On a class of first order congruences of lines*

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Abstract

We study a class of new examples of congruences of lines of order one, *i.e.* the congruences associated to the completely exceptional Monge-Ampère equations. We prove that they are in general not linear, and that through a general point of the focal locus there passes a planar pencil of lines of the congruence. In particular, the completely exceptional Monge-Ampère equations are of Temple type.

Introduction

In [AF01], Agafonov and Ferapontov introduce and study congruences of lines associated to hyperbolic systems of conservation laws. They prove that in $\mathbb{P}^N$, $N \leq 4$, these families of lines, if the systems are of Temple type, are, in fact, linear congruences.

Successively, they consider, for all $N \in \mathbb{N}$, the completely exceptional Monge-Ampère equations, studied in [Boi92], and state that these systems are of Temple type, and for $N \geq 5$, the associated congruence of lines is not linear.

In [DM07], we have studied the first interesting case, *i.e.* $N = 5$, giving a geometrical construction of the congruence of lines B_{MA} in $\mathbb{P}^5$ in this way obtained: it results to be a first order congruence and a smooth Fano 4-fold in $\mathbb{P}^{11}$ in the Plücker embedding; its focal locus is a sextic threefold X such that the lines of B_{MA} through a general point of X form a (planar) pencil. This confirms the fact that the considered system is of Temple type.

*This research was partially supported by MiUR, project "Geometria delle varietà algebriche e dei loro spazi di moduli" for both authors, and by Indam project "Birational geometry of projective varieties" for the second one

2000 *Mathematics Subject Classification :* Primary 14M15; Secondary 53A25, 14M07.

Key words and phrases : Congruences of lines, completely exceptional Monge-Ampère equations.

We have also determined the class of B_{MA} in the Chow ring of $\mathbb{G}(1,5)$, finding that the multidegree of B_{MA} is $(1,3,3)$. B_{MA} is indeed the residual in a quadratic congruence of the union of a subgrassmannian $\mathbb{G}(1,L)$ (where L is a $\mathbb{P}^3$) with a congruence of multidegree $(1,3,0)$ contained in a very special linear congruence. In particular B_{MA} is not a linear congruence. This brings us to some interesting conclusions on the focal variety $X \subset \mathbb{P}^5$, in particular that it is not 2-normal.

Here we describe the congruences associated to completely exceptional Monge-Ampère equations for all $N \geq 5$. The main difference, compared with the case of $\mathbb{P}^5$, is that, starting from $\mathbb{P}^9$, in the construction of B_{MA} it is necessary to consider congruences contained in hypersurfaces of degree bigger than 2. Precisely, B_{MA} is the residual in a congruence contained in a hypersurface of degree μ, with $4\mu - 3 \leq N \leq 4\mu$ of a special first order congruence B_C (that we describe in Section 4).

As for the (pure) focal locus X, we are able to deduce that its codimension in $\mathbb{P}^N$ is 2 and that it is not $(N-3)$-normal, *i.e.* $h^1(\mathcal{I}_X(N-3)) \neq 0$. This follows from the fact that a general hyperplane section of X is contained in a hypersurface of degree $N-2$, whereas X is not, because otherwise the lines of B_{MA}, which are $(N-1)$-secant to X, would be contained in this hypersurface. We also give an upper bound on the degree of X, and we show that the focal locus F contains a parasitical linear component of codimension two.

It would be interesting to know if X is irreducible and if B_{MA} is smooth for all N, as in the case $N = 5$.

This article is structured as follows: in Section 1, we recall some general definitions and results about congruences of lines. In Section 2 we consider higher secant varieties of Grassmann varieties, and we recall that these are defined by the rank of antisymmetric matrices. Then, we introduce a suitable generalisation of the classical Gauss map with respect to these varieties. Finally, we concentrate on linear spaces of antisymmetric matrices of constant rank four, in Subsection 2.1. Then in Section 3 we explain and investigate the association between completely exceptional Monge-Ampère type equations and congruences of lines. In particular, we show that all these congruences B_{MA} are of order one, and that through a general point of the focal locus of B_{MA} there passes a (planar) pencil of lines of the congruence. In particular, from general results of [AF01], we deduce that the completely exceptional Monge-Ampère type equations are of Temple type. Section 4 is devoted to the study of the congruences which arise naturally in the study of B_{MA}, and in particular to the geometrical description of B_{MA} itself.

1 Preliminaries on congruences of lines

We will work with schemes and varieties over the complex field $\mathbb{C}$, with standard notation and definitions as in [Har77]. In this article, a *variety* will always be projective and irreducible. Besides, we refer to [GH78] for notations about Schubert cycles.

We recall that a *congruence of lines* in $\mathbb{P}^N$ is a flat family (Λ, B, p) of lines of $\mathbb{P}^N$ obtained by a desingularisation of a subvariety B' of dimension $N-1$ of the Grassmannian $\mathbb{G}(1,N)$ of lines of $\mathbb{P}^N$. p is the restriction of the projection

$p_1 : B \times \mathbb{P}^N \to B$ to Λ, while we will denote the restriction of $p_2 : B \times \mathbb{P}^N \to \mathbb{P}^N$ by f. $\Lambda_b := p^{-1}(b)$, $(b \in B)$ will be an element of the family and $f(\Lambda_b) =: \Lambda(b)$ is a line of $\mathbb{P}^N$. We can summarise all these notations in the following two diagrams: the first one defines the family

$$\begin{array}{ccccc} \Lambda := \psi^*(\mathcal{H}_{1,N}) & \xrightarrow{\psi^*} & \mathcal{H}_{1,N} & \xrightarrow{p_2} & \mathbb{P}^N \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p_1}\big\downarrow & & \\ B & \xrightarrow{\psi} & B' \subset \mathbb{G}(1,N), & & \end{array}$$

where $\mathcal{H}_{1,N} \subset \mathbb{G}(1,N) \times \mathbb{P}^N$ is the incidence variety and ψ is the desingularisation map, and the second one explains the notation for the elements of the family

$$\begin{array}{ccc} \Lambda_b \subset \Lambda & \xrightarrow{f:=\psi^* p_2} & \mathbb{P}^N \supset \Lambda(b) := f(\Lambda_b) \\ {\scriptstyle p}\big\downarrow & & \\ b \in B. & & \end{array}$$

A point $y \in \mathbb{P}^N$ is called *fundamental* if its fibre $f^{-1}(y)$ has dimension greater than the dimension of the general one. In particular, if f is dominant, a point is fundamental if $\dim(f^{-1}(y)) > 0$. The *fundamental locus* is the set of the fundamental points. The *focal (sub)scheme* $V \subset \Lambda$ is the scheme of the ramification points of f. The *focal locus*, $\Phi := f(V) \subset \mathbb{P}^N$, is the set of the branch points of f. We endow this locus with the scheme structure given by considering it as the scheme-theoretic image of V under f (see, for example, [Har77]).

To a congruence is associated a *multidegree* $(a_0, \dots, a_\nu)$ if we write

$$[B] = \sum_{i=0}^{\nu} a_i \sigma_{N-1-i,i}$$

—where we put $\nu := \left[\frac{N-1}{2}\right]$—as a linear combination of Schubert cycles of the Grassmannian (recall that $\sigma_{N-1-i,i} := [\{\ell \in \mathbb{G}(1,N) \mid \ell \cap \mathbb{P}^i \neq \emptyset;\ \ell, \mathbb{P}^i \subset \mathbb{P}^{N-i}\}]$). In particular, the *order* a_0 is the number of lines of B passing through a general point of $\mathbb{P}^N$. The fundamental locus is contained in the focal locus and the two loci coincide in the case of a first order congruence, *i.e.* through a focal point there pass infinitely many lines of the congruence. An important result—independent of order and class—due to C. Segre is the following:

Proposition 1. *On every line $\Lambda_b \subset \Lambda$ of the family, the focal subscheme V either coincides with the whole Λ_b—in which case $\Lambda(b)$ is called* focal line*—or is a zero dimensional subscheme of Λ_b of length $N-1$. Moreover, in the latter case, if Λ is a first order congruence, $\Phi \cap \Lambda(b)$ has length $N-1$.*

See [CS] for a modern proof.

The following result gives the converse to the fact that fundamental and focal loci coincide, if the order is one:

Proposition 2. *Let B be a congruence whose focal locus F has codimension at least two. Then, B has order either zero or one.*

See [DP04] for a proof.

2 Grassmannians and higher secant varieties

Let us start by considering the Grassmann variety of lines in $\mathbb{P}^N$ with the Plücker embedding

$$\mathbb{G}(1,N) \subset \mathbb{P}(\wedge^2 V),$$

where $\mathbb{P}^N = \mathbb{P}(V)$, and its (higher) secant varieties $S_k\mathbb{G}(1,N)$, *i.e.*

$$S_k\mathbb{G}(1,N) := \overline{\{P \in \mathbb{P}(\wedge^2 V) \mid P \in \mathbb{P}^k = \langle P_0, \dots, P_k\rangle,\ P_i \in \mathbb{G}(1,N),\ \forall i\}}$$

if $k \le \binom{N}{2} - 1$ (set $S_k\mathbb{G}(1,N) := \mathbb{P}(\wedge^2 V)$ otherwise).

It is well known that we have the chain of inclusions

$$\mathbb{G}(1,N) \subset S_1\mathbb{G}(1,N) \subset S_2\mathbb{G}(1,N) \subset \cdots \subset S_{k_0}\mathbb{G}(1,N) \subsetneq \mathbb{P}(\wedge^2 V), \qquad (1)$$

where $S_{k_0}\mathbb{G}(1,N)$ is the last higher secant variety properly contained in $\mathbb{P}(\wedge^2 V)$. It has codimension one if N is odd, and three if N is even; moreover, if we put $N = 2m+1$ (respectively, $N = 2m$), then $k_0 = m-1$.

The stratification (1) of $\mathbb{P}(\wedge^2 V)$ is given by the tensorial rank. In fact, the points in $\mathbb{G}(1,N)$ are just the projectivisations of the *totally decomposable tensors*, *i.e.* of the form $[u \wedge v]$, with $u, v \in V$, those in $S_1\mathbb{G}(1,N)$ correspond to the tensorial rank two, *i.e.* of the form $[u_1 \wedge v_1 + u_2 \wedge v_2]$, $u_1, v_1, u_2, v_2 \in V$, and so on.

The Plücker coordinates $[p_{ij}]$ of a point in $\mathbb{P}(\wedge^2 V)$ can be put as the entries of an $(N+1)\times(N+1)$-skew-symmetric matrix; the points in $\mathbb{G}(1,N)$ correspond to matrices of rank two, those in $S_1\mathbb{G}(1,N)$ to matrices of rank at most four and, in general the points in $S_i\mathbb{G}(1,N)$, to matrices of rank at most $2(i+1)$. We note moreover that the singular locus of $S_{i+1}\mathbb{G}(1,N)$ is $S_i\mathbb{G}(1,N)$, for all i.

In the dual projective space $\mathbb{P}(\wedge^2 V^*)$, which parametrises the hyperplanes of $\mathbb{P}(\wedge^2 V)$, we have a stratification isomorphic to the one in (1).

The dual Grassmannian $\check{\mathbb{G}}(1,N)$, which parametrises the tangent hyperplanes to $\mathbb{G}(1,N)$, is isomorphic to $S_{k_0}\mathbb{G}(1,N)$: its general point represents a hyperplane which is tangent at a unique point of $\mathbb{G}(1,N)$, if N is odd, and at the points of a 2-plane in $\mathbb{G}(1,N)$, if N is even. More generally, in $\check{\mathbb{G}}(1,N)$ every subvariety of the chain parametrises the hyperplanes which are tangent at all the points of the subgrassmannians of a fixed dimension. In particular, hyperplanes which are tangent along a $\mathbb{G}(1,\mathbb{P}^{N-2})$ form a subvariety which is naturally isomorphic to $\mathbb{G}(N-2,N)$, and therefore isomorphic to $\mathbb{G}(1,N)$; its secant variety $S_i\mathbb{G}(N-2,N)$ parametrises the hyperplanes which are tangent along a $\mathbb{G}(1,N-2(i+1))$.

Besides the filtration, we have also a family of rational maps, φ_i:

$$\begin{array}{ccccccccc}
\mathbb{G}(N-2,N) \subset & S_1\mathbb{G}(N-2,N) & \subset & S_2\mathbb{G}(N-2,N) & \subset \cdots \subset & S_{k_0}\mathbb{G}(N-2,N) & \cong \check{\mathbb{G}}(1,N) \\
& \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & \cdots & \downarrow{\scriptstyle \varphi_{k_0}} & \\
& \mathbb{G}(N-4,N) & & \mathbb{G}(N-6,N) & \cdots & \mathbb{G}(1,N) &
\end{array}$$

where φ_i associates to the hyperplane, which is tangent to $\mathbb{G}(1,N)$ along $\mathbb{G}(1,L)$, precisely the subspace L. If $H \in S_i\mathbb{G}(N-2,N)$ has dual Plücker coordinates $[a_{ij}]$, entries of a skew-symmetric matrix A, the coordinates of $\varphi_i(H)$ result to be the principal Pfaffians of order $2(i+1)$ of A, which are homogeneous polynomials of degree $i+1$ in the a_{ij}'s. If $H \in S_i\mathbb{G}(N-2,N) \setminus S_{i-1}\mathbb{G}(N-2,N)$, these Pfaffians are not all zero, so φ_i is regular at the point H.

Remark 1. If $\Lambda \cong \mathbb{P}^d$ is a linear subvariety of $S_i\mathbb{G}(N-2,N)$, disjoint from $S_{i-1}\mathbb{G}(N-2,N)$, $\varphi_i \mid_\Lambda$ is a regular map defined by homogeneous polynomials of degree $i+1$, therefore $\varphi_i(\Lambda)$ results to be an $(i+1)$-tuple Veronese embedding of $\mathbb{P}^d$ in $\mathbb{G}(N-2(i+1),N)$ or a projection of it of the same degree.

Example 1. For $i=1$, $\varphi_1\colon S_1\mathbb{G}(N-2,N) \dashrightarrow \mathbb{G}(N-4,N)$ is a double Veronese embedding v_2, if it is restricted to a subspace Λ which is disjoint from $\mathbb{G}(N-2,N)$.

Example 2. For $i=k_0$ and N odd, $\varphi_{k_0}\colon \check{\mathbb{G}}(1,N) \to \mathbb{G}(1,N)$ is the *Gauss map* of the Pfaffian hypersurface $\check{\mathbb{G}}(1,N)$.

Remark 2. A linear subspace Λ of dimension d contained in $S_i\mathbb{G}(N-2,N) \setminus S_{i-1}\mathbb{G}(N-2,N)$ can be interpreted as a linear space of skew-symmetric matrices of order $N+1$ of constant rank $2(i+1)$.

2.1 Linear subspaces

We are interested in studying a particular linear space Λ of matrices of constant rank 4, or, better, its orbit under the natural action of the projective linear group $\mathbb{P}\mathrm{GL}(N+1)$. Λ is defined by the linear system of matrices of the form

$$\Lambda(a_1 : \cdots : a_{N-2}) := \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_{N-2} & 0 & 0 \\ & 0 & 0 & \cdots & & 0 & 0 \\ & & 0 & \cdots & & 0 & a_1 \\ & & & & & 0 & a_2 \\ & & & & & & \vdots \\ & & & & & 0 & a_{N-2} \\ & & & & & & 0 \end{pmatrix}, \tag{2}$$

varying $a_1, \dots, a_{N-2}$; it is therefore a linear subvariety of dimension $N-3$ of $S_1\mathbb{G}(N-2,N)$ and it is immediate to verify that it is disjoint from $\mathbb{G}(N-2,N)$.

Λ can be written as the linear span $\langle h_1, \dots, h_{N-2}\rangle$, where h_i is the point $\Lambda(0 : \cdots : 0 : 1 : 0 : \cdots : 0) \in \mathbb{P}(\wedge^2 V^*)$ obtained with $a_i \neq 0$ and $a_j = 0\ \forall j \neq i$. h_i represents a hyperplane $H_i \in \mathbb{P}(\wedge^2 V)$. The intersection

$$\Gamma := H_1 \cap \cdots \cap H_{N-2} \cap \mathbb{G}(1,N) \tag{3}$$

is a family of lines whose expected dimension is N; we will see later that it contains a subgrassmannian $\mathbb{G}(1,N-2)$, so it is reducible if $N > 5$. The equations defining Γ are

$$p_{0,1} + p_{2,N} = p_{0,2} + p_{3,N} = \cdots = p_{0,N-2} + p_{N-1,N} = 0 \tag{4}$$

plus the ones defining the Grassmannian.

For every $h \in \Lambda$, $\varphi_1(h)$ is—as we have seen above—the unique $(N-4)$-dimensional linear space M such that h is tangent to $\mathbb{G}(1,N)$ along $\mathbb{G}(1,M)$. The image $\varphi_1(\Lambda)$ is a double Veronese embedding of $\Lambda \cong \mathbb{P}^{N-3}$ in $\mathbb{G}(N-4,N)$. Indeed, we can say more:

Proposition 3. *Let φ_1 and Λ be as above; then*

1. *$\varphi_1(\Lambda)$ is a double Veronese embedding of $\Lambda \cong \mathbb{P}^{N-3}$ in $\mathbb{G}(N-4, L)$, where $L := V(x_0, x_N) \subset \mathbb{P}^N$;*

2. *geometrically, $\varphi_1(\Lambda) \subset \mathbb{G}(N-4, L)$ represents the family of the $\mathbb{P}^{N-4}$'s $(N-3)$-secants to a rational normal curve C (of degree $N-2$) contained in L;*

3. *$C \subset L$ is the curve defined by the 2×2-minors of the persymmetric matrix*

$$\begin{pmatrix} x_1 & \cdots & x_{N-2} \\ x_2 & \cdots & x_{N-1} \end{pmatrix}. \tag{5}$$

Proof. The fact that $\varphi_1(\Lambda) \subset \mathbb{G}(N-4, L)$ is immediate from the equations defining Λ.

The rest follows from [SU07], since—if $N > 5$—this is the unique double Veronese embedding of a linear space of dimension $N-3$ in $\mathbb{G}(N-4, N-2)$. The case $N = 5$ is in [DM07], and for $N \leq 4$ it is trivial. ∎

Extending the results of [DM07], we can describe for all $P \in \mathbb{P}^N$ the subfamily Γ_P of Γ of the lines passing through P:

Lemma 4. *Let us denote by Γ_P the set of lines of*

$$\Gamma = V(p_{0,1} + p_{2,N}, \dots, p_{0,N-2} + p_{N-1,N}) \cap \mathbb{G}(1, N)$$

passing through $P \in \mathbb{P}^N$; then

1. *if $P \in \mathbb{P}^N \setminus L$, Γ_P is a pencil of lines, contained in a plane α_P that intersects L in one point, belonging to C;*

2. *if $P \in L \setminus C$, Γ_P is the star of all lines through P contained in L;*

3. *if $P \in C$, then Γ_P is the star of all lines through P contained in a hyperplane of $\mathbb{P}^N$ which contains L.*

Proof. If $P = (a_0 : \cdots : a_N) \in \mathbb{P}^N$, then, by (4), and recalling that $p_{ij} = x_i a_j - x_j a_i$, the lines through P belonging to Γ are contained in the linear space defined by

$$\begin{aligned} a_1x_0 - a_0x_1 + a_Nx_2 - a_2x_N &= 0 \\ a_2x_0 - a_0x_2 + a_Nx_3 - a_3x_N &= 0 \\ &\vdots \\ a_{N-2}x_0 - a_0x_{N-2} + a_Nx_{N-1} - a_{N-1}x_N &= 0. \end{aligned}$$

Now, the matrix of the coefficients

$$A = \begin{pmatrix} a_1 & -a_0 & a_N & 0 & \cdots & 0 & -a_2 \\ a_2 & 0 & -a_0 & a_N & 0 & \cdots & -a_3 \\ \vdots & & & & & & \\ a_{N-2} & 0 & \cdots & 0 & -a_0 & a_N & -a_{N-1} \end{pmatrix} \tag{6}$$

has maximal rank $N-2$ if and only if $P \in \mathbb{P}^N \setminus L$, and the solutions form in fact a plane.

Then, it is an easy exercise to show that

$$\mathrm{rk}(A) < 3 \iff P \in L \tag{7}$$

and, if $a_0 = a_N = 0$

$$\mathrm{rk}(A) < 2 \iff P \in C, \tag{8}$$

so if $P \in L \setminus C$ the solutions form a $\mathbb{P}^{N-2} \subset L$ and if $P \in C$ the solutions, and so the lines of Γ_P, form the hyperplane H_P of equation $a_N x_0 - a_0 x_N = 0$ containing L. ■

Proposition 5. *If $N \geq 6$, the family of lines Γ contains the subgrassmannian $\mathbb{G}(1, L)$ as irreducible component. Let $\tilde{\Gamma}$ denote the closure of $\Gamma \setminus \mathbb{G}(1, L)$; then $\tilde{\Gamma}$ has dimension N, is irreducible and its class in the Chow ring of $\mathbb{G}(1, N)$ is $[\tilde{\Gamma}] = \sigma_{N-2} + (N-3)\sigma_{N-3,1} + (N-5)\sigma_{N-4,2} + \cdots$.*

Proof. That $\mathbb{G}(1, L)$ is contained in Γ follows from Lemma 4(2). Since $\mathbb{G}(1, L)$ has codimension 4 in $\mathbb{G}(1, N)$, it is an irreducible component of $\Gamma_4 := \mathbb{G}(1, N) \cap H_1 \cap \cdots \cap H_4$ and Γ_4 can be written as the union of two proper closed subsets:

$$\Gamma_4 = \tilde{\Gamma}_4 \cup \mathbb{G}(1, L).$$

Then we consider $\Gamma_5 := \tilde{\Gamma}_4 \cap H_5$; since $\mathbb{G}(1, L) \subset H_5$, again we can decompose it as union of two proper closed subsets:

$$\Gamma_5 = \tilde{\Gamma}_5 \cup (\tilde{\Gamma}_4 \cap \mathbb{G}(1, L)),$$

and so on until we get $\tilde{\Gamma} = \tilde{\Gamma}_{N-2}$. We observe that the lines of $\tilde{\Gamma}$ through a point $P \in \mathbb{P}^N \setminus L$ or $P \in C$ are all the lines of Γ through P, while if $P \in L \setminus C$, they form the join of P and C. The irreducibility of $\tilde{\Gamma}$ follows with a standard argument from this.

In order to calculate the class of $\tilde{\Gamma}$, we use Schubert calculus in $\mathbb{G}(1, N)$, recalling that σ_1 is a hyperplane section and the coefficient of $\sigma_{N-2-i,i}$ is the number of lines of $\tilde{\Gamma}$ contained in a $\mathbb{P}^{N-1-i}$ meeting a $\mathbb{P}^i$, linear spaces of a fixed flag of $\mathbb{P}^N$. The first two coefficients of $[\tilde{\Gamma}]$, as those of Γ, are the same as in σ_1^{N-2}, *i.e.* 1 and $N-3$. To compute the others, we fix a flag such that $\mathbb{P}^i$ is generated by $i+1$ points of C, $P_0, \ldots, P_i$. Then, in order to generate a suitable $\mathbb{P}^{N-1-i}$, we add to it $N-1-2i$ general lines of $\tilde{\Gamma}$ passing through the P_j's. Here "general" means that we first take $[\frac{N-1-2i}{i+1}]$ lines through each point, and then one more line through the first $N-1-2i-(i+1)[\frac{N-1-2i}{i+1}]$ points. With this choice, we deduce that the coefficient of $\sigma_{N-2-i,i}$ is $N-1-2i$. ■

Remark 3. The singular locus of $\tilde{\Gamma}$, as a subvariety of $\mathbb{G}(1, N) \subset \mathbb{P}(\wedge^2 V)$, is contained in its intersection with $\mathbb{G}(1, L)$, which has dimension (at most) $N-2$: in fact it is the union of the subgrassmannians $\mathbb{G}(1, M)$, as M varies in $\varphi_1(\Lambda)$. Note that if $\ell \in \mathbb{G}(1, L)$, then there exists an $(N-4)$-space M which is $(N-3)$-secant to the rational normal curve $C \subset L$ such that $\ell \in \mathbb{G}(1, M)$, therefore the singular locus is contained in $\tilde{\Gamma} \cap \mathbb{G}(1, L)$.

Remark 4. For $N \leq 5$ we have that $\mathbb{G}(1, L) \subset \Gamma$ also, but it is not a component, since it has lower dimension.

3 On the completely exceptional Monge-Ampère equations

For the definitions and generalities about systems of conservation laws and their correspondence with congruences of lines, we refer to [AF01], see also [DM05]. An important class of strictly hyperbolic PDE's of conservation laws are the *completely exceptional Monge-Ampère equations*, which are introduced and studied in [Boi92]. It is shown there that they are *linearly degenerate*, which means that the eigenvalues of the Jacobian matrix associated to the system are constant along the rarefaction curves. In [AF01], it is asserted that this class is even a *T-system*, *i.e.* that the rarefaction curves are lines, and it is suggested that these could be examples of non-linear T-systems. We will prove this fact. We need to divide the cases of order odd and even.

3.1 The even order case

The completely exceptional Monge-Ampère systems of the even order $2m$ are defined as follows (see [AF01], Section 5). Introduce the *Hankel (or persymmetric) matrix* of type $(m+1) \times (m+1)$

$$H := \begin{pmatrix} \frac{\partial^{2m} u}{\partial x^{2m}} & \frac{\partial^{2m} u}{\partial x^{2m-1}\partial t} & \cdots & \frac{\partial^{2m} u}{\partial x^{m}\partial t^{m}} \\ \frac{\partial^{2m} u}{\partial x^{2m-1}\partial t} & \frac{\partial^{2m} u}{\partial x^{2m-2}\partial t^{2}} & & \frac{\partial^{2m} u}{\partial x^{m-1}\partial t^{m+1}} \\ \vdots & & & \\ \frac{\partial^{2m} u}{\partial x^{m}\partial t^{m}} & \frac{\partial^{2m} u}{\partial x^{m-1}\partial t^{m+1}} & \cdots & \frac{\partial^{2m} u}{\partial t^{2m}} \end{pmatrix},$$

and consider the PDE of the $2m$-th order

$$a_{(0,\dots,m),(0,\dots,m)} \det(H) + \cdots + a = 0, \tag{9}$$

i.e. a linear combination of the minors of all orders of H (where we suppose $a_{(0,\dots,m),(0,\dots,m)} \neq 0$).

3.2 The odd order case

The completely exceptional Monge-Ampère systems of the odd order $2m-1$ are defined again via the Hankel matrix, which now is of type $m \times (m+1)$ and has the form

$$H := \begin{pmatrix} \frac{\partial^{2m-1} u}{\partial x^{2m-1}} & \frac{\partial^{2m-1} u}{\partial x^{2m-2}\partial t} & \cdots & \frac{\partial^{2m-1} u}{\partial x^{m-1}\partial t^{m}} \\ \frac{\partial^{2m-1} u}{\partial x^{2m-2}\partial t} & \frac{\partial^{2m-1} u}{\partial x^{2m-3}\partial t^{2}} & & \frac{\partial^{2m-1} u}{\partial x^{m-2}\partial t^{m+1}} \\ \vdots & & & \\ \frac{\partial^{2m-1} u}{\partial x^{m}\partial t^{m-1}} & \frac{\partial^{2m-1} u}{\partial x^{m-1}\partial t^{m}} & \cdots & \frac{\partial^{2m-1} u}{\partial t^{2m-1}} \end{pmatrix},$$

and consider the PDE of the $(2m-1)$-th order

$$\sum_i a_{(0,\dots,\hat{\imath},\dots,m),(1,\dots,m)} (h_{(0,\dots,\hat{\imath},\dots,m),(1,\dots,m)}) + \cdots + a = 0, \tag{10}$$

i.e. a linear combination of the minors of all orders of H (where we suppose that at least one of the $a_{(0,\dots,\hat{\imath},\dots,m),(1,\dots,m)}$'s is not zero).

3.3 The congruence

We now suppose that $N = 2m + 1$ in the even order case and $N = 2m$ in the odd case.

After the introduction of the new variables $(u_1, u_2, \ldots, u_N)$ such that

$$
\begin{aligned}
u_1 &= \frac{\partial^{N-1} u}{\partial x^{N-1}},\\
u_2 &= \frac{\partial^{N-1} u}{\partial x^{N-2}\partial t},\\
u_3 &= \frac{\partial^{N-1} u}{\partial x^{N-3}\partial t^2},\\
&\vdots\\
u_N &= \frac{\partial^{N-1} u}{\partial t^{N-1}},
\end{aligned}
$$

Equation (9) becomes

$$
a_{(0,\ldots,m),(0,\ldots,m)} \det \begin{pmatrix} u_1 & u_2 & \cdots & u_{m+1} \\ u_2 & u_3 & \cdots & u_{m+2} \\ \vdots & & & \\ u_{m+1} & u_{m+2} & \cdots & u_N \end{pmatrix} + \cdots + a = 0 \tag{11}
$$

(and Equation (10) has an analogous expression), moreover

$$
(u_1)_t = (u_2)_x, \quad (u_2)_t = (u_3)_x, \quad (u_3)_t = (u_4)_x, \quad \ldots, (u_{N-1})_t \quad = (u_N)_x. \tag{12}
$$

This is a system of conservation laws, and the corresponding congruence B_{MA} in $\mathbb{P}^N$, according to the Agafonov-Ferapontov construction [AF01] is (in non-homogeneous coordinates $(x_0, \ldots, x_{N-1})$)

$$
\begin{aligned}
x_1 &= u_1 x_0 - u_2\\
x_2 &= u_2 x_0 - u_3\\
x_3 &= u_3 x_0 - u_4\\
&\vdots\\
x_{N-1} &= u_{N-1} x_0 - u_N
\end{aligned}
$$

together with Equation (11). Using projective coordinates $(x_0 : \cdots : x_N)$ in $\mathbb{P}^N$ and $(u_0 : \cdots : u_N)$ as parameters for the lines in B_{MA}, it is given by

$$
\begin{aligned}
u_0 x_1 &= u_1 x_0 - u_2 x_N\\
u_0 x_2 &= u_2 x_0 - u_3 x_N\\
u_0 x_3 &= u_3 x_0 - u_4 x_N\\
&\vdots\\
u_0 x_{N-1} &= u_{N-1} x_0 - u_N x_N.
\end{aligned}
$$

Here we have to be careful, since in this way we introduce some extra components (having u_0 identically zero), which must not be considered. In fact, since B_{MA} is parametrised by the hypersurface in $\mathbb{P}^N$ defined in Equation (11), we deduce that

Proposition 6. *B_{MA} is an irreducible variety of dimension $N-1$.*

To obtain the equations of B_{MA} in the Plücker embedding, we can proceed as this: we notice that the line corresponding to the parameters $(u_0 : \cdots : u_N)$ joins the points $(u_0 : \cdots : u_{N-1} : 0)$ and $(0 : -u_2 : \cdots : -u_N : u_0)$, hence we can compute its Plücker coordinates taking the 2×2-minors of the matrix

$$M = \begin{pmatrix} u_0 & u_1 & \cdots & u_{N-1} & 0 \\ 0 & u_2 & \cdots & u_N & -u_0 \end{pmatrix}, \tag{13}$$

obtaining

$$\begin{aligned} p_{0,1} &= u_0 u_2 \\ &\vdots \\ p_{0,N-1} &= u_0 u_N \\ p_{1,N} &= -u_0 u_1 \\ &\vdots \\ p_{N-1,N} &= -u_0 u_{N-1}. \end{aligned}$$

From this, we get immediately that the points of B_{MA} satisfy Equations (4), so $B_{MA} \subset \Gamma$. We recall (see Proposition 5) that Γ is the union of $\tilde{\Gamma}$ with other components contained in $\mathbb{G}(1, L)$. Since a general line of B_{MA} is not in $\mathbb{G}(1, L)$, we have proven:

Proposition 7. *With notation as above, $B_{MA} \subset \tilde{\Gamma}$.*

Theorem 8. *Let B_{MA} be the congruence of lines associated to a completely exceptional Monge-Ampère equation; then*

1. *B_{MA} is a first order congruence;*
2. *the lines of B_{MA} passing through a focal point P not in L form a planar pencil of lines.*

Proof. Fix a general point $P = (y_0, \ldots, y_{N-1}) \in \mathbb{A}^N$ of the affine space $x_N \neq 0$, and set $u := u_1$. Then, using the above affine equations, we deduce

$$\begin{aligned} u_2 &= y_0 u - y_1 \\ u_3 &= y_0 u_2 - y_2 = y_0^2 u - y_0 y_1 - y_2 \\ u_4 &= y_0 u_3 - y_3 = y_0^3 u - y_0^2 y_1 - y_0 y_2 - y_3 \\ &\vdots \\ u_N &= y_0^{N-1} u - \cdots - y_{N-1}. \end{aligned}$$

If we substitute these equations in Equation (9) (or (10)), we get

$$a_{(0,\dots,m),(0,\dots,m)} \det \begin{pmatrix} u & y_0u-y_1 & \cdots & y_0^m u-\cdots-y_m \\ y_0u-y_1 & y_0^2u-y_0y_1-y_2 & \cdots & y_0^{m+1}u-\cdots-y_{m+1} \\ \vdots & & & \\ y_0^m u-\cdots-y_m & y_0^{m+1}u-\cdots-y_{m+1} & \cdots & y_0^{N-1}u-\cdots-y_{N-1} \end{pmatrix} + \cdots + a = 0. \tag{14}$$

Now, we observe that

$$\det \begin{pmatrix} u & y_0u-y_1 & \cdots & y_0^m u-\cdots-y_m \\ y_0u-y_1 & y_0^2u-y_0y_1-y_2 & \cdots & y_0^{m+1}u-\cdots-y_{m+1} \\ \vdots & & & \\ y_0^m u-\cdots-y_m & y_0^{m+1}u-\cdots-y_{m+1} & \cdots & y_0^{N-1}u-\cdots-y_{N-1} \end{pmatrix} =$$
$$= \det \begin{pmatrix} u & -y_1 & \cdots & -y_m \\ y_0u-y_1 & -y_2 & \cdots & -y_{m+1} \\ \vdots & & & \\ y_0^m u-\cdots-y_m & -y_{m+1} & \cdots & -y_{N-1} \end{pmatrix}$$

and similarly for each minor in Equation (14), therefore Equation (14) becomes a linear equation in u, and so B_{MA} is a first order congruence.

Then, from the general results cited in the introduction, since B_{MA} is a first order congruence, if $P \in \mathbb{P}^N$ is a focal point for B_{MA}, then there pass infinitely many lines of B_{MA} through it. But from Proposition 7, $B_{MA} \subset \tilde{\Gamma}$, and, by Lemma 4, if P is a focal point not in L, there passes a pencil of lines of B_{MA} through it.

Alternatively, if P is a focal point, we have that Equation (14) depends on two parameters (for example u_1 and u_2), but it is always a linear equation, and so it defines a plane. ∎

Moreover, if we multiply each entry of the matrices in the Equation (9) (or (10)) by u_0, and we homogenise the obtained expression by u_N, then, by the above parametric expressions of the $p_{0,j}$'s, we obtain

$$a_{(0,\dots,m),(0,\dots,m)} \det \begin{pmatrix} p_{1,N} & p_{2,N} & \cdots & p_{m+1,N} \\ p_{2,N} & p_{3,N} & \cdots & p_{m+2,N} \\ \vdots & & & \\ p_{m+1,N} & p_{m+2,N} & \cdots & -p_{0,N-1} \end{pmatrix} + \cdots + ap_{0,N-1}^m = 0. \tag{15}$$

We will give in the next section the precise algebraic construction of this congruence. From general results contained in [AF01], Section 2, we deduce immediately that

Corollary 9. *The completely exceptional Monge-Ampère systems, given by Equations* (11) *and* (12), *are T-systems.*

4 Congruences contained in $\tilde{\Gamma}$

We continue to use the notation of Section 3, assuming $N \geq 6$. In order to obtain a congruence B contained in $\tilde{\Gamma}$, we have (at least to start) to intersect $\tilde{\Gamma}$ with a hypersurface. Since we are interested in first order congruences, we may need that the congruence B splits in some components.

We note first that $\mathbb{G}(1, L)$ is the intersection of $\mathbb{G}(1, N)$ with the following space:

$$\Pi_L = V(p_{0,1}, p_{0,2}, \ldots, p_{0,N}, p_{1,N}, p_{2,N}, \ldots, p_{N-1,N}) \tag{16}$$

and that the hyperplane $H_L := V(p_{0N})$ is the only one which is tangent to $\mathbb{G}(1, N)$ along $\mathbb{G}(1, L)$.

Example 3. We first intersect $\tilde{\Gamma}$ with a general hyperplane H, in particular we request that $H \not\supset \mathbb{G}(1, L)$. In this case the congruence $B = \tilde{\Gamma} \cap H$ is irreducible, and by Proposition 5, the multidegree comes from

$$\begin{aligned}[\tilde{\Gamma}] \cdot \sigma_1 &= (\sigma_{N-2} + ((N-3)\sigma_{N-3,1} + (N-5)\sigma_{N-4,2} + \cdots) \cdot \sigma_1 \\ &= \sigma_{N-1} + (N-2)\sigma_{N-2,1} + 2(N-4)\sigma_{N-3,2} + \cdots\end{aligned}$$

so the multidegree is $(1, N-2, 2(N-4), 2(N-6), \ldots)$.

Finally, we observe that the lines of B, passing through a point P, not in L, of the focal locus X, form a planar pencil since B is contained in a linear congruence (as in Proposition 4.2 of [DM05]), or by Lemma 4(3).

Example 4. We consider now $B_L := \tilde{\Gamma} \cap H_L$. A line of $\tilde{\Gamma}$ belongs to B_L if it intersects L. Hence B_L contains the congruence B_C formed by the lines of $\tilde{\Gamma}$ meeting the rational normal curve C of degree $N-2$ contained in L.

B_C is an irreducible congruence of lines whose multidegree is $(1, N-2, 0, \ldots, 0)$: the first number is 1 because through a general point of $\mathbb{P}^N$ there passes one line of $\tilde{\Gamma}$ meeting C; the second number is $N-2$ because a hyperplane intersects C in $N-2$ points; the other numbers are 0 because a general linear space of codimension greater than 2 does not meet C. So we can write $B_L = B_C \cup B'$, where, by Example 3, B' is a congruence of class $[B'] = (0, 0, 2(N-4), 2(N-6), \ldots)$. Hence B' is formed by lines all contained in a linear subspace of codimension at least 2, which is necessarily equal to L.

For example, for $N = 6$ the multidegree of B' is $(0, 0, 4)$, and the one of B_C is $(1, 4, 0)$.

In terms of coordinates, let $\mathcal{I}(B_C)$ be the homogeneous ideal of B_C of Example 4. We have

Proposition 10. *If $N \geq 5$, $\mathcal{I}(B_C)$, modulo $\mathcal{I}(\mathbb{G}(1, N))$, contains at least one quadric which contains also $\mathbb{G}(1, L)$.*

If $N \geq 7$, $\mathcal{I}(B_C)$, modulo $\mathcal{I}(\mathbb{G}(1, N))$, contains at least one quadric which does not contain $\mathbb{G}(1, L)$.

Proof. We have that

$$\begin{aligned}\mathcal{I}(B_L) &\supset \mathcal{I}(\mathbb{G}(1, N)) + (p_{0,N}, p_{0,1} + p_{2,N}, \ldots, p_{0,N-2} + p_{N-1,N}) \\ &= (p_{0,i}p_{j,N} - p_{0,j}p_{i,N}, p_{a,b}p_{c,d} - p_{a,c}p_{b,d} + p_{a,d}p_{b,c}, p_{0,N}, p_{0,1} + p_{2,N}, \ldots), \\ &\quad 1 \leq i < j \leq N-1,\ 1 \leq a < b < c < d \leq N-1.\end{aligned}$$

In particular, the first relations, *i.e.* $p_{0,i}p_{j,N} - p_{0,j}p_{i,N}$ with $1 \leq i < j \leq N-1$, can be seen as the 2×2-minors of the matrix

$$\begin{pmatrix} p_{0,1} & p_{0,2} & \cdots & p_{0,N-1} \\ p_{1,N} & p_{2,N} & & p_{N-1,N} \end{pmatrix},$$

or, equivalently, using the relations in $\mathcal{I}(B_L)$, by the minors of

$$\begin{pmatrix} p_{2,N} & p_{3,N} & \cdots & -p_{0,N-1} \\ p_{1,N} & p_{2,N} & & p_{N-1,N} \end{pmatrix}.$$

Equivalently, these minors can be seen as the 2×2-minors of the persymmetric matrices M_N of one of the two types, which correspond to the cases N odd or even:

$$M_N = \begin{pmatrix} p_{1,N} & p_{2,N} & & p_{m+1,N} \\ p_{2,N} & p_{3,N} & \cdots & -p_{m+2,N} \\ \vdots & & & \\ p_{m,N} & p_{m+1,N} & & p_{N-1,N} \\ p_{m+1,N} & p_{m+2,N} & \cdots & -p_{0,N-1} \end{pmatrix},$$

if $N = 2m+1$ is odd, or

$$M_N = \begin{pmatrix} p_{1,N} & p_{2,N} & & p_{m+1,N} \\ p_{2,N} & p_{3,N} & \cdots & -p_{m+2,N} \\ \vdots & & & \\ p_{m-1,N} & p_{m,N} & & p_{N-1,N} \\ p_{m,N} & p_{m+1,N} & \cdots & -p_{0,N-1} \end{pmatrix},$$

if $N = 2m$ is even.

We deduce that $\mathcal{I}(B_L)$ contains also the minors of higher order of M_N, in particular, the determinant of M_N if N is odd, and the $m+1$ minors of order m if N is even.

Now, we observe that

$$\begin{aligned} \mathcal{I}(B_C) &\supset \mathcal{I}(B_L) : \mathcal{I}(\mathbb{G}(1,L))^2 \\ &\supset \mathcal{I}(B_L) : \mathcal{I}(\mathbb{G}(1,L)) \end{aligned}$$

moreover, every 3×3-minor of M_N gives an expression of the type $p_{0,N}Q$, where Q is a quadric, and one can prove, with simple computations, that $Q \in \mathbb{G}(1,L)$ and $Q \in \mathcal{I}(B_L) : \mathcal{I}(\mathbb{G}(1,L))$.

Instead, every 4×4-minor of M_N gives an expression of the type $p_{0,N}^2 Q$, where Q is again a quadric, this time such that $Q \notin \mathbb{G}(1,L)$ but $Q \in \mathcal{I}(B_L) : \mathcal{I}(\mathbb{G}(1,L))^2$. ■

Remark 5. The matrix M_N introduced in the proof of the preceding proposition, is nothing but the matrix whose minors give rise to Equation (15). In particular, the hypersurface of degree m defined by this equation contains always the congruence B_C of Example 4 with some multiplicity, and, of course, the congruence B_{MA}.

Then we can pass to consider quadratic complexes, always with the aim of finding first order congruences.

Example 5. Let us take $\tilde{\Gamma} \cap Q$, where Q is a quadratic complex. If Q is general, then $\tilde{\Gamma} \cap Q$ is an irreducible congruence of order 2. In order to get a reducible congruence, we require that Q contains B_C of Example 4.

If $N = 6$, by Proposition 10, such a Q contains also $\mathbb{G}(1, L)$, so we can proceed as in the proof of Proposition 5. We get that the irreducible components of $\tilde{\Gamma} \cap Q$ are B_C, $\tilde{\Gamma} \cap \mathbb{G}(1, L)$ and a congruence B_Q of multidegree $(1, 4, 7)$.

If $N > 6$, using Proposition 5 and Example 4, we get that $\tilde{\Gamma} \cap Q$, with Q general, contains B_C and a residual component B_Q, which is a congruence of type $2[\tilde{\Gamma}] \cdot \sigma_1 - [B_C]$, hence of multidegree $(1, N-2, 4(N-4), 4(N-6), \dots)$.

We compare B_Q with a linear congruence, whose multidegree is $(1, N-2, \binom{N-2}{2} - 1, \dots, \binom{N-2}{i} - \binom{N-2}{i-2}, \dots)$. The first two coefficients are the same and the difference of the third ones is 2 for $N = 6$ and $\frac{(N-9)(N-4)}{2}$ for $N \geq 7$, which is negative for $N < 9$. So for $N < 9$ B_Q is not contained in any linear congruence.

Let us consider again the congruences B_{MA} associated to the Monge-Ampère equations: the congruences just obtained in Example 5 are those obtained from Equation (15) if the only nonzero coefficients are those of the minors of order less than or equal to four. So, these are associated to the general Monge-Ampère equation if $N \leq 8$, but only to degenerate ones if $N \geq 9$.

In order to obtain the congruences associated to a non-degenerate Monge-Ampère equation, it is not sufficient to consider quadratic complexes, if $N \geq 9$. However, the result is similar:

Theorem 11. *Let B_{MA} be a general congruence in $\mathbb{P}^N$ defined by a Monge-Ampère type equation as in Section 3; then B_{MA} is an irreducible congruence contained in $\tilde{\Gamma} \cap V$, where V is a hypersurface of degree μ, if $4\mu - 3 \leq N \leq 4\mu$, and V contains the congruence B_C of Proposition 10 with multiplicity $\mu - 1$. Moreover if $N \geq 7$ the multidegree of B_{MA} is $(1, N-2, 2\mu(N-4), 2\mu(N-6), \dots)$. In particular, B_{MA} is a first order congruence which is not linear.*

Proof. For the sake of clearness, we only show this for the case $N = 9$: if we consider M_9, then $\det(M_9)$ gives $p_{0,9}^2 F$, with $\deg(F) = 3$. But $F \in \mathcal{I}(B_C)^2$, since if we develop the determinant with respect to the 2×2-minors of the first two rows and the 3×3-minors of the remaining three, every term of the development is the product of a quadratic polynomial and a cubic polynomial, each of them contained in $\mathcal{I}(B_C)$. Therefore, we can construct a component of order one taking $V(F) \cap \tilde{\Gamma}$: set-theoretically we obtain $B_C \cup \tilde{B}$, but B_C appears with multiplicity 2, hence $[\tilde{B}] = [B_{MA}]$ and its class is as stated. It is easy to see that the same conclusion – *i.e.* there is a polynomial F of degree three such that set-theoretically $V(F) \cap \tilde{\Gamma} = B_C \cup \tilde{B}$, and so $[\tilde{B}] = [B_{MA}]$ – holds for $10 \leq N \leq 12$, since, as we have seen in the proof of Proposition 10, the $k \times k$-minors, with $k \leq 4$, give polynomials in $\mathcal{I}(B_C)$.

If $N = 4\mu - h$, $0 \leq h \leq 3$, note that $\frac{(N-1)(N-4)}{2} < 2\mu(N-4)$, which proves last assertion. ∎

Remark 6. It is now easy to see that the focal locus X of B_{MA} has codimension two in $\mathbb{P}^N$, since B_{MA} has order one, and through a point of X not in L there passes a pencil of lines of the congruence, by Theorem 8.

Moreover, X cannot be contained in a hypersurface S of degree $N-2$, again by the fact that B_{MA} has order one. In fact, if there were such a hypersurface S, by Proposition 1, each line of the congruence would be contained in S.

On the other hand, the general hyperplane section of X, $X \cap H$, is contained in a hypersurface of degree $N-2$, since B_{MA} has second multidegree $N-2$, which in fact is the degree of the hypersurface of the lines of B_{MA} contained in H. Therefore, by the long exact cohomology sequence associated to the exact sequence

$$0 \to \mathcal{I}_X(N-3) \xrightarrow{\cdot H} \mathcal{I}_X(N-2) \to \mathcal{I}_{H\cap X}(N-2) \to 0,$$

we deduce that X is not $(N-3)$-normal.

We will compute now the degree of X, using a formula from [DP05]. To state it we need to introduce the parasitical components of the fundamental locus and the hypersurfaces V_Π.

Definition 1. Let B be a non-degenerate congruence of lines in $\mathbb{P}^N$ and F be an irreducible component of the fundamental locus of B of dimension d, with $2 \leq d \leq N-2$. F is called *i-parasitical* if:

1. through every point of F there pass infinitely many focal lines of B contained in F;
2. F is a component of the fundamental locus with multiplicity i;
3. a general line of B does not meet F.

The union of the non-parasitical components of the fundamental locus is called the *pure fundamental locus*.

Let Π be a general linear subspace of $\mathbb{P}^N$ of dimension $N-2$, let V_Π be the union of the lines of B meeting Π. Then V_Π is a hypersurface of degree $a_0 + a_1$, where a_0 and a_1 are the first two multidegrees of B.

Proposition 12. *(see [DP05]) Let B be a congruence of lines of order one in $\mathbb{P}^N$. Let F_j, $j = 1, \ldots, h$, be the irreducible components of dimension $N-2$ of the fundamental locus. Let m_j denote the degree of $(F_j)_{\mathrm{red}}$ and k_j the algebraic multiplicity of $(F_j)_{\mathrm{red}}$ on V_Π. Finally, let l_j denote the length of the intersection of $(F_j)_{\mathrm{red}}$ with a general line of B. Then the following formulae hold:*

$$\sum_{j=1}^{h} l_j k_j \leq a_0 + a_1; \tag{17}$$

$$(a_0 + a_1)^2 = \sum_{j=1}^{h} k_j^2 m_j + a_0 + 2a_1 + a_2 + x, \tag{18}$$

where x is a non-negative number, which vanishes only if there are no parasitical components of dimension $N-2$. Moreover, if the fundamental locus has pure dimension $N-2$, then equality holds in Formula (17).

Theorem 13. *In the above notations:*

1. *L is the only parasitical component of the congruence B_{MA} of dimension $N-2$;*
2. *the pure fundamental locus X has pure dimension $N-2$;*
3. *if $N=6$, $\deg(X)=9$; if $N \geq 7$ $\deg(X)=(N-2)^2-2\mu(N-4)-x$, where $N=4\mu-h$, $0 \leq h \leq 3$ and $x \geq 1$.*

Remark 7. If $P \in X \setminus L$, then $\alpha_P \cap X = \{P\} \cup C$, where C is a plane curve of degree $\delta \leq N-2$ (possibly $P \in C$), because on each line of the pencil of centre P in α_P there are $N-1$ foci including P (counting multiplicities).

Proof. Every irreducible component of the focal locus has dimension at most $N-2$ because B_{MA} has order one. If F_j is a non-parasitical component and if there exists a point $P \in F_j \setminus L$, the lines of B_{MA} through P form a planar pencil, so $\dim F_j = N-2$: this proves (2).

To prove that L is a parasitical component we remark that Q contains B_C, *i.e.* all lines of $\tilde{\Gamma}$ meeting C, and that the lines of $\tilde{\Gamma}$ through any point P of $L \setminus C$ are all the lines of the join of P and C.

On the other hand, a parasitical component F is necessarily contained in L: if $P \in F \setminus L$, through P there passes a focal line $l \subset F$, which is necessarily k-secant X, with $k > N-1$. So $\alpha_P \cap X$ contains P, a curve of degree $\delta \leq N-2$ and at least one other point $P' \neq P$. Hence P' is the centre of another pencil of lines of B_{MA}. Therefore α_P is a focal plane, which means that all lines of α_P belong to B_{MA}: this is impossible by Lemma 4, because $P \notin L$. We conclude that $F \subset L$.

To prove (3), it is enough to apply Formula 18, noticing that $k_j = 1$ for all indices j. In fact k_j, the algebraic multiplicity of F_j, is nothing else but the number of lines of B_{MA} passing through a general point P of F_j and meeting a general subspace Π of dimension $N-2$. Since $P \notin L$, there is only one such line, joining P with $\alpha_P \cap \Pi$. ∎

Remark 8. With a computation performed with Macaulay2, we have seen that for $N=7$ the constant x which appears in Theorem 13(3) is 1.

Interesting questions now arise. We can ask first of all whether the congruences B_{MA} are smooth for every N, as in the case of $N=5$. Other questions can be addressed about the pure focal locus X: if it is irreducible, and about the type and the dimension of its singular locus. Also for these last questions, we have answers only for $N=5$ (see [DM07]).

Acknowledgement. We wish to thank Frédéric Han for interesting discussions and remarks.

References

[AF01] S. I. Agafonov and E. V. Ferapontov, *Systems of conservation laws of Temple class, equations of associativity and linear congruences in $\mathbb{P}^4$*, Manuscripta Math. **106** (2001), no. 4, 461–488.

[Boi92] G. Boillat, *Sur l'équation générale de Monge-Ampère d'ordre supérieur*, C. R. Acad. Sci. Paris Sr. I Math. **315** (1992), no. 11, 1211–1214.

[CS] C. Ciliberto and E. Sernesi, *Singularities of the theta divisor and congruences of planes*, J. Algebraic Geom., **1** (1992), no. 2, 231–250.

[DP03] P. De Poi, *Threefolds in $\mathbb{P}^5$ with one apparent quadruple point*, Comm. Algebra, **31** (2003), no. 4, 1927–1947.

[DP04] ———, *Congruences of lines with one-dimensional focal locus*, Port. Math. (N.S.) **61** (2004), no. 3, 329–338.

[DP05] ———, *On first order congruences of lines in $\mathbb{P}^4$ with irreducible fundamental surface*, Math. Nachr. **278** (2005), no. 4, 363–378.

[DM05] P. De Poi and E. Mezzetti, *Linear congruences and systems of conservation laws*, Projective Varieties with Unexpected Properties (A Volume in Memory of Giuseppe Veronese. Proceedings of the international conference "Varieties with Unexpected Properties", Siena, Italy, June 8–13, 2004), de Gruyter, Berlin, 2005, pp. 209–230.

[DM07] ———, *Congruences of lines in $\mathbb{P}^5$, quadratic normality, and completely exceptional Monge-Ampère equations*, Geom. Dedicata **131** (2008), no. 1, 213–230.

[GH78] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, 1978.

[Har77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Vol. 52, Springer Verlag, 1977.

[SU07] J. C. Sierra and L. Ugaglia, *On globally generated vector bundles on projective spaces*, J. Pure Appl. Algebra, **213** (2009), no. 11, 2141–2146.

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