrow \mathbb{R}$ be the Euler functional for problem (1.1), namely

$$\varphi(x) \;=\; \frac{1}{p}\|Dx\|_p^p - \int_Z F(z,x(z))\,dz \quad \forall x \in W_0^{1,p}(Z). \tag{3.1}$$

Evidently $\varphi \in C^1(W_0^{1,p}(Z))$.

In the next proposition we produce the first nontrivial smooth solution for problem (1.1). In what follows

$$C_+ = \big\{x \in C_0^1(\overline{Z}) : \; x(z) \geqslant 0 \text{ for all } z \in \overline{Z}\big\}$$

(the positive cone of $C_0^1(\overline{Z})$) and

$$\begin{aligned} \operatorname{int} C_+ \;&=\; \big\{x \in C_0^1(\overline{Z}) : \; x(z) > 0 \text{ for all } z \in Z \\ &\qquad \text{and } \frac{\partial x}{\partial n}(z) < 0 \text{ for all } z \in \partial Z\big\}. \end{aligned}$$

Proposition 3.2. *If hypotheses $H(f)$ hold,*
then problem (1.1) has a solution $x_0 \in \operatorname{int} C_+$, which is a local minimizer of φ.

Proof. Because of hypothesis $H(f)(iv)$, for a given $\varepsilon > 0$, we can find $M_1 = M_1(\varepsilon) > 0$, such that

$$f(z,\zeta) \;\leqslant\; (\vartheta(z)+\varepsilon)\zeta^{p-1} \quad \text{for a.a. } z \in Z \text{ and all } \zeta \geqslant M_1. \tag{3.2}$$

Moreover, due to hypothesis $H(f)(iii)$, we can find $a_\varepsilon \in L^\infty(Z)_+$, $a_\varepsilon \neq 0$, such that

$$f(z,\zeta) \;\leqslant\; a_\varepsilon(z) \quad \text{for a.a. } z \in Z \text{ and all } 0 \leqslant \zeta \leqslant M_1. \tag{3.3}$$

Since

$$f_+(z,\zeta) \;=\; 0 \quad \text{for a.a. } z \in Z \text{ and all } \zeta \leqslant 0, \tag{3.4}$$

from (3.2), (3.3) and (3.4), we infer that

$$f_+(z,\zeta) \leqslant (\vartheta(z)+\varepsilon)|\zeta|^{p-1} + a_\varepsilon(z) \quad \text{for a.a. } z \in Z \text{ and all } \zeta \in \mathbb{R},$$

so

$$F_+(z,\zeta) \leqslant \frac{1}{p}(\vartheta(z)+\varepsilon)|\zeta|^p + a_\varepsilon(z)|\zeta| \quad \text{for a.a. } z \in Z \text{ and all } \zeta \in \mathbb{R}. \tag{3.5}$$

Then, using (3.5), (2.2) and Lemma 3.1, for a given $x \in W_0^{1,p}(Z)$, we have

$$\begin{aligned} \varphi_+(x) &= \frac{1}{p}\|Dx\|_p^p - \int_Z F_+(z,x(z))\,dz \\ &\geqslant \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \vartheta|x|^p\,dz - \frac{\varepsilon}{p}\|x\|_p^p - c_1\|Dx\|_p \\ &\geqslant \frac{1}{p}\left(\xi_0 - \frac{\varepsilon}{\lambda_1}\right)\|Dx\|_p^p - c_1\|Dx\|_p, \end{aligned} \tag{3.6}$$

for some $c_1 = c_1(\varepsilon) > 0$. If we choose $\varepsilon < \lambda_1\xi_0$, then from (3.6), we infer that φ_+ is coercive. Also exploiting the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we can easily check that φ_+ is weakly lower semicontinuous. Hence by virtue of the Weierstrass theorem, we can find $x_0 \in W_0^{1,p}(Z)$, such that

$$\varphi_+(x_0) = \inf\left\{\varphi_+(x) : x \in W_0^{1,p}(Z)\right\}.$$

First we show that $x_0 \neq 0$. To this end, note that hypothesis $H(f)(v)$ implies that there exists $c_2 > 0$, such that

$$F(z,\zeta) \geqslant c_2\zeta^\mu \quad \text{for a.a. } z \in Z \text{ and all } 0 \leqslant \zeta \leqslant \delta. \tag{3.7}$$

Since $u_1 \in \operatorname{int} C_+$ (by the nonlinear strong maximum principle of Vázquez [14, Theorem 5, p. 200]; see also Gasiński-Papageorgiou [6, Theorem 6.2.8, p. 738]), we can find $\sigma \in (0,1)$ small enough, such that

$$0 \leqslant \sigma u_1(z) \leqslant \delta \quad \forall z \in \overline{Z}.$$

Since $\sigma u_1 \in \operatorname{int} C_+$ and using (3.7), we have

$$\begin{aligned} \varphi_+(\sigma u_1) &= \frac{\sigma^p}{p}\|Du_1\|_p^p - \int_Z F(z,\sigma u_1(z))\,dz \\ &\leqslant \frac{\sigma^p}{p}\|Du_1\|_p^p - c_2\sigma^\mu\|u_1\|_\mu^\mu \\ &= c_3\sigma^p - c_4\sigma^\mu = \sigma^\mu(c_3\sigma^{p-\mu} - c_4), \end{aligned}$$

for some $c_3, c_4 > 0$.

Since $\mu < p$, by choosing $\sigma \in (0,1)$ even smaller if necessary, we can have $c_3\sigma^{p-\mu} < c_4$ and so $\varphi_+(\sigma u_1) < 0 = \varphi_+(0)$. Therefore $\varphi_+(x_0) < 0 = \varphi_+(0)$ and so $x_0 \neq 0$. Because x_0 is the minimizer of φ_+, we have

$$\varphi_+'(x_0) = 0$$

and so

$$A(x_0) = N_+(x_0), \tag{3.8}$$

where

$$N_+(x)(\cdot) \;=\; f_+(\cdot, x(\cdot)) \quad \forall x \in L^p(Z)$$

and $A\colon W_0^{1,p}(Z) \longrightarrow W^{-1,p'}(Z)$ (with $\frac{1}{p}+\frac{1}{p'}=1$) is the nonlinear operator defined by

$$\langle A(x), y\rangle \;=\; \int\limits_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N}\, dz \quad \forall x, y \in W_0^{1,p}(Z).$$

Here by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $\big(W^{-1,p'}(Z), W_0^{1,p}(Z)\big)$.

We take the duality brackets of (3.8) with the test function $-x_0^- \in W_0^{1,p}(Z)$ and obtain

$$\|Dx_0^-\|_p^p \;=\; 0,$$

i.e. $x_0^- = 0$ and so $x_0 \geqslant 0$, $x_0 \neq 0$.

From (3.8), we have

$$\begin{cases} -\operatorname{div}\big(\|Dx_0(z)\|^{p-2}Dx_0(z)\big) \;=\; f_+(z, x_0(z)) \;=\; f(z, x_0(z)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } z \in Z, \\ x_0|_{\partial Z} = 0. \end{cases} \tag{3.9}$$

Then, from (3.9) and nonlinear regularity theory (see Lieberman [9] or Gasiński-Papageorgiou [6, Theorem 6.2.7, p. 738]), we have $x_0 \in C_+$. Hypothesis $H(f)(v)$ implies that

$$f_+(z, x_0(z)) \;\geqslant\; 0 \quad \text{for a.a. } z \in Z$$

and so from (3.9), we have

$$\Delta_p x_0(z) \;\leqslant\; 0 \quad \text{for a.a. } z \in Z.$$

Therefore, by the nonlinear strong maximum principle of Vázquez [14], we have that $x_0 \in \operatorname{int} C_+$. Hence, we can find $r > 0$ small enough, such that

$$B_r^{C_0^1(\overline{Z})} \;=\; \big\{u \in C_0^1(\overline{Z}) :\; \|u\|_{C_0^1(\overline{Z})} \leqslant r\big\} \;\subseteq C_+.$$

It follows that

$$\varphi_+\big|_{B_r^{C_0^1(\overline{Z})}} \;=\; \varphi\big|_{B_r^{C_0^1(\overline{Z})}}$$

and so $x_0 \in \operatorname{int} C_+$ is a local $C_0^1(\overline{Z})$-minimizer of φ. Invoking Theorem 1.2 of Garcia Azorero-Manfredi-Peral Alonso [5, p. 387], we have that x_0 is a local minimizer in $W_0^{1,p}(Z)$ of φ. ∎

From the well known characterization of the critical groups at a local minimizer (see Chang [2, p. 33] or Mawhin-Willem [12, p. 175]), we have the following corollary.

Corollary 3.3. *If hypotheses $H(f)$ hold and $x_0 \in \operatorname{int} C_+$ is the solution obtained in Proposition 3.2, then*

$$C_k(\varphi, x_0) \;=\; \delta_{k,0}\mathbb{Z} \quad \forall k \geqslant 0.$$

Also because of hypothesis $H(f)(v)$ and Proposition 2.1 of Jiu-Su [8, p. 592], we have the following proposition.

Proposition 3.4. *If hypotheses $H(f)$ hold, then*

$$C_k(\varphi, 0) \;=\; 0 \quad \forall k \geqslant 0.$$

Now we are ready for the multiplicity result.

Theorem 3.5. *If hypotheses $H(f)$ hold, then problem (1.1) has at least two nontrivial solutions $x_0 \in \operatorname{int} C_+$ and $v_0 \in C_0^1(\overline{Z})$.*

Proof. From Proposition 3.2, we already have one solution $x_0 \in \operatorname{int} C_+$ being a local minimizer of φ. Because of hypothesis $H(f)(iv)$, we have

$$\varphi_1(-tu_1) \;\longrightarrow\; -\infty \quad \text{as } t \to +\infty,$$

and so the functional φ is unbounded below. Thus we can use the mountain pass theorem and obtain a second solution $v_0 \in W_0^{1,p}(Z)$. From Chang [2, p. 89], we know that

$$C_1(\varphi, v_0) \;\neq\; 0.$$

But from Proposition 3.4, we already know that all critical groups at the origin are zero. So v_0 is nontrivial. Finally nonlinear regularity theorem (see Lieberman [9]) implies that $v_0 \in C_0^1(\overline{Z})$. ∎

Remark 3.6. If $N = 1$ (ordinary differential equation), then we can replace hypotheses $H(f)(ii)$ and (iii) by the following more general ones:

$H(f)(ii)'$ for every $M > 0$ there exists $a_M \in L^1(0,1)_+$, such that

$$\big|f(z,\zeta) - f(z,\overline{\zeta})\big| \;\leqslant\; a_M(z)|\zeta - \overline{\zeta}| \quad \text{for a.a. } z \in Z \text{ and all } |\zeta|, |\overline{\zeta}| \leqslant M;$$

$H(f)(iii)'$ there exist $a \in L^1(0,1)$ and $c > 0$, such that

$$|f(z,\zeta)| \;\leqslant\; a(z) + x|\zeta|^{p-1} \quad \text{for a.a. } z \in Z \text{ and all } \zeta \in \mathbb{R};$$

Note that in the new hypotheses we do not need the generalized subdifferential of $f(z,\cdot)$. Indeed, now by virtue of hypothesis $H(f)(ii)'$ and exploit the embedding $W_0^{1,p}(Z) \subseteq C(Z)$, we can show that N_f is locally Lipschitz in (4.25).

4 Critical groups at infinity

In this section we compute the critical groups at infinity for the asymmetric Euler functional of the problem. We believe that our result here can be useful to people using Morse theoretic techniques in their study of multiple solutions. We start with an auxiliary result that is related to Lemma 2.4 of Perera-Schechter [13, p. 365].

Lemma 4.1. *If X is a Banach space, $(t,x) \longmapsto \varphi_t(x)$ is a function which belongs to $C^1([0,1] \times X;\mathbb{R})$, such that:*
(i) *$x \longmapsto \partial_t\varphi_t(x)$ and $x \longmapsto \varphi_t'(x)$ are both locally Lipschitz;*
(ii) *there exists $R > 0$, such that*

$$\inf\left\{\|\varphi_t'(x)\| : \ t \in [0,1], \ \|x\| > R\right\} \ > \ 0 \tag{4.1}$$

and

$$\inf\left\{\varphi_t(x) : \ t \in [0,1], \ \|x\| \leqslant R\right\} \ > \ -\infty; \tag{4.2}$$

(iii) *φ_1 is unbounded below and*

$$C_k(\varphi_0,\infty) \ = \ 0 \quad \forall k \geqslant 0, \tag{4.3}$$

then

$$C_k(\varphi_1,\infty) \ = \ 0 \quad \forall k \geqslant 0.$$

Proof. Note that by (4.1) we have that $K^\varphi \subseteq \overline{B}_R^X$.

For every $t \in [0,1]$, let $v_t \colon X \setminus K^\varphi \longrightarrow \mathbb{R}$ be the pseudogradient vector field corresponding to φ_t Gasiński-Papageorgiou [6, Definition 5.1.16, p. 614 and Theorem 5.1.19, p. 616]). By definition it is locally Lipschitz, so the map

$$X \setminus K^\varphi \ni x \ \longmapsto \ -\frac{|\partial_t\varphi_t(x)|}{\|\varphi_t'(x)\|^2}v_t(x) \in X$$

is locally Lipschitz (see Clarke [3]).

Let

$$\alpha \ < \ \inf\left\{\varphi_t(x) : \ t \in [0,1], \ \|x\| \leqslant R\right\}$$

and let $u \in \varphi_0^\alpha$. We consider the Cauchy problem

$$\begin{cases} \dot\eta(t) = -\dfrac{|\partial_t\varphi_t(\eta(t))|}{\|\varphi_t'(\eta(t))\|^2}v_t(\eta(t)) & \text{for a.a. } t \in [0,1] \\ \eta(0) = u. \end{cases} \tag{4.4}$$

By the local existence theorem (see e.g. Gasiński-Papageorgiou [6, Theorem 5.1.21, p. 618])), there exists local flow $\eta(t)$. Then

$$\begin{aligned} \frac{d}{dt}\varphi_t(\eta(t)) \ &= \ \langle \varphi_t'(\eta(t)), \dot\eta(t)\rangle + \partial_t\varphi_t(\eta(t)) \\ &= \ -\langle \varphi_t'(\eta(t)), v_t(\eta(t))\rangle \frac{|\partial_t\varphi_t(\eta(t))|}{\|\varphi_t'(\eta(t))\|^2} + \partial_t\varphi_t(\eta(t)) \\ &\leqslant \ -\|\varphi_t'(\eta(t))\|^2\frac{|\partial_t\varphi_t(\eta(t))|}{\|\varphi_t'(\eta(t))\|^2} + \partial_t\varphi_t(\eta(t)) \\ &= \ -|\partial_t\varphi_t(\eta(t))| + \partial_t\varphi_t(\eta(t)) \ \leqslant \ 0. \end{aligned}$$

Therefore the function $t \longmapsto \varphi_t(\eta(t))$ is decreasing and we have

$$\varphi_t(\eta(t)) \ \leqslant \ \varphi_0(\eta(0)) \ = \ \varphi_0(u) \ \leqslant \ \alpha.$$

Thus $\|\eta(t)\| > R$ and the global solution $\eta(t)$ of (4.4) exists in $X \setminus \overline{B}_R^X$. Hence $\eta(1)$ is a homeomorphism of φ_0^α onto a subset of φ_1^α. Let $\varepsilon > 0$ and let us set $C = \varphi_1^{a-\varepsilon} \setminus \varphi_0^a$. Then

$$\overline{C} \;=\; \overline{\varphi_1^{a-\varepsilon} \setminus \varphi_0^a} \;\subseteq\; \operatorname{int} \varphi_1^a.$$

By virtue of the excision property of singular homology, we have

$$H_k(X \setminus C, \varphi_1^{a-\varepsilon} \setminus C) \;=\; H_k(X, \varphi_1^{a-\varepsilon}),$$

so

$$H_k(X \setminus C, \varphi_0^a) \;=\; H_k(X, \varphi_1^{a-\varepsilon}) \;=\; H_k(X, \varphi_1^a) \;=\; C_k(\varphi, \infty).$$

Note that

$$\eta(1)(\varphi_0^a) \;\subseteq\; \varphi_1^a \;\subseteq\; X.$$

Let

$$i\colon (\varphi_1^a, \eta(1)(\varphi_0^a)) \longrightarrow (X, \eta(1)(\varphi_0^a))$$

and

$$j\colon (X, \eta(1)(\varphi_0^a)) \longrightarrow (X, \varphi_1^a)$$

be the inclusion maps. We have the following exact sequence:

$$\begin{aligned} &C_{k+1}(\varphi_1, \infty) \;=\; H_{k+1}(X, \varphi_1^a) \;\xrightarrow{\partial}\; H_k(\varphi_1^a, \eta(1)(\varphi_0^a)) \;\xrightarrow{i_*} \\ &\xrightarrow{i_*}\; H_k(X, \eta(1)(\varphi_0^a)) \;\xrightarrow{j_*}\; H_k(X, \varphi_1^a). \end{aligned}$$

Since $\eta(1)$ is a homeomorphism and using hypothesis (4.3), we have

$$H_k(X, \eta(1)(\varphi_0^a)) \;=\; H_k(X, \varphi_0^a) \;=\; C_k(\varphi_0, \infty) \;=\; 0.$$

So, it follows that

$$C_k(\varphi_1, \infty) \;=\; 0 \quad \forall k \geqslant 1.$$

Moreover, since φ_1 is unbounded below, we have that $\varphi_1^a \neq \emptyset$ and so

$$C_0(\varphi_1, \infty) \;=\; H_0(X, \varphi_1^a) \;=\; 0.$$

Therefore, finally we obtain that

$$C_k(\varphi_1, \infty) \;=\; 0 \quad \forall k \geqslant 0 \qquad \blacksquare$$

Now we are ready to compute the critical groups of the Euler functional φ at infinity.

Theorem 4.2. *If $p \geqslant 2$, hypotheses $H(f)$ hold and φ is defined by (3.1), then*

$$C_k(\varphi, \infty) \;=\; 0 \quad \forall k \geqslant 0.$$

Proof. Let $h \in L^\infty(Z)_+$, $h \neq 0$ and consider the one parameter family of C^1-maps $\varphi_t\colon W_0^{1,p}(Z) \longrightarrow \mathbb{R}$, defined by

$$\begin{aligned}\varphi_t(x) &= \frac{1}{p}\|Dx\|_p^p - t\int_Z F(z,x(z))\,dz \\ &\quad -\frac{1-t}{p}\int_Z \beta(x^-)^p\,dz + (1-t)\int_Z hx\,dz \quad \forall t \in [0,1].\end{aligned}$$

Claim 1. There exists $R > 0$, such that

$$\inf\big\{\|\varphi_t'(x)\| : t \in [0,1],\ \|x\| > R\big\} > 0. \tag{4.5}$$

To prove the Claim we argue by contradiction. So suppose that the Claim is not true. We can find sequences $\{t_n\}_{n\geqslant 1} \subseteq [0,1]$ and $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$, such that

$$t_n \longrightarrow t \in [0,1], \quad \|x_n\| \longrightarrow +\infty \quad \text{and} \quad \varphi_{t_n}'(x_n) \longrightarrow 0 \text{ in } W^{-1,p'}(Z).$$

We have

$$\begin{aligned}\Big|\langle A(x_n),v\rangle - t_n\int_Z f(z,x_n(z))v(z)\,dz + (1-t_n)\int_Z \beta(x_n^-)^{p-1}v\,dz \\ +(1-t_n)\int_Z hv\,dz\Big| \leqslant \varepsilon_n\|v\| \qquad \forall v \in W_0^{1,p}(Z),\end{aligned} \tag{4.6}$$

with $\varepsilon_n \searrow 0$.

In (4.6), we let $v = x_n^+ \in W_0^{1,p}(Z)$. We obtain

$$\|Dx_n^+\|_p^p - t_n\int_Z f(z,x_n(z))x_n^+(z)\,dz + (1-t_n)\int_Z hx_n^+\,dz \leqslant \varepsilon_n\|x_n^+\|. \tag{4.7}$$

From hypotheses $H(f)(iii)$ and (iv), we know that for each $\varepsilon > 0$, there is $a_\varepsilon \in L^\infty(Z)_+$, such that

$$f(z,\zeta) \leqslant (\vartheta(z)+\varepsilon)\zeta^{p-1} + a_\varepsilon(z) \quad \text{for a.a. } z \in Z \text{ and all } \zeta \geqslant 0.$$

Hence (4.7) becomes

$$\|Dx_n^+\|_p^p - \int_Z \vartheta(x_n^+)^p\,dz - \varepsilon\|x_n^+\|_p^p \leqslant \varepsilon_n\|x_n^+\|$$

(recall that $t_n \in [0,1]$, $h \geqslant 0$), so using (2.2) and Lemma 3.1, we have

$$\Big(\xi_0 - \frac{\varepsilon}{\lambda_1}\Big)\|Dx_n^+\|_p^p \leqslant \varepsilon_n'\|Dx_n^+\|_p, \tag{4.8}$$

with $\varepsilon_n' \searrow 0$.

So, if we choose $\varepsilon < \lambda_1 \xi_0$, then from (4.8) we infer that the sequence $\{x_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded. Because $\|x_n\| \longrightarrow +\infty$, we must have that $\|x_n^-\| \longrightarrow +\infty$. We set

$$y_n = \frac{x_n^-}{\|x_n^-\|} \quad \forall n \geqslant 1.$$

By passing to a suitable subsequence if necessary, we may assume that

$$\begin{aligned} y_n &\xrightarrow{w} y \quad \text{in } W_0^{1,p}(Z), \\ y_n &\longrightarrow y \quad \text{in } L^p(Z), \\ y_n(z) &\longrightarrow y(z) \quad \text{for a.a. } z \in Z, \\ |y_n(z)| &\leqslant k(z) \quad \forall n \geqslant 1, \end{aligned}$$

with $k \in L^p(Z)_+$.

In (4.6), we take $v = y_n - y \in W_0^{1,p}(Z)$ and we find

$$\begin{aligned} &\Big| \frac{1}{\|x_n^-\|^{p-1}} \langle A(x_n^+), y_n - y\rangle - \langle A(y_n), y_n - y\rangle \\ &-t_n \int\limits_Z \frac{f(z,x_n(z))}{\|x_n^-\|^{p-1}}(y_n - y)\,dz + (1-t_n)\int\limits_Z \beta y_n^{p-1}(y_n-y)\,dz \\ &+(1-t_n)\int\limits_Z \frac{h}{\|x_n^-\|^{p-1}}(y_n-y)\,dz \Big| \leqslant \varepsilon_n \|y_n - y\|. \end{aligned} \tag{4.9}$$

Due to the boundedness of $\{x_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$, we have

$$\frac{1}{\|x_n^-\|^{p-1}} \langle A(x_n^+), y_n - y\rangle \longrightarrow 0. \tag{4.10}$$

We have

$$\begin{aligned} &\int\limits_Z \frac{f(z,x_n(z))}{\|x_n^-\|^{p-1}}(y_n-y)\,dz \\ &= \int\limits_{\{x_n>0\}} \frac{f(z,x_n(z))}{\|x_n^-\|^{p-1}}(y_n-y)\,dz + \int\limits_{\{x_n<0\}} \frac{f(z,x_n(z))}{\|x_n^-\|^{p-1}}(y_n-y)\,dz. \end{aligned} \tag{4.11}$$

Evidently, we have

$$\int\limits_{\{x_n>0\}} \frac{f(z,x_n(z))}{\|x_n^-\|^{p-1}}(y_n-y)\,dz \longrightarrow 0. \tag{4.12}$$

Also from hypotheses $H(f)(iii)$ and (iv), we have

$$\frac{|N(-x_n^-)|}{\|x_n^-\|^{p-1}} \leqslant \frac{\widehat{a}(z)}{\|x_n^-\|^{p-1}} + \widehat{c}|y_n(z)|^{p-1} \quad \text{for a.a. } z \in Z \text{ and all } n \geqslant 1, \tag{4.13}$$

with $\widehat{a} \in L^\infty(Z)_+$ and $\widehat{c} > 0$. Thus the sequence $\{\frac{N(-x_n^-)}{\|x_n^-\|^{p-1}}\} \subseteq L^{p'}(Z)$ is bounded.

Therefore, passing to a subsequence if necessary, we may assume that

$$\frac{N(-x_n^-)}{\|x_n^-\|^{p-1}} \xrightarrow{w} g \quad \text{in } L^{p'}(Z), \tag{4.14}$$

for some $g \in L^{p'}(Z)$. For every $\varepsilon > 0$ and $n \geqslant 1$, we introduce the set

$$C_{\varepsilon,n} = \left\{ z \in Z : \, x_n^-(z) > 0, \, \beta(z) - \varepsilon \leqslant \frac{f(z, -x_n^-(z))}{-x_n^-(z)^{p-1}} \leqslant \widehat{\beta}(z) + \varepsilon \right\}.$$

Note that

$$x_n^-(z) \longrightarrow +\infty \quad \text{for a.a. } z \in \{y > 0\},$$

so from hypothesis $H(f)(iv)$, we have

$$\chi_{C_{\varepsilon,n}}(z) \longrightarrow 1 \quad \text{for a.a. } z \in \{y > 0\}.$$

From this and (4.14), it follows that

$$\chi_{C_{\varepsilon,n}} \frac{N(-x_n^-)}{\|x_n^-\|^{p-1}} \xrightarrow{w} g \quad \text{in } L^{p'}(\{y > 0\}). \tag{4.15}$$

From the definition of the set $C_{\varepsilon,n}$, we have that

$$\begin{aligned} -\chi_{C_{\varepsilon,n}}(z)\big(\widehat{\beta}(z) + \varepsilon\big) y_n(z)^{p-1} &\leqslant \chi_{C_{\varepsilon,n}}(z) \frac{N(-x_n^-)(z)}{\|x_n^-\|^{p-1}} \\ = \chi_{C_{\varepsilon,n}}(z) \frac{f(z, -x_n^-(z))}{-x_n^-(z)^{p-1}} y_n(z)^{p-1} & \\ \leqslant -\chi_{C_{\varepsilon,n}}(z)\big(\beta(z) - \varepsilon\big) y_n(z)^{p-1} & \quad \text{for a.a. } z \in Z. \end{aligned}$$

Passing to the limit as $n \to +\infty$ and using (4.15) together with Mazur's lemma, we obtain

$$-\big(\widehat{\beta}(z) - \varepsilon\big) y(z)^{p-1} \leqslant g(z) \leqslant -\big(\beta(z) + \varepsilon\big) y(z)^{p-1} \quad \text{for a.a. } z \in \{y > 0\}.$$

Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \searrow 0$, to obtain

$$-\widehat{\beta}(z) y(z)^{p-1} \leqslant g(z) \leqslant -\beta(z) y(z)^{p-1} \quad \text{for a.a. } z \in \{y > 0\}. \tag{4.16}$$

Moreover, from (4.13), it is clear that

$$g(z) = 0 \quad \text{for a.a. } z \in \{y = 0\}. \tag{4.17}$$

Since $Z = \{y > 0\} \cup \{y = 0\}$, from (4.16) and (4.17), we infer that

$$g(z) = -\widehat{g}(z) y(z)^{p-1} \quad \text{for a.a. } z \in Z, \tag{4.18}$$

with $\widehat{g} \in L^\infty(Z)_+$ and $\beta(z) \leqslant \widehat{g}(z) \leqslant \widehat{\beta}(z)$ for almost all $z \in Z$. Note that

$$\int_{\{x_n < 0\}} \frac{f(z, x_n(z))}{\|x_n^-\|^{p-1}} (y_n - y) \, dz \longrightarrow 0. \tag{4.19}$$

Clearly, we also have

$$\int\limits_Z \frac{h}{\|x_n^-\|^{p-1}}(y_n - y)\,dz \;\longrightarrow\; 0 \quad \text{and} \quad \int\limits_Z \beta y_n^{p-1}(y_n - y)\,dz \;\longrightarrow\; 0. \tag{4.20}$$

So, if we pass to the limit as $n \longrightarrow +\infty$ in (4.9) and use (4.10), (4.11), (4.12), (4.19) and (4.20), we obtain

$$\lim_{n\to+\infty} \langle A(y_n), y_n - y\rangle \;=\; 0. \tag{4.21}$$

But A is an operator of type $(S)_+$ (see e.g. Gasiński-Papageorgiou [6, Definition 3.2.55(b), p. 338]). So from (4.21), it follows that

$$y_n \;\longrightarrow\; y \quad \text{in } W_0^{1,p}(Z).$$

We write (4.6) in the following form:

$$\begin{aligned} \Bigg| \frac{1}{\|x_n^-\|^{p-1}}\langle A(x_n^+), v\rangle - \langle A(y_n), v\rangle - t_n \int\limits_Z \frac{f(z, x_n(z))}{\|x_n^-\|^{p-1}} v\,dz \\ + (1 - t_n)\int\limits_Z \beta y_n^{p-1} v\,dz + (1 - t_n)\int\limits_Z \frac{h}{\|x_n^-\|^{p-1}} v\,dz \Bigg| \;\leqslant\; \frac{\varepsilon_n}{\|x_n^-\|^{p-1}}\|v\|. \end{aligned}$$

So, in the limit as $n \to +\infty$, because of (4.14) and (4.18), we have

$$\langle A(y), v\rangle \;=\; \int\limits_Z \big(t\widehat{g} + (1-t)\beta)\big)y^{p-1} v\,dz \quad \forall v \in W_0^{1,p}(Z),$$

so

$$A(y) \;=\; \overline{g} y^{p-1},$$

with $\overline{g} = t\widehat{g} + (1-t)\beta \in L^\infty(Z)_+$ and thus

$$\begin{cases} -\operatorname{div}\big(\|Dy(z)\|^{p-2}Dy(z)\big) \;=\; \overline{g}(z)|y(z)|^{p-2}y(z) & \text{for a.a. } z \in Z, \\ y|_{\partial Z} = 0,\; y \neq 0. \end{cases} \tag{4.22}$$

From the strict monotonicity of the principal eigenvalue on the weight (see (2.3)), we have

$$\widehat{\lambda}_1(\overline{g}) \;\leqslant\; \widehat{\lambda}_1(\beta) \;<\; \widehat{\lambda}_1(\lambda_1) \;=\; 1. \tag{4.23}$$

Combining (4.22) and (4.23), we infer that y must change sign, a contradiction. So Claim 1 is true.

Clearly, we also have

$$\inf\big\{\varphi_t(x) :\; t \in [0,1],\; \|x\| \leqslant R\big\} \;>\; -\infty. \tag{4.24}$$

Obviously $\partial_t\varphi_t$ is locally Lipschitz. We need to check that also φ_t' is locally Lipschitz. We have

$$\varphi_t'(x) \;=\; A(x) - N_f(x) + (1-t)\beta(x^-)^{p-1} + (1-t)h, \tag{4.25}$$

where $N_f(x) = f(\cdot, x(\cdot))$. Obviously the first (since $p \geqslant 2$), the third and the fourth terms on the right hand side are locally Lipschitz. Because of hypotheses $H(f)(ii)$ and (iii) we also have that N_f is locally Lipschitz (see Clarke [3]). So φ_t' is locally Lipschitz.

Note that

$$\begin{aligned}
\varphi_0(x) &= \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \beta(x^-)^p\,dz + \int_Z hx\,dz \quad \forall x \in W_0^{1,p}(Z),\\
\varphi_1(x) &= \frac{1}{p}\|Dx\|_p^p - \int_Z F(z,x(z))\,dz \;=\; \varphi(x) \quad \forall x \in W_0^{1,p}(Z).
\end{aligned}$$

Suppose that φ_0 has a critical point. Then

$$A(x) \;=\; -\beta(x^-)^{p-1} - h. \tag{4.26}$$

We take duality brackets with $x^+ \in W_0^{1,p}(Z)$ and obtain

$$\|Dx^+\|_p \;=\; 0,$$

i.e. $x^+ = 0$ and so $x \leqslant 0$, $x \neq 0$ (see (4.26)).

Hence from (4.26), we have

$$A(-x) \;=\; \beta(-x)^{p-1} + h,$$

so

$$\begin{cases} -\mathrm{div}\left(\|D(-x)(z)\|^{p-2}D(-x)(z)\right) \;=\; \beta(z)|(-x)(z)|^{p-2}(-x)(z) + h(z) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } z \in Z, \\ x|_{\partial Z} = 0, \end{cases}$$

Without any loss of generality, we may assume that

$$\lambda_1 \;\leqslant\; \beta(z) \;\leqslant\; \lambda_1 + \varepsilon \quad \text{for a.a. } z \in Z,$$

with $\varepsilon > 0$ small enough (depending on h). So invoking the antimaximum principle of Godoy-Gossez-Paczka [7, Theorem 5.1], we infer that $x \in \operatorname{int} C_+$, a contradiction. Therefore, φ_0 has no critical points and so

$$C_k(\varphi_0, \infty) \;=\; 0 \quad \forall k \geqslant 0. \tag{4.27}$$

Moreover, due to hypothesis $H(f)(iv)$, we have

$$\varphi_1(-tu_1) \;\longrightarrow\; -\infty \quad \text{as } t \to +\infty,$$

and so φ_1 is unbounded below. Thus, using also (4.5), (4.24) and (4.27), we can invoke Lemma 4.1, to conclude that

$$C_k(\varphi, \infty) \;=\; 0 \quad \forall k \geqslant 0. \qquad \blacksquare$$

Remark 4.3. Because of the Theorem 4.2 we can obtain the second nontrivial solution in Theorem 3.5 arguing in another way. Suppose that 0 and x_0 are the only critical points of φ. From Corollary 3.3, Propositions 3.4, Theorem 4.2 and Poincaré-Hopf formula (see (2.5)), we have that $(-1)^0 = 0$, a contradiction. This implies that we can find $v_0 \in W_0^{1,p}(Z)$, $v_0 \neq 0$, a third critical point of φ.

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