(n)-pairing with axes in rational homotopy

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Abstract

Let $f : X \to Z$ and $g : Y \to Z$ be maps between connected pointed CW-complexes. Recall the definition of pairing with axes f and g due to N.Oda [16]. In this paper, we introduce *(n)-pairing*, which is a generalization of *H(n)*-space due to Y.Félix and D.Tanré [5] and define a family of subsets of the homotopy set of maps. We give some rational characterizations of it and illustrate some examples in Sullivan models. Also we consider about the $G(n)$-sequence of a fibration which is a generalization of G-sequence [11],[13].

1 Introduction

Let $f : X \to Z$ and $g : Y \to Z$ be based maps between pointed connected CW-complexes. Recall

Definition 1.1 ([16]). There is a pairing with axes f and g and denote as $f \perp g$ if there is a map $F : X \times Y \to Z$ (called a pairing) with the commutative diagram:

$$(1) \qquad \begin{tikzcd} X \times Y \arrow[dr, dotted, "F"] & Y \arrow[l, "i_Y"'] \arrow[d, "g"] \\ X \arrow[u, "i_X"] \arrow[r, "f"'] & Z. \end{tikzcd}$$

Here i_X and i_Y are the maps with $i_X(x) = (x, *)$ and $i_Y(y) = (*, y)$.

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This is a generalization of a cyclic map [18], i.e., a map $f : X \to Z$ is a cyclic map when $Y = Z$ and $g = id_Z$. Also it a generalization of an H-space, i.e., a space X is an H-space when $X = Y = Z$ and $f = g = id_X$ and F is the multiplication. For $n > 0$, put $X(n)$ the n-th Ganea space for X for a positive integer n and $p_n^X : X(n) \to X$ the n-th Ganea fibration in the fiber-cofiber construction [6]([1, Definition 1.59], [4, p.357]). Y.Félix and D.Tanré [5] relaxed the definition of 'H-space' by using Ganea space and defined '$H(n)$-space'. In this paper, we will relax Definition 1.1. By using the following Definition 1.2, we can define a family (see Theorem 1.5 below) of subsets of the homotopy set of maps and also it is possible to calculate the family by using the model of Ganea fibration in rational homotopy.

Definition 1.2. We say there is an (n)-pairing with axes f and g and denote as $f \underset{n}{\perp} g$ if there is a map $F(n) : (X \times Y)(n) \to Z$ (called an (n)-pairing) with the commutative diagram:

$$
(2) \qquad
\begin{tikzcd}
(X \times Y)(n) \arrow[dr, dotted, "F(n)"] & Y(n) \arrow[l, "i_Y(n)"'] \arrow[d, "g\circ p_n^Y"] \\
X(n) \arrow[u, "i_X(n)"] \arrow[r, "f\circ p_n^X"'] & Z.
\end{tikzcd}
$$

Here $i_X(n)$ and $i_Y(n)$ are canonical maps respectively obtained from maps i_X and i_Y in (1) [1, Proposition 1.60].

For $n \leq n'$, we have $f \underset{n}{\perp} g$ if $f \underset{n'}{\perp} g$ since there is the map $F(n) : (X \times Y)(n) \to Z$ by $F(n) := F(n') \circ p_{n,n'}^{X\times Y}$ in the commutative diagram:

$$
\begin{tikzcd}
X(n) \arrow[r, "i_X(n)"] \arrow[d, "p^X_{n,n'}"'] \arrow[ddr, "f\circ p_n^X"] & (X \times Y)(n) \arrow[d, "p^{X\times Y}_{n,n'}"] & Y(n) \arrow[l, "i_Y(n)"'] \arrow[d, "p^Y_{n,n'}"] \arrow[ddl, "g\circ p_n^Y"'] \\
X(n') \arrow[r, "i_X(n')"'] \arrow[dr, "f\circ p^X_{n'}"'] & (X \times Y)(n') \arrow[d, dotted, "F(n')"] & Y(n') . \arrow[l, "i_Y(n')"] \arrow[dl, "g\circ p^Y_{n'}"] \\
& Z. &
\end{tikzcd}
$$

Since the direct limit $X(\infty) := \varinjlim_n X(n)$ of the maps $p_{n,n'}^X : X(n) \to X(n')$ for $n \leq n'$ has the homotopy type of X [6], we see $f \perp g$ if and only if $f \underset{\infty}{\perp} g$. If $f \underset{n}{\perp} g$ and $h : Z \to U$ is a map, then $h \circ f \underset{n}{\perp} h \circ g$. If $f \underset{n}{\perp} g$ and $h : X' \to X$ is a map, then $f \circ h \underset{n}{\perp} g$. Recall that X is an $H(n)$-space [5] if and only if $X = Y = Z$ and $f = g = id_X$ in (2). For example, the complex projective space $\mathbb{C}P^3$ is an $H(2)$-space but not $\mathbb{C}P^2$ [5, Ex.5].

Also denote the set of the homotopy classes of the axes of a map $g : Y \to Z$ by $g^{\perp}(X,Z) := \{[f] \in [X,Z] | f \perp g\}$ [15]. This is a generalization of a generalized Gottlieb set due to Varadarajan [18]. Especially, $G_n(Z,X;f) := f^{\perp}(S^n, Z)$ is said as the n-th evaluation subgroup of a map f [20]. We will relax the set $g^{\perp}(X,Z)$ as

Definition 1.3. For a map $g : Y \to Z$, the set of homotopy classes of axes of (n)-pairing with g is denoted as

$$g^{\frac{\perp}{n}}(X, Z) := \{[f] \in [X, Z] \mid f \underset{n}{\perp} g\}.$$

Especially, denote $G_m^{(n)}(Z, X; f) := f^{\frac{\perp}{n}}(S^m, Z)$ as the m-th (n)-evaluation subgroup of $f : X \to Z$. Also $G_m^{(n)}(Z) := id_X^{\frac{\perp}{n}}(S^m, Z)$ as the m-th (n)-Gottlieb group of Z. An element $a \in \pi_m(Z)$ is in $G_m^{(n)}(Z, X; f)$ if there is a map $F(n) : (X \times S^m)(n) \to Z$ such that the following diagram commutes:

$$(3) \qquad \begin{tikzcd} (X \times S^m)(n) \arrow[dr, dotted, "F(n)"] & S^m(n) \arrow[l, "i_{S^m(n)}"'] \arrow[d, "a \circ p_n^{S^m}"] & S^m \arrow[l, "s"'] \arrow[dl, bend left, "a"] \\ X(n) \arrow[u, "i_X(n)"] \arrow[r, "f \circ p_n^X"'] & Z, & \end{tikzcd}$$

where s is a section [6] induced from the fact $cat(S^m) = 1$ (see below).

Here $cat(X)$ is the L-S (Lusternik-Schnirelmann) category of X, that is the least integer n such that X can be covered by $n + 1$ open subsets contractible in X [14]. Then $cat(X) \leq n$ if and only if the n-th Ganea fibration $p_n^X : X(n) \to X$ has a section [6]. Recall the product formula [9, §4] of Ganea space: the inclusion $(X \times Y)(n) \subset (X \times Y)(\infty) = X(\infty) \times Y(\infty)$ can be deformed into $\cup_{i+j=n} X(i) \times Y(j) \subset X(\infty) \times Y(\infty)$, which is induced by the universality of the canonical A_∞-structure of loop space due to Stasheff [17]. Thus there exists the commutative diagram ([5, p.716])

$$(4) \qquad \begin{tikzcd} (X \times Y)(n) \arrow[rr, shift left, "\alpha"] \arrow[dr, "p_n^{X \times Y}"'] & & \cup_{i+j=n} X(i) \times Y(j) \arrow[ll, shift left, "\beta"] \arrow[dl, "\cup_{i+j=n} p_i^X \times p_j^Y"] \\ & X \times Y, & \end{tikzcd}$$

in particular,

$$\begin{tikzcd} (X \times S^m)(n) \arrow[rr, shift left, "\alpha"] \arrow[dr, "p_n^{X \times S^m}"'] & & X(n) \cup X(n-1) \times S^m(1) \arrow[ll, shift left, "\beta"] \arrow[dl, "p_n^X \cup p_{n-1}^X \times p_1^{S^m}"] \\ & X \times S^m & \end{tikzcd}$$

when $Y = S^m$. Especially note that there exists the commutative diagram for maps α and β in (4)

$$(5) \qquad \begin{tikzcd} & X(n) \vee Y(n) \arrow[dl, "i_X(n) \vee i_Y(n)"'] \arrow[dr, "incl."] & \\ (X \times Y)(n) \arrow[rr, shift left, "\alpha"] & & \cup_{i+j=n} X(i) \times Y(j) \arrow[ll, shift left, "\beta"] \end{tikzcd}$$

in this paper. Then we have

Proposition 1.4. *In $\pi_m(Z)$,*

$$G_m^{(n)}(Z,X;f) = G_m(Z,X(n-1);f\circ p_{n-1}^X).$$

In particular, $G_m^{(n)}(X) = G_m(X,X(n-1);p_{n-1}^X)$.

Then we can calculate the (rational) (n)-evaluation subgroups. Especially $G_m^{(n)}(Z, X; f)$ is a subgroup of $\pi_m(Z)$.

Theorem 1.5. *There is a decreasing sequence of sets*

$$[X,Z] = g^{\underset{1}{\perp}}(X,Z) \supset \cdots \supset g^{\underset{n}{\perp}}(X,Z) \supset g^{\underset{n+1}{\perp}}(X,Z) \supset \cdots \supset g^{\perp}(X,Z)$$

for a map $g : Y \to Z$. Especially, there is a sequence of subgroups in the m-th homotopy group $\pi_m(Z)$:

$$\pi_m(Z) = G_m^{(1)}(Z,X;f) \supset \cdots \supset G_m^{(n)}(Z,X;f) \supset G_m^{(n+1)}(Z,X;f) \supset \cdots \supset G_m(Z,X;f)$$

for a map $f : X \to Z$. In particular

$$\pi_m(Z) = G_m^{(1)}(Z) \supset \cdots \supset G_m^{(n)}(Z) \supset G_m^{(n+1)}(Z) \supset \cdots \supset G_m(Z).$$

If $cat(X) \leq n$ and Z be an $H(n)$-space, then $\mathcal{F}_*(X,Z,*)$ is an H-space [5, Proposition 1]. Here $\mathcal{F}_*(X,Z,*)$ is the function space of based maps in the component of the trivial map. In this paper, we give

Corollary 1.6. *If $cat(X) < n$, then $G_m^{(n)}(Z,X;f) = G_m(Z,X;f)$ and $G_m^{(n)}(X) = G_m(X)$.*

Finally define $m_\perp(f,g)$ as the greatest integer n such that $f \underset{n}{\perp} g$ for maps $f : X \to Z$ and $g : Y \to Z$. Denote $m_\perp(f,g) = \infty$ if $f \perp g$. It is equal to $m_H(X)$ [5, §3] if $X = Y = Z$ and $f = g = id_X$. Recall that there is an useful equation: $m_H(X_{\mathbb{Q}}) = dl(X) - 1$ [5, Proposition 8], where $dl(X)$ is the differential length of the model of X [10]. We know that $m_\perp(f,g) \geq n$ if Z is an $H(n)$-space from the map $F(n)\circ(f\times g)(n) : (X\times Y)(n) \to (Z\times Z)(n) \to Z$. Also $m_\perp(f,g) \leq m_\perp(f\circ h, g)$ for any map $h : X' \to X$. We propose a

Problem A. *Estimate $m_\perp(f,g)$ for given maps $f : X \to Z$ and $g : Y \to Z$.*

In this paper, we consider in the homotopy category. So, for example, a commutative diagram means a homotopy commutative diagram. In the followings, we suppose that spaces have the homotopy type of nilpotent CW complexes when we rationalize them or consider the Sullivan models of them. Put $e_Z : Z \to Z_{\mathbb{Q}}$ and $f_{\mathbb{Q}} = e_Z \circ f : X \to Z_{\mathbb{Q}}$ the rationalizations of Z and $f : X \to Z$, respectively [8]. Then $\pi_n(Z_{\mathbb{Q}}) \cong \pi_n(Z)_{\mathbb{Q}} := \pi_n(Z)\otimes\mathbb{Q}$ for $n > 1$. By the universality of localization, $f_{\mathbb{Q}}$ equivalent to $\tilde{f}_{\mathbb{Q}} : X_{\mathbb{Q}} \to Z_{\mathbb{Q}}$, often we do not distinguish from $f_{\mathbb{Q}}$. We prepare the Sullivan minimal model [4] in Section 2. Whether or not there is a pairing with axes f and g, (i.e., $f \perp g$), relates with the rational Toomer invariants [1] of f, g and Z as we can see in Section 2. We give some examples of the sets of rational axes of (n)-pairing in Section 3. In the examples, we focus in the cases that $H^*(Z;\mathbb{Q})$ is monogenic for the target space Z in Definition 1.3. We see, even in the cases, that it may not be easy to determine $m_\perp(f_\mathbb{Q}, g_\mathbb{Q})$. Finally, we consider about the (n)-version of Gottlieb group [7],[19],[20], G-sequence and Gottlieb homology of a fibration [11], [13] in Section 4.

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2 Sullivan model, Toomer invariant and LS-category

In the first half of this section, we use the *Sullivan minimal model* $M(X)$ of a space X. It is a free $\mathbb{Q}$-commutative differential graded algebra (DGA) $(\Lambda V, d)$ with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i\geq 1} V^i$ where $\dim V^i < \infty$ and a decomposable differential and $d \circ d = 0$. Denote the degree of an element x of a graded algebra as $|x|$, the $\mathbb{Q}$-vector space of basis $\{v_i\}_i$ as $\mathbb{Q} < v_i >_i$ and the ideal in ΛV generated by V as $\Lambda^+ V$. Also $\Lambda^i V := \Lambda^+ V \cdot ... \cdot \Lambda^+ V$ (i-times). A map $f : X \to Z$ has a minimal model which is a DGA-map $M(f) : M(Z) \to M(X)$. Notice that $M(X)$ determines the rational homotopy type of X, especially $H^*(X;\mathbb{Q}) \cong H^*(M(X))$ and $\pi_i(X) \otimes \mathbb{Q} \cong Hom(V^i, \mathbb{Q})$. See [4] for a general introduction and the standard notations.

Definition 2.1 ([1])**.** The (rational) Toomer invariant of a space X is $e_0(X) := max\{n|$ there is a non-exact cocycle $\alpha \in \Lambda^{\geq n} V$ such that $0 \neq [\alpha] \in H^*(X;\mathbb{Q})\}$ for $M(X) = \Lambda V$. The (rational) Toomer invariant of a map $f : X \to Z$ is $e_0(f) := max\{n|$ there is a non-exact cocycle $\alpha \in \Lambda^{\geq n} W$ such that $0 \neq [M(f)(\alpha)] \in H^*(X;\mathbb{Q})\}$ for $M(Z) = \Lambda W$.

Then there are relations: $e_0(X \times Y) = e_0(X) + e_0(Y)$, $e_0(f) \leq min(e_0(X), e_0(Z))$ and $e_0(X) \leq cat(X_\mathbb{Q}) \leq cat(X)$ [1], [2], [4]. First we note that the Toomer invariant of map plays a natural part for a necessary condition for the existence of a pairing with axes as follows.

Lemma 2.2. *If F is a pairing with axes $f : X \to Z$ and $g : Y \to Z$, then $e_0(f) \leq e_0(F)$ and $e_0(g) \leq e_0(F)$.*

Proof. Put $e_0(F) = m$. Suppose that $e_0(f) > e_0(F)$ or $e_0(g) > e_0(F)$. Put $M(Z) = (\Lambda W, d)$. Then $M(f)(a) \not\sim 0$ (not cohomologous to zero) or $M(g)(a) \not\sim 0$ for some cocyle $a \in \Lambda^{>m} W$. From the following commutative diagram:

$$(1)' \quad \begin{tikzcd} M(X \times Y) = M(X) \otimes M(Y) \arrow[r, "proj."] \arrow[d, "proj."'] & M(Y) \\ M(X) & M(Z), \arrow[l, "M(f)"] \arrow[u, "M(g)"'] \arrow[ul, dotted, "M(F)"'] \end{tikzcd}$$

we have
$$0 \sim M(F)(a) = M(f)(a) \otimes 1 + 1 \otimes M(g)(a) + b \not\sim 0$$
for some $b \in \Lambda^+ V \otimes \Lambda^+ U$ for $M(X) = \Lambda V$ and $M(Y) = \Lambda U$. It is a contradiction. ∎

Recall from [3] that a model of the n-th Ganea fibration p_n^X is given by the composition
$$(\Lambda V, d) \overset{p_n}{\to} (\Lambda V/\Lambda^{>n} V, \overline{d}) \hookrightarrow (\Lambda V/\Lambda^{>n} V, \overline{d}) \oplus S$$
where p_n is the natural projection and the second map is the canonical injection together with $S \cdot S = S \cdot V = 0$ and $dS = 0$. As p_n is functorial and the second map admits a left inverse over $(\Lambda V, d)$, we may use the realization of $p_n : (\Lambda V, d) \to (\Lambda V/\Lambda^{>n} V, \overline{d})$ as substitute for the Ganea fibration (see [5]).

Proposition 2.3. *If F is an (n)-pairing with axes $f : X \to Z$ and $g : Y \to Z$, then $min(n, e_0(f)) \leq e_0(F)$ and $min(n, e_0(g)) \leq e_0(F)$.*

Proof. We have the following commutative diagram:

$$(2)' \quad \begin{tikzcd}
(\Lambda(V \oplus U)/\Lambda^{>n}(V \oplus U), \overline{d_X \otimes 1 + 1 \otimes d_Y}) \arrow[r, "\overline{proj.}"] \arrow[d, "\overline{proj.}"'] & (\Lambda U/\Lambda^{>n} U, \overline{d_Y}) \\
(\Lambda V/\Lambda^{>n} V, \overline{d_X}) & M(Z) \arrow[l, "p_n \circ M(f)"] \arrow[u, "p_n \circ M(g)"'] \arrow[ul, dotted, "M(F)"']
\end{tikzcd}$$

for $M(X) = (\Lambda V, d_X)$ and $M(Y) = (\Lambda U, d_Y)$. Then the proof is similar to the proof of Lemma 2.2. ∎

Proposition 2.4. *Suppose $H^*(Z; \mathbb{Q}) \cong \mathbb{Q}[z]/(z^n)$. If $e_0(f) + e_0(g) < n$ or $N < n$, then $f_{\mathbb{Q}} \underset{N}{\perp} g_{\mathbb{Q}}$.*

Proof. Define the (N)-pairing map $F(N) : (X \times Y)(N)_{\mathbb{Q}} \to Z_{\mathbb{Q}}$ by $M(F(N))(z) := M(f)(z) \otimes 1 + 1 \otimes M(g)(z)$. Then it is well defined since
$$M(F(N))(z^n) = \sum_{i=0}^{n} \binom{n}{i} (M(f)(z))^i \otimes (M(g)(z))^{n-i} \sim 0$$
from the assumption. ∎

Example 2.5. Suppose X, Y and Z are simply connected rational spaces. Let $H^*(X; \mathbb{Q}) = \mathbb{Q}[x]/(x^l)$, $H^*(Y; \mathbb{Q}) = \mathbb{Q}[y]/(y^m)$, $H^*(Z; \mathbb{Q}) = \mathbb{Q}[z]/(z^n)$ and $|x| = i, |y| = j, |z| = k$ all even with $i|k$ and $j|k$. Then there is a non-trivial (N)-pairing if and only if $li + mj \leq nk$ or $N < n$. Indeed, when $li + mj \leq nk$ or $N < n$, we have the commutative diagram:

$$\begin{tikzcd}
\mathbb{Q}[x,y]/(x^l, \{x^s y^t\}_{s+t=N+1}, y^m) \arrow[r, "proj."] \arrow[d, "proj."'] & \mathbb{Q}[y]/(y^m, y^{N+1}) \\
\mathbb{Q}[x]/(x^l, x^{N+1}) & \mathbb{Q}[z]/(z^n) \arrow[l, "f^*"] \arrow[u, "g^*"'] \arrow[ul, dotted, "h"']
\end{tikzcd}$$

where $f^*(z) := x^{k/i}$, $g^*(z) := y^{k/j}$ and $h(z) := x^{k/i} + y^{k/j}$. It is well defined, that is, $h(z^n) = 0 = (x^{k/i} + y^{k/j})^n$. Then there is an (N)-pairing map $F : (X \times Y)(N) \to Z$ with $F^*(z) = h$. Conversely, if $li + mj > nk$ and $N \geq n$, there is no map h in the above diagram.

Corollary 2.6. *When Z is a simply connected rational space with $\pi_i(Z) = \mathbb{Q}$ for $i = k,\ kn-1$ and zero for the other $i > 0$, there are the following two sequences:*

$$(1) \quad \mathbb{Q} = \pi_k(Z) = G_k^{(1)}(Z) = \cdots = G_k^{(n-1)}(Z) \supset G_k^{(n)}(Z) = \cdots = G_k(Z) = 0$$

$$(2) \quad \mathbb{Q} = \pi_k(Z) = G_k^{(1)}(Z) = \cdots = G_k^{(n)}(Z) = \cdots = G_k(Z),$$

where Z satisfies (1) iff $H^*(Z;\mathbb{Q}) \cong \mathbb{Q}[z]/(z^n)$ with $|z| = k$ and Z satisfies (2) iff $Z \simeq K(\mathbb{Q},k) \times K(\mathbb{Q},kn-1)$.

Remark 2.7. (1) Recall that $cat(f)$ for $f : X \to Z$ is the least integer n such that X can be covered by $n+1$ open subsets whose images by f are contractible in Z [1]. If F is a pairing with axes $f : X \to Z$ and $g : Y \to Z$, then $cat(f) \leq cat(F)$. Indeed, from [6], $cat(F)$ is the least n such that there is a lifting $\tilde{F}$ of F in the following diagram:

$$\begin{tikzcd}
 & Z(n) \arrow[d, "p_n^Z"] \\
X \times Y \arrow[ur, dotted, "\tilde{F}"] \arrow[r, "F"'] & Z.
\end{tikzcd}$$

Then we can put $\tilde{f} := \tilde{F} \circ i_X : X \to Z(n)$ which satisfies $f \simeq p_n^Z \circ \tilde{f}$. Also we have $cat(g) \leq cat(F)$.

(2) For maps $f : X \to Z$ and $g : Y \to Z$, we have $f \perp g$ if $cat(X) + cat(Y) = n$ and $f \underset{n}{\perp} g$. Indeed, there is a pairing with axes f and g: $X \times Y \underset{s}{\to} (X \times Y)(n) \underset{F_n}{\to} Z$ since there is a section s from $cat(X \times Y) \leq cat(X) + cat(Y)$. See [5, Example 4] as a special case.

Proof of Proposition 1.4. Recall the maps α and β in (4) of §1. For an element a of $G_m^{(n)}(Z, X; f)$, there is a commutative diagram from (5):

$$\begin{tikzcd}
 & & & & S^m \arrow[dllll, bend right, "i_{S^m}"'] \arrow[dl, "s"'] \arrow[ddl, bend left, "a"] \\
X(n-1) \times S^m \arrow[r, "id \times s"'] & X(n) \cup X(n-1) \times S^m(1) \arrow[r, "\beta"] & (X \times S^m)(n) \arrow[dr, "F(n)"] & S^m(n) \arrow[l, "i_{S^m}(n)"'] \arrow[d, "a \circ p_n^{S^m}"] & \\
X(n-1) \arrow[u, "i_{X(n-1)}"] \arrow[r, "p_{n-1,n}^X"'] & X(n) \arrow[ur, "i_X(n)"] \arrow[r, "p_n^X"'] & X \arrow[r, "f"'] & Z, &
\end{tikzcd}$$

where $F(n)$ is an (n)-pairing (see (3)). Then $F(n) \circ \beta \circ (id_{X(n-1)} \times s) : X(n-1) \times S^m \to Z$ is a pairing with axes $f \circ p_{n-1}^X$ and a. Thus $a \in G_m(Z, X(n-1); f \circ p_{n-1}^X)$.

Conversely, for an element a of $G_m(Z, X(n-1); f \circ p^X_{n-1})$, there is a commutative diagram:

$$\begin{tikzcd}
X(n-1) \arrow[r, "p^X_{n-1,n}"] \arrow[d, "i_{X(n-1)}"'] \arrow[dr, phantom, "\text{push out}"] & X(n) \arrow[d, "i"] \arrow[ddr, bend left, "f \circ p^X_n"] & \\
X(n-1) \times S^m(1) \arrow[r] \arrow[d, "id_{X(n)} \times p^{S^m}_1"'] & X(n) \cup X(n-1) \times S^m(1) \arrow[dr, dotted, "\gamma"] & \\
X(n-1) \times S^m \arrow[rr, "F_a"] & & Z,
\end{tikzcd}$$

where F_a is a pairing with axes a and $f \circ p^X_{n-1}$. Note that $i \simeq \alpha \circ i_X(n) : X(n) \to (X \times S^m)(n) \to X(n) \cup X(n-1) \times S^m(1)$ from (5). Also $\gamma \circ \alpha \circ i_{S^m}(n) \simeq \gamma \circ incl. \simeq a \circ p^{S^m}_n : S^m(n) \to Z$ from (5). Thus we see that $\gamma \circ \alpha : (X \times S^m)(n) \to Z$ is an (n)-pairing with axes f and a. Thus $a \in G^{(n)}_m(Z, X; f)$. ∎

Proof of Theorem 1.5. It is sufficient to show that $[X, Z] = g^{\perp}_{1}(X, Z)$. It follows from the commutative diagram:

$$\begin{tikzcd}
 & (X \times Y)(1) \arrow[d, "\alpha"] & \\
X(1) \arrow[ur, "i_X(1)"] \arrow[dr] \arrow[ddr, "f \circ p^X_1"'] & X(1) \times * \cup * \times Y(1) \arrow[d, equal] & Y(1) \arrow[ul, "i_Y(1)"'] \arrow[dl] \arrow[ddl, "g \circ p^Y_1"] \\
 & X(1) \vee Y(1) \arrow[d, dotted] & \\
 & Z &
\end{tikzcd}$$

from (5). ∎

Remark 2.8. Let $j : S^m \to S^m \vee S^m$ denote the usual pinching comultiplication, where the sum $a + b$ is the composition $(a|b) \circ j := \bigtriangledown \circ a \vee b \circ j$ for $a, b \in \pi_m(Z)$. By using the maps α and β in (4), we can see directly that $G^{(n)}_m(Z, X; f)$ is a subgroup of $\pi_m(Z)$ by the commutative diagram induced from (5):

$$\begin{tikzcd}
X(n) \vee S^m(n) \arrow[r, "i_X(n) \vee i_{S^m}(n)"] \arrow[d, "id(n) \vee id(n)"'] & (X \times S^m)(n) \arrow[d, "(id_X \times j)(n)"] \\
(X \vee S^m)(n) \arrow[d, "(id_X \vee j)(n)"'] & (X \times (S^m_a \vee S^m_b))(n) \arrow[d, "\alpha"] \\
(X \vee S^m_a \vee S^m_b)(n) \arrow[d, equal] & X(n) \cup X(n-1) \times (S^m_a \vee S^m_b)(1) \arrow[d, "id_n \cup id_{n-1} \times (p_a(1) \vee p_b(1))"] \\
(S^m_a \vee X \vee S^m_b)(n) \arrow[d, "p_n^{S \vee X \vee S}"'] & X(n) \cup X(n-1) \times (S^m_a(1) \vee S^m_b(1)) \arrow[d, equal] \\
S^m_a \vee X \vee S^m_b \arrow[d, "(a|f|b)"'] & S^m_a(1) \times X(n-1) \cup X(n) \cup X(n-1) \times S^m_b(1) \arrow[d, "\beta_a \cup \beta_b"] \\
Z & (S^m_a \times X)(n) \cup_{X(n)} (X \times S^m_b)(n) \arrow[l, "F_a(n) \cup F_b(n)"]
\end{tikzcd}$$

for $a, b \in G_m^{(n)}(Z, X; f)$ and some (n)-pairings $F_a(n)$ and $F_b(n)$.

Definition 2.9. Define $cat_X(f)$ as the least n such that there is a map s_n in the following commutative diagram:

$$\begin{tikzcd}
X(n) \arrow[dr, "f \circ p_n^X"] & \\
X \arrow[u, dotted, "s_n"] \arrow[r, "f"'] & Z.
\end{tikzcd}$$

Then we have $cat(f) \leq cat_X(f) \leq cat(X)$. Indeed, if $cat_X(f) \leq n$, there is a map $s_n : X \to X(n)$ with $f \simeq f \circ p_n^X \circ s_n \simeq p_n^Z \circ f(n) \circ s_n$ for $f(n) : X(n) \to Z(n)$. Then we have $cat(f) \leq n$ since $f(n) \circ s_n$ is a lifting of f:

$$\begin{tikzcd}
& Z(n) \arrow[d, "p_n^Z"] \\
X \arrow[ur, dotted, "f(n) \circ s_n"] \arrow[r, "f"'] & Z.
\end{tikzcd}$$

Example 2.10. (1) Let $f : S^3 \to S^3 \times S^3$ be the map with $f(x) = (x, *)$. Then $cat(f_{\mathbb{Q}}) = cat_X(f_{\mathbb{Q}}) = cat(X_{\mathbb{Q}}) = 1$.

(2) Let $f : X = S^3 \times S^3 \to S^3$ be the map with $f(x, y) = x$. Then $cat(f_{\mathbb{Q}}) = cat_X(f_{\mathbb{Q}}) = 1 < cat(X_{\mathbb{Q}}) = 2$.

(3) Let $f : X = S^3 \times S^3 \times S^3 \to S^9$ be the map given by collapsing $(S^3 \times S^3 \times S^3)^{<9}$. Then $cat(f_{\mathbb{Q}}) = 1 < cat_X(f_{\mathbb{Q}}) = cat(X_{\mathbb{Q}}) = 3$.

(4) Let $f = h \circ g : X = S^3 \times S^3 \times S^3 \to S^3 \times S^3 \to S^6$ be the map with $g(x, y, z) = (x, y)$ and the collapsing map $h(S^3 \vee S^3) = *$. Then $cat(f_{\mathbb{Q}}) = 1 < cat_X(f_{\mathbb{Q}}) = 2 < cat(X_{\mathbb{Q}}) = 3$.

From Proposition 1.4, we have

Proposition 2.11. *If* $cat_X(f) < n$, *then* $G_m^{(n)}(Z, X; f) = G_m(Z, X; f)$.

Proof. We have $G_m(Z, X(n-1); f \circ p^X_{n-1}) \subset G_m(Z, X; f)$ from the commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[d, dotted, "s_{n-1}"'] \arrow[ddr, "f"] & X \times S^m \arrow[d, dotted, "s_{n-1} \times id_{S^m}"] & S^m \arrow[l, "i_{S^m}"'] \arrow[d, equal] \\
X(n-1) \arrow[r] \arrow[dr, "f \circ p^X_{n-1}"'] & X(n-1) \times S^m \arrow[d, "F_a"] & S^m \arrow[l] \arrow[dl, "a"] \\
 & Z &
\end{tikzcd}$$

for $a \in G_m(Z, X(n-1); f \circ p^X_{n-1})$. ∎

Remark 2.12. For $f, f' : X \to Z$, define $f \underset{(n)}{\sim} f'$ (say (n)-*homotop*) if there is a map $H(n) : X(n) \times I \to Z$ in the commutative diagram:

$$\begin{tikzcd}
X(n) \arrow[r, "i_0"] \arrow[d, "p_n^X"'] & X(n) \times I \arrow[d, dotted, "H(n)"] & X(n) \arrow[l, "i_1"'] \arrow[d, "p_n^X"] \\
X \arrow[r, "f"] & Z & X, \arrow[l, "f'"']
\end{tikzcd}$$

where $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$ for $I = [0, 1]$. If $f \sim f'$, then $f \underset{(n)}{\sim} f'$ for all n. If $f \underset{n}{\perp} g$ for a map $g : Y \to Z$ and $f \underset{(n)}{\sim} f'$, then $f' \underset{n}{\perp} g$.

3 Examples of the set of rational axes

We denote the set of the homotopy classes of rational axes of (N)-pairing of a map $g : Y \to Z$ by

$$g^{\underset{N}{\perp}}(X, Z)_{\mathbb{Q}} := \{[f_{\mathbb{Q}}] \in [X_{\mathbb{Q}}, Z_{\mathbb{Q}}] \mid f_{\mathbb{Q}} \underset{N}{\perp} g_{\mathbb{Q}}\}.$$

Example 3.1. Suppose X, Y and Z are simply connected rational spaces where $H^*(X;\mathbb{Q}) = \mathbb{Q}[x]/(x^l)$, $H^*(Y;\mathbb{Q}) = \mathbb{Q}[y]/(y^m)$, $H^*(Z;\mathbb{Q}) = \mathbb{Q}[z]/(z^n)$ and $|x| = i, |y| = j, |z| = k$ with $i|k$ and $j|k$. Put $g^*(z) = y^{k/j}$. When $li + mj \leq nk$, $g^{\perp}(X, Z) = \mathbb{Q}$. When $li + mj > nk$, $g^{\underset{N}{\perp}}(X, Z) = \mathbb{Q}$ for $N < n$ and $g^{\underset{N}{\perp}}(X, Z) = *$ for $N \geq n$.

Example 3.2. Put $X = S^2 \times (\mathbb{C}P^3 \vee S^6)$ with $H^*(X;\mathbb{Q}) = \mathbb{Q}[x_1, x_2, x_3]/(x_1^2, x_2^4, x_2x_3, x_3^2)$; $|x_1| = |x_2| = 2$, and $|x_3| = 6$, $Y = \mathbb{C}P^n$ with $H^*(Y;\mathbb{Q}) = \mathbb{Q}[y]/(y^{n+1})$; $|y| = 2$, $H^*(Z;\mathbb{Q}) = \mathbb{Q}[z]/(z^3)$; $|z| = 8$ and $g^*(z) = y^4$ ($n < 12$). Then the map

$$h : \mathbb{Q}[z]/(z^3) \to \mathbb{Q}[x_1, x_2, x_3]/(x_1^2, x_2^4, x_2x_3, x_3^2) \otimes \mathbb{Q}[y]/(y^{n+1})$$

is given by

$$h(z) = a_1x_1x_3 + a_2x_1x_2^3 + b_1x_3y + b_2x_1x_2y^2 + b_3x_2^2y^2 + b_4x_1y^3 + b_5x_2y^3 + \\ b_6x_2^3y + b_7x_1x_2^2y + y^4$$

with $a_i, b_i \in \mathbb{Q}$. Therefore $g^{\perp}_{N}(X,Z)_{\mathbb{Q}} \subset \mathbb{Q} < a_1, a_2 >$. Since h is an algebra map, we have

$$\begin{aligned} 0 &= h(z^3) \\ &= (3a_1x_1x_3 + 3a_2x_1x_2^3 + 6b_2b_3x_1x_2^3 + 6b_1b_4x_1x_3 + 6b_4b_6x_1x_2^3 + 6b_5b_7x_1x_2^3)y^8 \\ &+ (3b_1x_3 + 6b_2b_5x_1x_2^2 + 6b_3b_4x_1x_2^2 + 6b_3b_5x_2^3 + 3b_6x_2^3 + 3b_7x_1x_2^2)y^9 \\ &+ (3b_2x_1x_2 + 3b_3x_2^2 + 6b_4b_5x_1x_2 + 3b_5^2x_2^2)y^{10} \\ &+ 3(b_4x_1 + b_5x_2)y^{11}. \end{aligned}$$

Then we have the following table of equations of the coefficients $a_1, a_2, b_1, .., b_7$:

$g^{\perp}_{N}(X,Z)_{\mathbb{Q}}$	$N \leq 9$	$N = 10$	$N = 11$	$N = 12$	$\cdots$	$N = \infty$
$n \leq 8$	ϕ	(1)	(1)	(1), (2)	$\cdots$	(1), (2)
$n = 9$	ϕ	(1), (3)	(1), (3)	(1) $\sim$ (4)	$\cdots$	(1) $\sim$ (4)
$n \geq 10$	ϕ	(1), (3)	(1), (3)	(1) $\sim$ (6)	$\cdots$	(1) $\sim$ (6)

where

$$\begin{array}{l} (1)\ \ a_1 + 2b_1b_4 = 0 \quad (2)\ \ a_2 + 2b_2b_3 + 2b_4b_6 + 2b_5b_7 = 0 \\ (3)\ \ b_1 = 0 \quad (4)\ \ 2b_2b_5 + 2b_3b_4 + b_7 = 0,\ 2b_3b_5 + b_6 = 0 \\ (5)\ \ b_2 + 2b_4b_5 = 0,\ b_3 + b_5^2 = 0 \\ (6)\ \ b_4 = 0,\ b_5 = 0. \end{array}$$

Therefore we have $g^{\perp}(X,Z)_{\mathbb{Q}} \cong \mathbb{Q} < a_1, a_2 >$ if $n \leq 8$ since the equation $(1), (2)$ does not restrict a_1 and a_2. Also $g^{\perp}(X,Z)_{\mathbb{Q}} \cong \mathbb{Q} < a_2 >$ if $n = 9$ from $(1) \sim (4)$ and $g^{\perp}(X,Z)_{\mathbb{Q}} = *$ if $n \geq 10$ from $(1) \sim (6)$. When $n \geq 9$, for example, we have $g^{\perp}_{N}(X,Z)_{\mathbb{Q}} \cong \mathbb{Q} < a_2 >$ for $N = 10$. In fact, we have $a_1 = 0$ from the equation $(1), (3)$ of the coefficients of elements in $(\mathbb{Q}[x_1, x_2, x_3]/(x_1^2, x_2^4, x_2x_3, x_3^2) \otimes \mathbb{Q}[y]/(y^{n+1}))^{\leq 10}$ but a_2 is free. Thus we have the following table:

$g^{\perp}_{N}(X,Z)_{\mathbb{Q}}$	$N \leq 9$	$N = 10$	$N = 11$	$N = 12$	$\cdots$	$N = \infty$
$n \leq 8$	$\mathbb{Q} \oplus \mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Q}$	$\cdots$	$\mathbb{Q} \oplus \mathbb{Q}$
$n = 9$	$\mathbb{Q} \oplus \mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\cdots$	$\mathbb{Q}$
$n \geq 10$	$\mathbb{Q} \oplus \mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$*$	$\cdots$	$*$

where $\mathbb{Q} \oplus \mathbb{Q} = \mathbb{Q} < a_1, a_2 >$ and $\mathbb{Q} = \mathbb{Q} < a_2 >$.

Example 3.3. Consider an 11-dimensional manifold X such that

$$M(X) = (\Lambda(x_1, x_2, x_3), d); \quad |x_1| = |x_2| = 3,\ |x_3| = 5$$

with $dx_1 = dx_2 = 0$ and $dx_3 = x_1x_2$. Note that X is realized as the total space of the fibration $S^5 \to X \to S^3 \times S^3$ with classifying map the generator of $H^6(S^3 \times S^3)$. Put $Y = \mathbb{C}P^n$ with $H^*(Y;\mathbb{Q}) = \mathbb{Q}[y]/(y^{n+1})$; $|y| = 2$, $H^*(Z;\mathbb{Q}) = \mathbb{Q}[z]/(z^3)$; $|z| = 8$ and $g^*(z) = y^4$ ($n < 12$). Here $M(Z) = (\Lambda(z, z'), d)$ with $dz = 0$ and $dz' = z^3$ and $H^*(\Lambda(z, z'), d) \cong \mathbb{Q}[z]/(z^3)$. Then the DGA-map

$$h : (\Lambda(z, z'), d) \to (\Lambda(x_1, x_2, x_3), d) \otimes (\mathbb{Q}[y]/(y^{n+1}), 0)$$

is given by

$$h(z) = a_1x_1x_3 + a_2x_2x_3 + y^4 \quad a_i \in \mathbb{Q} \ \textit{and} \ h(z') = 0$$

Then we have $h(z^3) = 3a_1x_1x_3y^8 + 3a_2x_2x_3y^8$. For $a_1, a_2 \in \mathbb{Q} - \{0\}$, we see $f_{\mathbb{Q}} \perp g_{\mathbb{Q}}$ when $n < 8$ and $f_{\mathbb{Q}} \underset{N}{\perp} g_{\mathbb{Q}}$ when $N < 10$ and $8 \geq n$. Here the model of a map f is given by $M(f)(z) = a_1x_1x_3 + a_2x_2x_3$.

Next consider, for the above $g : Y \to Z$, the connected sum of products of two spheres

$$X' = (S^3 \times S^8)\sharp(S^3 \times S^8)$$

with $H^*(X';\mathbb{Q}) \cong \Lambda(x_1, x_2, v_1, v_2)/(x_1x_2, x_1v_1 - x_2v_2, x_1v_2, x_2v_1, v_1^2, v_1v_2, v_2^2)$ where $|x_1| = |x_2| = 3$ and $|v_1| = |v_2| = 8$. Note that X' is formal [4] but X is not formal, with same cohomologies; $H^*(X;\mathbb{Q}) \cong H^*(X';\mathbb{Q})$. Then the algebra map

$$h' : \mathbb{Q}[z]/(z^3) \to \Lambda(x_1, x_2, v_1, v_2)/(x_1x_2, x_1v_1 - x_2v_2, x_1v_2, x_2v_1, v_1^2, v_1v_2, v_2^2) \otimes \mathbb{Q}[y]/(y^{n+1})$$

is given by

$$h'(z) = a_1v_1 + a_2v_2 + y^4 \quad a_i \in \mathbb{Q}.$$

Then we have $h'(z^3) = 3a_1v_1y^8 + 3a_2v_2y^8$. For $a_1, a_2 \in \mathbb{Q} - \{0\}$, we see $f_{\mathbb{Q}} \perp g_{\mathbb{Q}}$ when $n < 8$ and $f_{\mathbb{Q}} \underset{N}{\perp} g_{\mathbb{Q}}$ when $N < 9$ and $8 \geq n$. Here the model of a map f is given by $M(f)(z) = a_1v_1 + a_2v_2$.

Notice that $g^{\overset{\perp}{9}}(X, Z)_{\mathbb{Q}} = \mathbb{Q} \oplus \mathbb{Q}$ but $g^{\overset{\perp}{9}}(X', Z)_{\mathbb{Q}} = *$, when $n \geq 8$. Thus the set of the homotopy classes of axes of (N)-pairing is not determined by cohomology in general.

4 rational $G(n)$-group and $G(n)$-sequence

Let A be a DGA $A = (A^*, d_A)$ with $A^* = \oplus_{i\geq 0} A^i$, $A^0 = \mathbb{Q}$, $A^1 = 0$ and the augmentation $\epsilon : A \to \mathbb{Q}$. Define $Der_i A$ the vector space of $\mathbb{Q}$-derivations of A decreasing the degree by $i > 0$, where $\theta(xy) = \theta(x)y + (-1)^{i|x|}x\theta(y)$ for $\theta \in Der_i A$. We denote $\oplus_{i>0} Der_i A$ by $Der A$. The boundary operator $\delta : Der_* A \to$

$Der_{*-1}A$ is defined by $\delta(\sigma) = d_A \circ \sigma - (-1)^{|\sigma|}\sigma \circ d_A$. For a DGA-map $\phi : A \to B$, define a ϕ-derivation of degree n to be a linear map $\theta : A^* \to B^{*-n}$ with $\theta(xy) = \theta(x)\phi(y) + (-1)^{n|x|}\phi(x)\theta(y)$ and $Der(A, B; \phi)$ the vector space of ϕ-derivations. The boundary operator $\delta_\phi : Der_*(A, B; \phi) \to Der_{*-1}(A, B; \phi)$ is defined by $\delta_\phi(\sigma) = d_B \circ \sigma - (-1)^{|\sigma|}\sigma \circ d_A$. Note $Der_*(A, A; id_A) = Der_*(A)$.

Recall the method of derivations in [12] and [13]. For $\phi : A = (\Lambda U, d_A) \to B$, the composition with the augmentation $\epsilon' : B \to \mathbb{Q}$ induces a chain map $\epsilon'_* : Der_n(A, B; \phi) \to Der_n(A, \mathbb{Q}; \epsilon)$. Define

$$G_n(A, B; \phi) := Im(H_n(\epsilon'_*) : H_n(Der(A, B; \phi)) \to Hom_n(U, \mathbb{Q})).$$

Especially

$$G_n(\Lambda U, d_A) := Im(H_n(\epsilon_*) : H_n(Der(\Lambda U, d_A)) \to Hom_n(U, \mathbb{Q})),$$

that is, $G_*(A, A; id_A) = G_*(A)$. Note that $z^* \in Hom_n(U, \mathbb{Q})$ (z^* is the dual of the basis element z of U^n) is in $G_n(A, B; \phi)$ if and only if z^* extends to a ϕ-derivation cycle θ of $Der_n(A, B; \phi)$, i.e., $\delta_\phi(\theta) = 0$ [4, Sec.29(d)]. Let $\xi : X \xrightarrow{j} E \to B$ be a fibration of simply connected CW complexes. Put the KS-extension of ξ [4] as $(\Lambda W, d_B) \to (\Lambda W \otimes \Lambda V, D) \underset{J}{\to} (\Lambda V, d) \cong M(X)$ in the followings.

Lemma 4.1. *[13]* $G_m(E_{\mathbb{Q}}, X_{\mathbb{Q}}; j_{\mathbb{Q}}) \cong G_m(\Lambda W \otimes \Lambda V, \Lambda V; J)$.

From Proposition 1.4, we have

Proposition 4.2. *Let $\xi : X \xrightarrow{j} E \to B$ be a fibration of simply connected CW complexes. Then*

$$G^{(n)}_m(E_{\mathbb{Q}}, X_{\mathbb{Q}}; j_{\mathbb{Q}}) \cong G_m(\Lambda W \otimes \Lambda V, \Lambda V/\Lambda^{\geq n}V,\ \overline{\delta_J}) \ \text{ and}$$
$$G^{(n)}_m(X_{\mathbb{Q}}) \cong G_m(\Lambda V, \Lambda V/\Lambda^{\geq n}V,\ \overline{\delta})$$

for all m. Here $\overline{\delta} = p_n \circ \delta$ and $\overline{\delta_J} = p_n \circ \delta_J$ for the projection $p_n : \Lambda V \to \Lambda V/\Lambda^{\geq n}V$.

The Gottlieb sequence (simply G-sequence) [11],[13] of a fibration $\xi : X \xrightarrow{j} E \xrightarrow{p} B$ is the restriction of the homotopy exact sequence of ξ:

$$\cdots \to G_m(X) \xrightarrow{J_m} G_m(E, X; j) \xrightarrow{P_m} \pi_m(B) \xrightarrow{\partial_m} G_{m-1}(X) \to \cdots .$$

The (n)-version:

$$\cdots \to G^{(n)}_m(X) \xrightarrow{J^{(n)}_m} G^{(n)}_m(E, X; j) \xrightarrow{P^{(n)}_m} \pi_m(B) \xrightarrow{\partial_m} G^{(n)}_{m-1}(X) \to \cdots$$

is called the *G(n)-sequence* of ξ. Then we can define the m-th *(n)-Gottlieb homology group* of ξ as $G^{(n)}H_m(\xi) := Ker\ P^{(n)}_m / Im\ J^{(n)}_m$.

Example 4.3. (Compare with [13, Ex.4.5]) Put $M(X) = (\Lambda(v_1, v_2, \cdots, v_{n+1}, v), d)$ ($n > 1$ is odd) with $dv_* = 0$, $dv = v_1 v_2 \cdots v_{n+1}$ of $|v_1| = \cdots = |v_{n+1}| = 3$ and $M(B) = (\Lambda w, 0)$. Consider the fibration $\xi_{\mathbb{Q}} : X_{\mathbb{Q}} \to E_{\mathbb{Q}} \to B_{\mathbb{Q}}$ where $Dv = v_1 v_2 \cdots v_{n+1} + w v_{n+1}$ and $Dv_* = 0$. Then $G_3(X_{\mathbb{Q}}) = 0$ and $GH_3(\xi_{\mathbb{Q}}) = \mathbb{Q} < v_1^*, \cdots, v_n^* >$. In fact, $\delta_J(v_i^* + (-1)^i w^* \otimes v_1 \cdots \check{v}_i \cdots v_n) = 0$ for $i \leq n$. On the other hand, $G^{(n)}_3(X_{\mathbb{Q}}) = \mathbb{Q} < v_1^*, \cdots, v_{n+1}^* >$ since $\overline{\delta}(v_i^*) = 0$ in $Der(\Lambda V, \Lambda V/\Lambda^{\geq n}V)$. Therefore $G^{(n)}H_3(\xi_{\mathbb{Q}}) = 0$.

Recall that ξ is G-trivial if $0 \to G_m(X) \stackrel{J_m}{\to} G_m(E,X;j) \stackrel{P_m}{\to} \pi_m(B) \to 0$ is exact for all m [13]. By analogy, we say that ξ is *G(n)-trivial* if $0 \to G_m^{(n)}(X) \stackrel{J_m^{(n)}}{\to} G_m^{(n)}(E,X;j) \stackrel{P_m^{(n)}}{\to} \pi_m(B) \to 0$ is exact for all m.

Corollary 4.4. *If $\xi_{\mathbb{Q}} : X_{\mathbb{Q}} \to E_{\mathbb{Q}} \to B_{\mathbb{Q}}$ is G-trivial for a finite complex X, it is G(n)-trivial.*

Proof. Denote by $h_{\mathbb{Q}} : B_{\mathbb{Q}} \to Baut_1 X_{\mathbb{Q}}$ the classifying map of $\xi_{\mathbb{Q}}$. From [13, Thorem 4.2], we have $\pi_*(h_{\mathbb{Q}}) = 0$ under this condition. Then as the proof of [13, Thorem 4.2 (1)⇒ (2)] we have

$$H_m(Der(\Lambda W \otimes \Lambda V, \Lambda V/\Lambda^{\geq n}V)) \cong H_m(Der(\Lambda V, \Lambda V/\Lambda^{\geq n}V)) \oplus Hom_m(W,\mathbb{Q}).$$

for all m. Thus we have the corollary from Proposition 4.2. ∎

The converse is not true. For example, put $M(X) = (\Lambda(v_1, v_2, \cdots, v_{n+2}, v), d)$ (n is odd) with $dv_* = 0$, $dv = v_{n+1}v_{n+2}$ and $M(B) = (\Lambda w, 0)$. Consider the fibration $\xi_{\mathbb{Q}} : X_{\mathbb{Q}} \to E_{\mathbb{Q}} \to B_{\mathbb{Q}}$ where $Dv = v_{n+1}v_{n+2} + wv_1 \cdots v_n$ and $Dv_* = 0$. Then $\xi_{\mathbb{Q}}$ is G(n)-trivial from $\overline{\delta_J}(w^*) = 0$, but it is not G-trivial since $\pi_*(h_{\mathbb{Q}})(w^*) \neq 0$ for the classifying map $h_{\mathbb{Q}}$ [13, Theorem 3.2]. Note that 'G-trivial' and 'G(n)-trivial' are equivalent for a general fibration ξ when $catX < n$ from Corollary 1.6.

Remark 4.5. For a fibration $\xi : X \to E \to B$, there is a decomposition of $G(E_{\mathbb{Q}}) = \oplus_m G_m(E_{\mathbb{Q}})$ as $G(E_{\mathbb{Q}}) = S \oplus T \oplus U$ [21] induced from the G-sequence. From the G(n)-sequence above, we have a similar decomposition

$$G^{(n)}(E_{\mathbb{Q}}) = S(n) \oplus T(n) \oplus U(n) \ \subset \ G^{(n)}(X_{\mathbb{Q}}) \oplus G^{(n)}H(\xi_{\mathbb{Q}}) \oplus G^{(n)}(B_{\mathbb{Q}}, E_{\mathbb{Q}}; p_{\mathbb{Q}}).$$

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