A topological vector space is Fréchet-Urysohn if and only if it has bounded tightness

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Abstract

We prove that a topological vector space E is Fréchet-Urysohn if and only if it has a bounded tightness, i.e. for any subset A of E and each point x in the closure of A there exists a bounded subset of A whose closure contains x. This answers a question of Nyikos on $C_p(X)$ (personal communication). We also raise two related questions for topological groups.

1 Preliminaries and theorem

The Fréchet-Urysohn property has been intensively studied in both topology and functional analysis, see [9] and [12]. A topological space X is *Fréchet-Urysohn* if for every subset A of X and every $x \in \overline{A}$ (the closure of A in X) there exists a sequence in A which converges to x. Examples of such spaces are metrizable spaces and one point compactifications of discrete spaces. Countable tightness of X is a natural generalization of the Fréchet-Urysohn property: X has *countable tightness* if for every subset A of X and every $x \in \overline{A}$ there exists a countable subset B of A with $x \in \overline{B}$, cf. [1].

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Pytkeev [13] and independently Gerlits [7], see also [1] and [10], proved that, for a completely regular Hausdorff space X the space $C_p(X)$ of continuous real-valued functions on X is Fréchet-Urysohn if and only if $C_p(X)$ is sequential iff $C_p(X)$ is a k-space. In general these three properties are different. As an instance, the space (φ) with countably infinite dimension endowed with its strongest locally convex topology is sequential, therefore a k-space, yet not Fréchet-Urysohn [12]. The space $C_p([0,1])$ is an example of a space with countable tightness which is not sequential, hence not Fréchet-Urysohn [1, II.1.4, II.3.5]. For compact X the $C_p(X)$ is Fréchet-Urysohn iff X is *scattered*, cf. [1, III.1.2] and [11] (see also [4] for recent improvements of this result). More generally, $C_p(X)$ is Fréchet-Urysohn if and only if X is an ω-space (defined later, also see [10]). $C_p(X)^{\mathbb{N}}$ is Fréchet-Urysohn for Fréchet-Urysohn $C_p(X)$ [1, II.3.2], but there exist Fréchet-Urysohn spaces $C_p(X)$ and $C_p(Y)$ whose product does not even have countable tightness [14, Theorem 7].

There is another interesting tightness-type condition formally weaker than the Fréchet-Urysohn property.

Let E be a (Hausdorff) topological vector space (tvs). If for every subset A of E and $x \in \overline{A} \subset E$ there is a bounded set $B \subset A$ such that $x \in \overline{B}$, then E is said to have *bounded tightness*, see [3] and [6]. Recall that a subset of a tvs E is called *bounded* if it is absorbed by each neighborhood of zero in E. In [6, Corollary 1] it is shown that if a locally convex space E has bounded tightness then E is bornological, and this fact has been used in [6, Theorem] to show that every locally convex space in class $\mathfrak{G}$ introduced in [2] has bounded tightness if and only if it is metrizable. Especially interesting seems to be the following question posed by Nyikos (personal communication):

Are the bounded tightness and Fréchet-Urysohn properties equivalent for spaces $C_p(X)$?

The affirmative answer of the theorem is also motivated by the partial positive results in [3, 5, 6].

Theorem 1. *For a tvs E the following assertions are equivalent:*

(1) E is Fréchet-Urysohn.

(2) For every subset A of E such that $0 \in \overline{A}$ *there exists a bounded subset B of A such that* $0 \in \overline{B}$.

(3) For any sequence $(A_n)_n$ *of subsets of E, each with* $0 \in \overline{A_n}$*, there exists a sequence* $B_n \subset A_n$*,* $n \in \mathbb{N}$*, such that* $\bigcup_n B_n$ *is bounded and* $0 \in \overline{\bigcup_{n\leq k} B_k}$ *for each* $n \in \mathbb{N}$.

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2 The proof and some observations and questions

Clearly (1) implies (2). Now assume (2). It is obvious that (3) holds if $0 \in A_n$ for infinitely many n. Therefore, we assume that $0 \in \overline{A_n} \backslash A_n$, for each $n \in \mathbb{N}$. Consequently, there exists a null sequence $(x_n)_n$ in $E \backslash \{0\}$. For each $n \in \mathbb{N}$ there exists a closed neighbourhood U_n of zero such that $0 \notin U_n + x_n$. Let each $C_n =$

$U_n \cap A_n$. Clearly 0 is in each $\overline{C_n} \backslash C_n$ and not in the set

$$A := \bigcup_n (C_n + x_n).$$

However, $0 \in \overline{A}$: For U, an open neighborhood of 0, there exist $k \in \mathbb{N}$ with $x_k \in U$ and, V, a neighborhood of 0 with $V + x_k \subset U$. As there is $y \in V \cap C_k$ we also have $y + x_k \in U \cap A$. Thus $0 \in \overline{A} \backslash A$.

By hypothesis, there is $B \subset A$ with B bounded and $0 \in \overline{B}$. There exists subsets $B_n \subset C_n = U_n \cap A_n$ such that

$$B = \bigcup_n (B_n + x_n).$$

By construction, 0 does not belong to the closed sets

$$\bigcup_{k<n} (U_k + x_k).$$

Therefore 0 is not in any $\overline{\bigcup_{k<n} (B_k + x_k)}$. This and $0 \in \overline{B}$ imply that

$$0 \in \overline{\bigcup_{n \leq k} (B_k + x_k)},$$

for each $n \in \mathbb{N}$.

Let W and V be any balanced neighborhoods of 0 with $V - V \subset W$. Fix $n \in \mathbb{N}$. There exists $m \geq n$, in $\mathbb{N}$, such that $x_k \in V$ for all $k \geq m$. From

$$0 \in \overline{\bigcup_{m \leq k} (B_k + x_k)},$$

it follows that there exist $k \geq m$ and $y \in B_k$ with $y + x_k \in V$. From $y \in V - x_k \subset V - V \subset W$, we see, for each $n \in \mathbb{N}$, the set W meets $\bigcup_{n \leq k} B_k$. As any neighborhood of 0 contains W and V as above, 0 is in the closure of each $\bigcup_{n \leq k} B_k$. Note also that $\bigcup_n B_n$ is bounded. Indeed, as

$$B = \bigcup_n (B_n + x_n)$$

and $C = \{x_m : m \in \mathbb{N}\}$ are bounded and since

$$\bigcup_n B_n \subset \bigcup_n (B_n + x_n) - \{x_m : m \in \mathbb{N}\} = B - C,$$

then $\bigcup_n B_n$ is also bounded too. We have proved that (2) implies (3).

(3) implies (1): Assume that $0 \in \overline{A}$, and set $A_n := nA$, for each $n \in \mathbb{N}$. Since 0 is in each $\overline{A_n}$, there exist $B_n \subset A_n$, , as in (3). So each $\bigcup_{n \leq k} B_k$ is nonempty, and, consequently, there exists a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ with B_{n_k} nonempty. For each k, let $y_k \in B_{n_k}$. There exists a sequence $(a_k)_k$ in A such that $y_k = n_k a_k$ for each $k \in \mathbb{N}$. Since $(n_k)_k$ is strictly increasing and $(y_k)_k = (n_k a_k)_k$ is bounded, the sequence $(a_k)_k$ in A converges to zero in E. The proof is complete. □

Gerlits-Nagy, McCoy and Pytkeev, see [7, 8, 10, 13], proved independently that for a completely regular Hausdorff space X the space $C_p(X)$ is Fréchet-Urysohn iff X is an ω-space. Recall [10, p. 58 and Theorem 4.7.4] that a space X is an

ω-space if every open ω-cover of X (*i.e.*, a cover $\mathfrak{U}$ of open subsets of X such that every finite subset of X is contained in some element of $\mathfrak{U}$) contains an ω-sequence $(U_n)_n$ (*i.e.*, such that every finite subset F of X is eventually in the terms of the sequence). Therefore $C_p(X)$ is not Fréchet-Urysohn iff there exists an open ω-cover $\mathfrak{U}$ such that (†) each sequence in $\mathfrak{U}$ frequently misses some point(s) of some fixed finite subset of X. To see this we can also apply (3): Assume $\mathfrak{U}$ satisfies condition (†). For each nonempty finite set $F \subset X$, choose $U_F \in \mathfrak{U}$ containing F and $f_F \in C_p(X)$ satisfying $\frac{1}{|F|} \le f_F \le 1$, $f_F[F] = \{\frac{1}{|F|}\}$, and $f_F[X \backslash U_F] = \{1\}$. With each $A_n = \{|F| \cdot f_F : \text{ finite } F \subset X,\ |F| > n\}$, 0 is in each $\overline{A_n} \setminus A_n$. For any sequence $(F_n)_n$ of nonempty finite sets with corresponding $U_n = U_{F_n} \in \mathfrak{U}$ and $f_n = f_{F_n} \in C_p(X)$, there is a finite set $F \subset X$ and a subsequence $(n_k)_k$ of $\mathbb{N}$ with all $F \backslash U_{n_k} \neq \emptyset$, so that each

$$|F_{n_k}| \in |F_{n_k}| \cdot f_{n_k}[F \backslash U_{n_k}].$$

Thus, for unbounded $(|F_{n_k}|)_k$ one gets that $\{|F_{n_k}| \cdot f_{n_k}\}_k \subset \{|F_n| \cdot f_n\}_n \subset C_p(X)$ is unbounded as it is so on the finite set F. Using (3) with $B_n \subset A_n$ and each $\bigcup_{n<k} B_k$ accumulating at zero, we see that each sequence $f_n \in \bigcup_{n<k} B_k$ is not bounded. By Theorem 1 $C_p(X)$ is not Fréchet-Urysohn. Also this $C_p(X)$ does not have bounded tightness, because if $A := \bigcup_n A_n$, clearly $0 \in \overline{A} \setminus A$ and if B is a subset of A accumulating at zero then each sequence $f_n \in B \cap (\bigcup_{n<k} A_k)$ is unbounded.

In as much as the concept of boundedness for a tvs may be defined strictly in terms of its additive group structure, it now appears that such boundedness may be a concept model for obtaining the Fréchet-Urysohn property for some topological groups. Of course, this is possible for many subgroups of the additive group of a tvs. We end with two variant questions raised by this. Suppose G is a topological group and P is a property of subsets of G. By *P-tightness* for G we mean that for all $A \subset G$ and all $x \in \overline{A}$ there is $B \subset A$ satisfying property P with $x \in \overline{B}$. (1) What P-tightness properties ensure that G is Fréchet-Urysohn? And, more specifically, (2) what boundedness properties for topological groups satisfy the equivalences of the above theorem?

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