Poincaré map in fractal analysis of spiral trajectories of planar vector fields

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Abstract

We study the box dimension and Minkowski content of spiral trajectories of planar vector fields, using information about the asymptotic behaviour of iterates of the Poincaré map. An auxilliary tool is a flow-sector theorem near the weak focus, which is of a similar nature as the well known flow-box theorem. Applications include Hopf bifurcation and Liénard systems. We obtain all possible values of box dimensions of spiral trajectories around weak focus, associated with polynomial vector fields.

1 Introduction

We deal with planar vector fields, and consider their spiral trajectories of limit cycle and focus types. Spiral trajectories of planar systems have been studied in [18] and [19], while spatial spirals have been studied in [20]. Concentrating on planar systems, we derive fractal properties of spirals using only an information about the asymptotic behaviour of the sequence of iterates of the corresponding Poincaré map near the origin, associated with the system

$$\begin{cases} \dot{x} &= -y + p(x,y) \\ \dot{y} &= \quad x + q(x,y), \end{cases} \tag{1}$$

where $p(x,y)$ and $q(x,y)$ are given C^1 functions such that $|p(x,y)| \le C(x^2+y^2)$ and $|q(x,y)| \le C(x^2+y^2)$ for some positive constant C and for (x,y) near the origin. The main result dealing with box dimension of spiral trajectories near the weak focus of (1) is contained in Theorem 1. It enables us to apply it to the Hopf bifurcation,

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Liénard systems, see Theorems 5 and 6. We obtain that analytic systems as in (1) have spirals near the origin with box dimensions only from a discrete set of the form $\{\frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \dots\}$, see Theorem 7, and all these values can be attained for example for simple Liénard systems (1) in which $p(x, y) = x^{2k+1}$, $q(x, y) \equiv 0$, $k \in \mathbb{N}$, see Theorem 6.

The paper is organized as follows. In Section 2 we introduce the notion of spirals of power $\beta > 0$, and state the main result. In Section 3 we prove the flow-sector theorem, which is of a similar nature as the flow-box theorem. In Section 4 we prove the main result using flow-box and flow-sector theorems, combined with a dimension result from [5] dealing with discrete systems. In Section 5 we provide applications.

Let us introduce some notation. For $A \subset \mathbb{R}^N$ bounded we define the *Minkowski sausage* of radius ε around A as the ε-neighbourhood of A (the notion has been introduced by B. Mandelbrot): $A_\varepsilon := \{y \in \mathbb{R}^N : d(y, A) < \varepsilon\}$. By *lower s-dimensional Minkowski content* of A, $s \geq 0$ we mean

$$\mathcal{M}_*^s(A) := \liminf_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}},$$

and analogously for the *upper s-dimensional Minkowski content* $\mathcal{M}^{*s}(A)$. Now we can introduce the lower and upper box dimensions of A by

$$\underline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}_*^s(A) = 0\}$$

and analogously $\overline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}^{*s}(A) = 0\}$. If these two values coincide, we call it simply the box dimension of A, and denote by $\dim_B A$. If $0 < \mathcal{M}_*^d(A) \leq \mathcal{M}^{*d}(A) < \infty$ for some d, then we say that A is *Minkowski nondegenerate*. In this case obviously $d = \dim_B A$. In the case when lower or upper d-dimensional Minkowski contents of A are 0 or ∞, where $d = \dim_B A$, we say that A is *degenerate*. For more details on these definitions see e.g. Falconer [6], [18], [19] and [20].

For any two sequences of positive real numbers (a_k) and (b_k) converging to zero we write $a_k \simeq b_k$ as $k \to \infty$ if there exist positive real numbers $A < B$ such that $a_k/b_k \in [A, B]$ for all k. Also if $f, g : (0, r) \to (0, \infty)$ are two functions converging to zero as $s \to 0$ and $f(s)/g(s) \in [A, B]$, we write $f(s) \simeq g(s)$ as $s \to 0$. We write $f(s) \sim g(s)$ if $f(s)/g(s) \to 1$ as $s \to 0$. Also, for two real functions f and g we shall sometimes write only $f(x) \simeq g(x)$ meaning that for some positive A and B we have $Ag(x) \leq f(x) \leq Bg(x)$ for all x. For example, for a function $F : U \to V$ with $U, V \subset \mathbb{R}^2$, $V = F(U)$, the condition $|F(x_1) - F(x_2)| \simeq |x_1 - x_2|$ means that f is a bilipschitz mapping, i.e., both F and F^{-1} are Lipschitzian. We also say that F is *lipeomorphism*, and that the sets U and V are lipeomorphic. If a real function function $p(x, y)$ is such that for some positive constants k and C we have $|p(x, y)| \leq C \cdot r^k$ for $r = \sqrt{x^2 + y^2}$ sufficiently small, we write $p(x, y) = O(r^k)$ when $r \to 0$. By $B_\varepsilon(a)$ we denote an open disk of radius ε centered at a.

2 Poincaré map and box dimension of spiral trajectories

If Γ is a spiral of limit cycle type (tending to a limit cycle Γ_0, say from inside), to each point x of Γ_0 we attach an axis $\sigma = \sigma(x)$ through this point, perpendicular to the limit cycle, oriented inwards, and with origin at x. The set of all such axes

σ will be denoted by Σ_c. Let $P_\sigma : (0,\varepsilon_\sigma)\cap\Gamma \to (0,\varepsilon_\sigma)\cap\Gamma$ be the Poincaré map corresponding to any axis $\sigma\in\Sigma_c$ and defined by Γ. We have $P_\sigma(x_k)=x_{k+1}$, where the sequence $(x_k)_k=(0,\varepsilon_\sigma)\cap\Gamma$ in σ is arranged in decreasing order. By P_σ^k we denote k-fold composition of P_σ. We take $\varepsilon_\sigma>0$ small enough, so that $P_\sigma^k(s)\to 0$ as $k\to\infty$ for all $s\in(0,\varepsilon_\sigma)\cap\Gamma$. If the family of Poincaré maps $\{P_\sigma : \sigma\in\Sigma_c\}$ is such that there exists $\beta>0$ satisfying

$$d_k(s_\sigma) := P_\sigma^k(s_\sigma) - P_\sigma^{k+1}(s_\sigma) \simeq k^{-1-\beta}, \quad k\to\infty, \tag{2}$$

where $s_\sigma=\max(\Gamma\cap(0,\varepsilon_\sigma))$ (here the maximum is taken on the σ-axis), we say that Γ is the *limit cycle spiral of power* β. In other words, there exist two positive constants $A_\sigma<B_\sigma$ such that for any axis $\sigma\in\Sigma_c$ there holds $A_\sigma k^{-1-\beta}\le d_k(s_\sigma)\le B_\sigma k^{-1-\beta}$ for all k.

Remark 1. It is easy to see that condition (2) implies that the family of Poincaré maps has the following property for all $\sigma\in\Sigma_c$:

$$P_\sigma^k(s_\sigma) \simeq k^{-\beta}, \quad k\to\infty. \tag{3}$$

Indeed, $P_\sigma^k(s_\sigma)=\sum_{i=k}^\infty d_i(s_\sigma)\simeq\sum_{i=k}^\infty i^{-1-\beta}\simeq\int_k^\infty x^{-1-\beta}dx\simeq k^{-\beta}$, as $k\to\infty$. This is the reason why the limit cycle spirals satisfying (2) is said to be of power β.

Similarly, if Γ is a spiral of focus type, we consider the set Σ_0 of all axes σ through the focus. Let $P_\sigma : (0,\varepsilon_\sigma)\cap\Gamma\to(0,\varepsilon_\sigma)\cap\Gamma$ be the Poincaré map corresponding to any fixed axis σ (we assume that $s=0$ corresponds to the focus for any axis σ). If the family of Poincaré maps $\{P_\sigma : \sigma\in\Sigma_0\}$ is such that there exists $\beta>0$ such that for all $\sigma\in\Sigma_0$ there holds (2) for any fixed $\sigma\in\Sigma_0$, we say that Γ is the *focus spiral of power* β.

The main result of this paper is the following theorem. We postpone its proof until Section 4. Let us recall that the function $d(s)=P(s)-s$, where $P(\cdot)$ is the Poincaré map and s small enough, is the displacement function. Note that if the Poincaré map corresponds to a limit cycle spiral with respect to an axis $\sigma\in\Sigma_c$ oriented inwards (if the spiral is inside the limit cycle), or outwards (if the spiral is outside the limit cycle) then in both cases we have $d(s)<0$. The same holds for spirals of focus type.

Theorem 1. *Let Γ be a spiral trajectory of a planar vector field of class C^1. Let $P_\sigma(s)$ be the Poincaré map with respect to an axis σ, and assume that it has the form $P_\sigma(s)=s+d_\sigma(s)$ for each σ, where the displacement function $d_\sigma(\cdot):(0,r_\sigma)\to(-\infty,0)$ is monotonically nonincreasing, such that $-d_\sigma(s)\simeq s^\alpha$ as $s\to 0$, for a constant $\alpha>1$ independent of σ. Then Γ is the spiral of power $1/(\alpha-1)$. Furthermore,*

(a) if Γ is a focus spiral associated with a system (1) such that $p(x,y)=O(r^2)$ and $q(x,y)=O(r^2)$ as $r=\sqrt{x^2+y^2}\to 0$, then

$$\dim_B\Gamma = \begin{cases} 2-\frac{2}{\alpha} & \text{for } \alpha>2, \\ 1 & \text{for } 1<\alpha\le 2, \end{cases} \tag{4}$$

and Γ is Minkowski nondegenerate for $\alpha\ne 2$, and Minkowski degenerate for $\alpha=2$;

(b) if Γ is a limit cycle spiral, then

$$\dim_B\Gamma = 2-\frac{1}{\alpha},$$

and it is Minkowski nondegenerate.

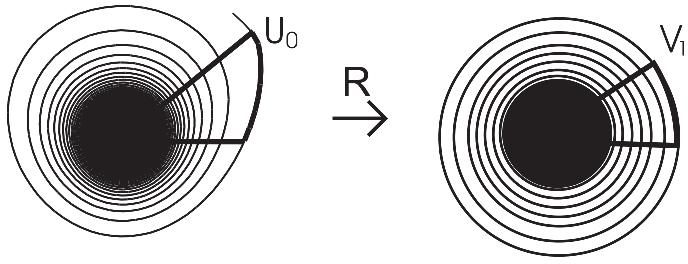

Figure 1: Flow-sector theorem: weak focus flow in sectors near the singular point is lipeomorphically equivalent to the annulus flow.

3 Flow-sector theorem

In the proof of Theorem 1(b) we shall need the following version of the well known *flow-box* theorem, dealing with *diffeomorphic equivalence* of phase portraits, i.e. mapping trajectories onto trajectories. In particular, the function realizing the equivalence is lipeomorphism. Recall that for two closed sets U and V we say to be diffeomorphic if there exists a diffeomorphism between their open neighbourhoods.

Theorem 2. (Flow-box theorem, see e.g. Dumortier, Llibre, Artés [4, Theorem 1.12], or Kuznetsov [12, p. 75]) *Let us consider a planar vector field of class C^1. Assume that $U \subset \mathbb{R}^2$ is a closed set the boundary of which is the union of two trajectories and two curves transversal to trajectories. If U is free of singularities and of periodic orbits, then the dynamical system restricted to U is diffeomorphically equivalent to the system*

$$\begin{cases} \dot{y}_1 &= 0 \\ \dot{y}_2 &= 1, \end{cases}$$

defined on the unit square $V = \{(y_1, y_2) : |y_1| \leq 1,\ |y_2| \leq 1\}$.

In the proof of Theorem 1(a) we shall need the following analog of flow-box theorem that we call *flow-sector theorem.* It shows that in any sufficiently small sector with vertex at the weak focus the dynamics is lipeomorphically equivalent to that of the annulus flow in a sector, see Figure 1. Here the *annulus flow* is defined by $\dot{r} = 0$, $\dot{\varphi} = 1$ in $\mathbb{R}^2 \setminus \{0\}$ in polar coordinates (r, φ). The result seems to be new even for analytic systems (1) such that near the singularity the flow is of spiral type, see Figure 1 on the left.

Theorem 3. (Flow-sector theorem) *Let $U_0 \subset \mathbb{R}^2$ be an open sector with the vertex at the origin, such that its opening angle is in $(0, 2\pi)$, and the boundary of U_0 consists of a part of a trajectory and of intervals on two rays emanating from the origin (see Figure 1 on the left). Assume that*

$$p(x,y) = O(r^2),\ q(x,y) = O(r^2) \quad \text{as } r = \sqrt{x^2+y^2} \to 0. \tag{5}$$

If the diameter of U_0 is sufficiently small, then system (1) restricted to U_0 is lipeomorphically equivalent to the system

$$\begin{cases} \dot{r} &= 0 \\ \dot{\varphi} &= 1, \end{cases} \tag{6}$$

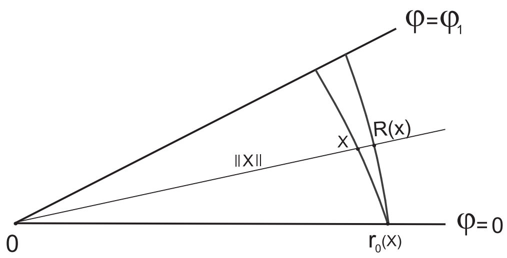

Figure 2: Definition of the mapping R

defined on the sector $V_0 = \{(r,\varphi) : 0 < r < 1,\ 0 < \varphi < \pi/2\}$ *in polar coordinates* (r,φ).

Proof. It will be convenient in this proof to denote points in the plane by $x = (x_1,x_2)^\top$. Let us assume without loss of generality that the angles corresponding to two rays from the definition of U_0 are $\varphi_0 = 0$ and $\varphi_1 \in (0,2\pi)$. Now we define a mapping $R : U_0 \to V_1 = R(U_0)$ for which we will show to be a lipeomorphic equivalence. For a given $x \in U_0$ let Γ_x be the trajectory passing through x, and let x_0 be the point of intersection of Γ with the ray $\{\varphi = 0\}$. Let $r_0(x) = \|x_0\|$, where $\|\cdot\|$ is the Euclidean norm. We define R by

$$R(x) = r_0(x)\,\hat{x} = \frac{r_0(x)}{\|x\|}\,x, \tag{7}$$

see Figure 2. It is well defined since the opening angle of sector U_0 is in $(0,2\pi)$, and since the rays are transversal to trajectories, and even almost perpendicular. Indeed,

$$\cos\angle(x,F(x)) = \frac{|x\cdot F(x)|}{\|x\|\,\|F(x)\|} \simeq \frac{|x_1p(x)+x_2q(x)|}{r^2} = \frac{O(r^3)}{r^2} \to 0,$$

as $r \to 0$, where $F(x) = (-x_1 + p(x), x_2 + q(x))^\top$ for $x = (x_1,x_2)^\top \in \overline{U}_0$, so that when U_0 has a sufficiently small diameter we have $\angle(x,F(x)) \approx \pi/2$. Note that the vector field (6) on sector V_1 is lipeomorphically equivalent to the system (6) viewed on the sector V_0.

Using (1) we obtain that $x\,p + y\,q = r\,\dot{r}$, hence, $\dot{r} = O(r^2)$. Using also $\dot{\varphi} = 1 + O(r)$ we obtain that

$$\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = \frac{O(r^2)}{1+O(r)} = O(r^2).$$

Dividing $-Cr^2 \le \frac{dr}{d\varphi} \le Cr^2$ by r^2 and integrating from $\varphi = 0$ to $\varphi(x)$, it follows that

$$|\,\|x\| - r_0(x)| \le \varphi(x)\,O(r_0(x)^2). \tag{8}$$

Hence, since $r_0(x) \to 0$ as $x \to 0$, we have

$$\|x\| \sim r_0(x) \quad \text{as } x \to 0. \tag{9}$$

Now we show that R is the lipeomorphic for x small enough. Applying the Lagrange mean value theorem to both components of $R = (R_1, R_2)^\top$, we obtain that for any $x, y \in U_0$ we have

$$R(x) - R(y) = T(u,v)(x-y), \tag{10}$$

where $T(u,v) = (\nabla R_1(u), \nabla R_2(v))^\top$ is a 2×2-matrix with raws $\nabla R_1(u)$ and $\nabla R_2(v)$, and $u, v \in [x,y] \subset U_0$. Here $[x,y]$ denotes the interval in the plane with vertices in x and y. To compute $T(u,v)$ we need the Fréchet derivative $R'(x)$, since $R'(x) = (\nabla R_1(x), \nabla R_2(x))^\top$. A direct computation shows that

$$R'(x) = \hat{x} \cdot \nabla r_0(x)^\top + \frac{r_0(x)}{\|x\|}(I - \hat{x} \cdot \hat{x}^\top). \tag{11}$$

In order to find $\nabla r_0(x)$, let us go back to the differential equation $\frac{dr}{d\varphi} = \frac{x\,p + y\,q}{r(1+O(r))} =: F(r,\varphi)$. By integration from $\varphi = 0$ to $\varphi = \varphi(x)$, we obtain $\|x\| - r_0(x) = \int_0^{\varphi(x)} F(r(\varphi), \varphi)\, d\varphi$. Taking the gradient and using $\varphi(x) = \arctan(x_2/x_1)$, we obtain that

$$\hat{x} - \nabla r_0(x) = F(r, \varphi(x))\, \nabla\varphi(x) = F(\|x\|, \varphi(x))\, r^{-2} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix},$$

where $r = \|x\|$. Exploiting (11) it follows that

$$R'(x) = \frac{r_0(x)}{\|x\|}\, I + \left(1 - \frac{r_0(x)}{\|x\|}\right) A(x) + B(x), \tag{12}$$

where

$$A(x) = \hat{x} \cdot \hat{x}^\top = \begin{bmatrix} \hat{x}_1^2 & \hat{x}_1\hat{x}_2 \\ \hat{x}_1\hat{x}_2 & \hat{x}_2^2 \end{bmatrix},$$

with $\hat{x} = \frac{x}{\|x\|} = (\hat{x}_1, \hat{x}_2)^\top$, and

$$B(x) = \frac{F(\|x\|, \varphi(x))}{\|x\|} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} [\sin\varphi \quad -\cos\varphi].$$

Condition (5) implies that $B(x) \to 0$ as $x \to 0$.

Let us now consider the max-norm of vectors. Using (12) in (10) and denoting vector raws of matrices A and B by A_i and B_i respectively, $i = 1, 2$, we obtain that for any $x, y \in \mathbb{R}^2$:

$$\begin{aligned} \|R(x) - R(y)\|_\infty &= \|T(u,v)(x-y)\|_\infty \\ &\geq \min\left\{\frac{r_0(u)}{\|u\|}, \frac{r_0(v)}{\|v\|}\right\} \|x-y\|_\infty \\ &\quad -M(u,v)\|(A_1(u), A_2(v))^\top (x-y)\|_\infty \\ &\quad -\|(B_1(u), B_2(v))^\top (x-y)\|_\infty, \end{aligned}$$

where $M(u,v) = \max_{z\in\{u,v\}} \left|1 - \frac{r_0(z)}{\|z\|}\right|$. Note that

$$\begin{aligned} \|(A_1(\hat{u}), A_2(\hat{v}))^\top (x-y)\|_\infty &\leq \max_{z\in\{u,v\}} \|A(\hat{z})\|_\infty \|x-y\|_\infty \\ &\leq 2\|x-y\|_\infty, \end{aligned}$$

since $\|A(\hat{z})\|_\infty = \hat{z}_1^2 + |\hat{z}_1\hat{z}_2| \le 2$ for $\|\hat{z}\|_\infty \le 1$, and

$$\|(B_1(u), B_2(v))^\top (x-y)\|_\infty \le \max_{z\in\{u,v\}} \|B(z)\|_\infty \|x-y\|_\infty.$$

Here $\max \|B(z)\|_\infty$ can be made arbitrarily small by taking the diameter of U_0 small enough. Therefore we have that $\|R(x) - R(y)\|_\infty \ge c_1\|x-y\|_\infty, \quad \forall x, y \in U_0$ for some positive constant c_1, provided

$$\min_{z\in\{u,v\}} \frac{r_0(z)}{\|z\|} - \max_{z\in\{u,v\}} \left|1 - \frac{r_0(z)}{\|z\|}\right| \cdot 2 > c_1 > 0. \tag{13}$$

This is satisfied due to (9), assuming that U_0 is of sufficiently small diameter. The inequality $\|R(x)-R(y)\|_\infty \le c_2\|x-y\|_\infty$ for some positive constant c_2 follows easily from (10) and (12). This proves that $R : U_0 \to V_1$ is a lipeomorphic equivalence. ∎

Remark 2. Let p and q be analytic functions such that

$$p(x,y) = \sum_{k=2}^{\infty} p_k(x,y), \quad q(x,y) = \sum_{k=2}^{\infty} q_k(x,y), \tag{14}$$

where p_k and q_k are homogeneous polynomials of degree k. Then the conditions of Theorem 3 are satisfied, and hence the flow-sector property for (1) holds.

Remark 3. It is easy to see that the function R constructed in the proof of Theorem 3 is a diffeomorphism. Indeed, $R^{-1}(x) = r_1(x)\hat{x}$, where $r_1(x)$ can be obtained similarly as $r_0(x)$ in the proof of Theorem 3. Thus, R is a lipeomorphic diffeomorphism. Furthermore, the Lipschitz constants c_1 and c_2 can be taken arbitrarily close to 1, if the diameter of the sector U_0 is sufficiently small, see (13). The reason is intuitively clear: for small x the function R is close to the identity, and R' is close to the identity matrix, see (7), (11), and (9).

4 Proof of the main result

In the following theorem we establish the connection between the asymptotic behaviour of iterates of the Poincaré map associated with a spiral, and the box dimension of the spiral. It complements our results in [19, Theorems 1 and 2]. The main Theorem 1 will be a consequence of Theorem 4 below dealing with continuous systems, and of a result from [5] dealing with one-dimensional discrete systems.

Theorem 4. *Let Γ be a spiral trajectory of a planar vector field of class C^1.*

(a) If Γ is a focus spiral trajectory of power $\beta > 0$, associated with the system described by (1), such that $p(x,y) = O(r^2)$ and $q(x,y) = O(r^2)$ as $r = \sqrt{x^2+y^2} \to 0$, then

$$\dim_B \Gamma = \begin{cases} \frac{2}{1+\beta} & \text{for } \beta \in (0,1), \\ 1 & \text{for } \beta \ge 1. \end{cases} \tag{15}$$

Furthermore, Γ is Minkowski nondegenerate when $\beta \ne 2$, and it is Minkowski degenerate for $\beta = 1$.

(b) If Γ is a limit cycle spiral trajectory of power $\beta > 0$, then

$$\dim_B \Gamma = \frac{2+\beta}{1+\beta}, \tag{16}$$

and the spiral is Minkowski nondegenerate.

Proof of Theorem 4. (a) For each $\sigma \in \Sigma_0$ there exists an open sector $U_0(\sigma, \delta_\sigma) \subset B_\varepsilon(0)$ as in Theorem 3, of sufficiently small diameter, with vertex at the focus, where $\delta_\sigma \in (0, 2\pi)$ denotes the opening angle of the sector, and σ is the middle ray corresponding to the opening angle in the sector. It is clear that we may take ε independent of σ. Using Theorem 3, we see that the set $\Gamma \cap U_0(\sigma, \delta_\sigma)$ is lipeomorphic to the set $B'_\sigma = C'_\beta \cap V_1(\sigma, \delta_\sigma)$, where C'_β is the union of concentric circles with centers at the origin, having radii $P^k_\sigma(s_\sigma)$, $k \in \mathbb{N}$, and $V_1(\sigma, \delta_\sigma, 1)$ is the corresponding sector in the unit ball $B_1(0)$. Now, we show that B'_σ is lipeomorphic to the set $B_\sigma = C_\beta \cap V_1(\sigma, \delta_\sigma)$, where C_β is the union of concentric circles of radii $k^{-\beta}$, $k \in \mathbb{N}$, centered at the origin. Indeed, we first define a piecewise linear function $f : \sigma \to \sigma$ by $F(P^k_\sigma(s_\sigma)) = k^{-\beta}$ defined in a neighbourhood of the origin, which is clearly lipeomorphic due to $P^k_\sigma(s_\sigma) - P^{k+1}_\sigma(s_\sigma) \simeq k^{-\beta} - (k+1)^{-\beta}$ as $k \to \infty$, see (2). Then we define the desired lipeomorphism $F : B'_\sigma \to B_\sigma$ by radial extension, that is, by $F(x) = f(\|x\|)\,\hat{x}$, where $\hat{x} = x/\|x\|$.

The family of open sectors $\{U_0(\sigma, \delta_\sigma, \varepsilon) : \sigma \in \Sigma_0\}$ is not an open cover of of the whole of $\overline{\Gamma}$, since for example $0 \in \overline{\Gamma}$, but not in any of the sectors. Therefore, for fixed σ we consider a part A_σ the set of all points x on the unit circle $S^1 = \{x : |x| = 1\}$ of the plane, such that the ray spanned by x and the origin intersects the sector $U_0(\sigma, \delta_\sigma)$. It is clear that the family $\{A_\sigma : \sigma \in \Sigma_0\}$ is an open cover of the circle, hence there exists a finite subcover: $S^1 = A_{\sigma_1} \cup \dots \cup A_{\sigma_n}$. From this we see that $\Gamma' \subset U_0(\sigma_1, \delta_1) \cup \dots \cup U_0(\sigma_n, \delta_n)$, where we have denoted $\delta_i = \delta_{\sigma_i}$, and Γ' is the part of Γ inside U. Similarly as in (a) we have that $\dim_B \Gamma = \dim_B \Gamma'$, since $\Gamma'' = \Gamma \setminus \Gamma'$ is rectifiable. Let us define $\Gamma'_i = \Gamma' \cap U_0(\sigma_i, \delta_i)$, $i = 1, \dots, n$. Then $\Gamma' = \Gamma'_1 \cup \dots \cup \Gamma'_n$, and due to finite stability of box dimension,

$$\dim_B \Gamma' = \max_{i=1,\dots,n} (\dim_B \Gamma'_i) = \max_{i=1,\dots,n} (\dim_B B_{\sigma_i}).$$

Now we show that each $\dim_B B_{\sigma_i}$ is equal to the right-hand side in (15). The sets B_{σ_i}, $i = 1, \dots, n$ are lipeomorphic (and moreover, isometric). Hence, using finite stability of box dimension we obtain that

$$\dim_B \Gamma' = \max_{i=1,\dots,n} (\dim_B B_{\sigma_i}) = \dim_B \left(\bigcup_{i=1}^{n} B_{\sigma_i} \right) = \dim_B C_\beta,$$

and the claim follows from $\dim_B C_\beta = 2/(1+\beta)$ for $0 < \beta < 1$ and $\dim_B C_\beta = 1$ for $\beta \geq 1$, see [19, Remarks 8 and 2].

The second claim follows from Minkowski nondegeneracy of B_σ (this follows easily from [19, Remarks 2 and 8]), combined with the fact that Minkowski nondegeneracy is preserved under lipeomorphisms, see [20] or [21].

(b) Let us fix $\sigma \in \Sigma_c$, and define U_σ as a closed set containing interval $(0, \varepsilon_\sigma) \subset \sigma$ in its interior, such that the boundary of U_σ is the union of a part of the limit cycle, of two closed intervals $I(\sigma') \subset \sigma'$ and $I(\sigma'') \subset \sigma''$, $\sigma', \sigma'' \in \Sigma_c$, and of a part of spiral Γ. Let $I(\sigma') \cap \Gamma = \{(x_k, 0) : k \in \mathbb{N}\} \subset \sigma'$. Assume that the sequence (x_k) is decreasing, converging to 0. We first show that the set $\Gamma \cap U_\sigma$ is lipeomorphic to the following Cartesian product:

$$A_\beta = \{k^{-\beta} : k \in \mathbb{N}\} \times [-1, 1]. \tag{17}$$

In order to show this, we first note that by the flow-box Theorem 2, the initial dynamical system on U_σ is diffeomorphically equivalent with the system $\dot{y}_1 = 0$, $\dot{y}_2 = 1$, on the square $[-1, 1]^2$. Let us show that $\Gamma \cap U_\sigma$ is lipeomorphic to $V' = \{x_k : k \in \mathbb{N}\} \times [-1, 1]$. Since the interval $I(\sigma')$ is mapped onto $[-1, 1] \times \{-1\}$, due to (2) we conclude that

$$x_k - x_{k+1} \simeq P^k_{\sigma'}(s_{\sigma'}) - P^{k+1}_{\sigma'}(s_{\sigma'}) = d_k(\sigma') \simeq k^{-1-\beta}. \tag{18}$$

Similarly as in (a), using flow-box theorem, we can show that V' and A_β are lipeomorphic.

The second claim follows as in (a), using the fact that Minkowski nondegeneracy of a set is preserved under Cartesian product of the set with $[-1, 1]$, see [18, Proposition 4.3]. Minkowski nondegeneracy of the set $\{k^{-\beta}\}$ has been established in Lapidus, Pomerance [13]. ■

Now we can prove the main result of this paper.

Proof of Theorem 1. The fact that the spirals are of power $\beta = 1/(\alpha - 1)$ follows immediately from [5, Theorem 1], with $f(s) = -d(s)$ there. The claims in (a) and (b) then follow from Theorem 4. ■

5 Applications

We consider a planar analytic system (1) with a weak focus at the origin, where $p(x, y)$ and $q(x, y)$ are as in (14). In polar coordinates (r, φ) system (1) reduces to

$$\frac{dr}{d\varphi} = \sum_{k=2}^{\infty} s_k(\varphi)\, r^k. \tag{19}$$

Let $r(\varphi, r_0)$ be the solution of (19) such that $r = r_0$ for $\varphi = 0$. For r small enough we can write

$$r(\varphi, r_0) = r_0 + \sum_{k=2}^{\infty} u_k(\varphi)\, r_0^k, \tag{20}$$

with $u_k(0) = 0$ for $k \geq 2$. The Poincaré map for (1) near the focus is defined by

$$P(r_0) = r(2\pi, r_0) = r_0 + \sum_{k=2}^{\infty} u_k(2\pi)\, r_0^k.$$

The coefficient $u_k(2\pi)$ in the above expansion is called *k-th Lyapunov coefficient* of the weak focus, $k \geq 2$. We denote the first nonzero Lyapunov coefficient by V_k. It can be shown that in such V_k the index k is always odd, see e.g. Dumortier, Llibre, Artés [4, p. 124], or Roussarie [15, Lemma 8].

Lyapunov coefficients were first introduced in [14]. There are some other names for the Lyapunov coefficient, like the Lyapunov number, Lyapunov constant or Lyapunov quantity. Also, there are different definitions, but all of them are mutually equivalent and the coefficients are equal up to some positive constant, see [4, p. 126]. Lyapunov coefficients are important in solving the stability problem of a plane system which is a perturbation of a linear focus at the origin, but computation of the Lyapunov coefficients is quite a difficult task. There exist different ways of their computation. A new algorithm for the computation of the Lyapunov coefficients, based on Gasull, Torregrosa [11] and relying on ideas of Françoise [8], is exposed in [4]. There one can find some numerical examples, see [4, pp. 144–145], to which our dimension result in Theorem 1 applies. In [4, Section 4.7] there is a list of articles dealing with the computation of Lyapunov coefficients.

5.1 Hopf bifurcaton

In [19] we dealt with the normal form called the standard model where the Hopf-Takens bifurcation occurs:

$$\begin{aligned} X_{\pm}^{(l)} \quad := \quad & \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \pm \Big((x^2+y^2)^l + \\ & a_{l-1}(x^2+y^2)^{l-1} + \cdots + a_0\Big)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right). \end{aligned} \tag{21}$$

Here $(a_0,\dots,a_{l-1}) \in \mathbb{R}^l$ is fixed. We established the relation between the exponent of the first nonzero monomial of (21) and the box dimension of the spiral trajectory tending to focus or to a limit cycle, see [19, Theorems 9 and 19].

It is known that according to Takens [16], any analytic vector field X of the form (1) satisfying (14) is locally diffeomorphic to its normal form $X_{\pm}^{(l)}$, see also Caubergh, Dumortier [1, p. 3]. It follows that the Hopf-Takens bifurcation of codimension l, assuming additionally that $k = l$ (i.e. when we have the birth of l limit cycles), occurs with box dimension equal to $2(1-\frac{1}{2l+1})$. For $l = 1$ we have a classical Hopf bifurcation. In [19] we have shown that if $l = 1$ and $a_0 = 0$ in (21), then any spiral trajectory tending to the origin has box dimension equal to 4/3. We hope that the approach undertaken in this paper via the Poincaré mapping could solve analogous problems for the case of nilpotent focus and degenerate focus (i.e. with zero linear part). It could be possible also to deal with spirals near a saddle-loop, and even 2-saddle cycle.

The next theorem deals with analytic systems of the form

$$\left\{\begin{array}{lcl} \dot{x} & = & ax - y + p(x,y) \\ \dot{y} & = & x + ay + q(x,y). \end{array}\right. \tag{22}$$

We reprove the fact that at parameter of Hopf bifurcation the spirals near the weak focus always have box dimension equal to 4/3. We have proved analogous result in [5] for saddle-node and period-doubling bifurcations of one-dimensional discrete systems.

Theorem 5. (Hopf bifurcation) *Assume that $p(x,y)$ and $q(x,y)$ are analytic functions as in (14). Let Γ be a spiral trajectory near the origin of system (22), where $a=0$. If the first nonzero Lyapunov coefficient is V_3, then the Hopf bifurcation occurs at the origin of the system (22) at $a=0$, the spiral is of power $1/2$, and*

$$\dim_B \Gamma = \frac{4}{3}.$$

Furthermore, Γ is Minkowski nondegenerate.

Proof. The claim follows from Theorem 1(a) with $\alpha = 3$. ∎

It is possible to extend this theorem for $l > 1$, which corresponds to Hopf-Takens bifurcation. Box dimension of a spiral trajectory depends on the nonzero Lyapunov coefficient.

5.2 Liénard systems

In the simple model (21) the box dimension depends only on the exponents of the system, see [19], but in general it is not true because the Lyapunov coefficients depend on the coefficients of the system. In Caubergh, Dumortier [1, Theorem 5] a relation between Lyapunov coefficients and Hopf-Takens bifurcation has been established. Application of the results proved in [1] for Liénard systems

$$\left\{\begin{array}{rcl} \dot{x} &=& -y + \sum_{i=1}^{N} a_{2i}x^{2i} + \sum_{i=k}^{N} a_{2i+1}x^{2i+1} \\ \dot{y} &=& x, \end{array}\right. \tag{23}$$

and generalised Liénard systems can be found in Caubergh, Françoise [2]. There is a great interest to consider Liénard systems, see for instance Françoise [7], [9], [10]. For such systems it is very simple to compute Lyapunov coefficients in terms of the coefficients of the system. As an example we will extend [2, Proposition 8] by a dimensional result. We recall that [2, Proposition 8] is a result cited from Christopher, Lloyd [3]. For the sake of simplicity we state the following result under less general conditions than in [2, Proposition 8].

Theorem 6. (Liénard system) *Let $a_{2k+1} \neq 0$ in (23), that is, a_{2k+1} is the first nonzero coefficient corresponding to an odd exponent of x. Then any spiral trajectory Γ, viewed near the origin, is of power $\frac{1}{2k}$ and has box dimension equal to*

$$\dim_B \Gamma = 2\left(1 - \frac{1}{2k+1}\right). \tag{24}$$

Furthermore, Γ is Minkowski nondegenerate.

Proof. The first nonzero Lyapunov coefficient in our notation is given by $V_{2k+1} = c_{2k+1}a_{2k+1} \neq 0$ for a certain rational number $c_{2k+1} > 0$, see [2, Proposition 8]. The claim follows from Theorem 1(a) with $\alpha = 2k+1$, since $p(x,y) = O(x^2)$ and $q(x,y) = 0$. ∎

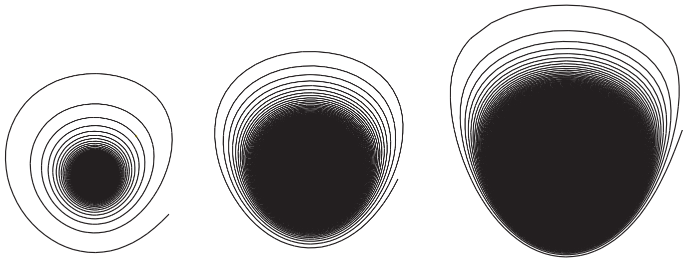

Figure 3: Three spirals of Liénard system (25) for $k = 1, 2, 3$, with box dimensions 4/3, 8/5, 12/7, see (24).

An illustration of Theorem 6 is provided by Figure 3, where three spiral trajectories are shown, corresponding to the Liénard system

$$\begin{cases} \dot{x} &= -y + \left(\sum_{i=1}^{k} x^{2i}\right) + x^{2k+1} \\ \dot{y} &= \quad x, \end{cases} \tag{25}$$

with $k = 1, 2, 3$ respectively. The terms with even exponents do not influence the box dimension of spirals.

The Liénard system in Theorem 6 concerns a class of planar systems in which $p(x, y)$ is a polynomial in x and $q(x, y) = 0$. One can ask if it is possible to construct polynomials $p(x, y)$ and $q(x, y)$ as in (14) such that a spiral trajectory of (1) near the origin does not have its box dimension of the form $2\left(1 - \frac{1}{2k+1}\right)$ or 1, that is, from the set

$$D_0 = \{\frac{4k}{2k+1} : k \in \mathbb{N}\} = \left\{\frac{4}{3}, \frac{8}{5}, \frac{12}{7}, \frac{16}{9}, \frac{20}{11}, \dots\right\}.$$

The answer is no. Moreover, even for analytic functions $p(x, y)$ and $q(x, y)$ as in (14), only dimensions from the set D_0 can be obtained. This is a consequence of the following theorem, which in turn follows immediately from Theorem 1(a). It extends Theorems 5 and 6.

Theorem 7. (Analytic systems) *Let Γ be a spiral trajectory near the origin of system (1), where $p(x, y)$ and $q(x, y)$ are analytic functions as in (14). If the first nonzero Lyapunov coefficient is V_{2k+1}, then*

$$\dim_B \Gamma = 2\left(1 - \frac{1}{2k+1}\right).$$

Furthermore, Γ is Minkowski nondegenerate.

Remark 4. From [19, Theorem 10] we know that for analytic fields with normal form (21) each spiral trajectory Γ of limit cycle type has box dimension of the form $\dim_B \Gamma = 2 - \frac{1}{m}$, where m is algebraic multiplicity of the spiral, that is, from the set

$$D_1 = \{2 - \frac{1}{m} : m \in \mathbb{N}\} = \left\{1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \dots\right\}.$$

References

[1] M. Caubergh, F. Dumortier, Hopf-Takens bifurcations and centers, J. Differential Equations, 202 (2004), no. 1, 1–31.

[2] M. Caubergh, J.P. Françoise, Generalized Liénard equations, cyclicity and Hopf-Takens bifurcations, Qualitative Theory of Dynamical Systems Vol. 5, No. 2 (2004), 195–222.

[3] C.J. Christopher, N. G. Lloyd, Small-amplitude limit cycles in Liénard systems, Nonlinear Differential Equations Appl. 3, No 2, (1996), 183-190.

[4] F. Dumortier, J. Llibre, J.C. Artés, *Qualitative Theory of Planar Differential Systems*, Springer-Verlag Berlin Heidelberg, (2006).

[5] N. Elezović, V. Županović, D. Žubrinić, Box dimension of trajectories of some discrete dynamical systems, Chaos, Solitons & Fractals Vol. 34, 2 (2007), 244-252.

[6] K. Falconer, *Fractal Geometry*, Chichester: Wiley, 1990.

[7] J.-P. Françoise, Théorie des singularités et systémes dynamiques, Press Universitaires de France, Paris (1995).

[8] J.-P. Françoise, Successive derivatives of a first return map, application to the study of quadratic vector fields, Ergodic Theory Dyn. Syst. 16 (1996), 87–96.

[9] J.-P. Françoise, Analytic properties of the return mapping of Liénard equation, Mathematical Research Letters, 9, (2002), 255-266.

[10] J.-P. Françoise, C.C. Pugh, Keeping track of Limit cycles, Journal of Differential Equations 65, (1986), 139-157

[11] A. Gasull, J. Torregrosa, A new approach to the computation of the Lyapunov Constants, Computational and Applied Mathematics, Vol. 20, N. 1-2, (2001), 1-29.

[12] Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer Verlag, 1998.

[13] Lapidus M.L., Pomerance C., The Riemann zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums, Proc. London Math. Soc. (3) **66** (1993), no. 1, 41–69.

[14] A.M. Lyapunov, *The General Problem of the Stability of Motion*, Taylor and Francis, (1992).

[15] R. Roussarie, *Bifurcations of Planar Vector Fields and Hilbert's Sixteenth Problem*, Birkhäuser, 1998.

[16] F. Takens, Unfoldings of certain singularities of vector fields, Generalized Hopf bifurcations, J. Differential Equations, 14 (1973) 476–493.

[17] C. Tricot, *Curves and Fractal Dimension*, Springer–Verlag, (1995).

[18] D. Žubrinić, Analysis of Minkowski contents of fractal sets and applications, Real Anal. Exchange, Vol 31(2), 2005/2006, 315-354.

[19] D. Žubrinić, V. Županović, Fractal analysis of spiral trajectories of some planar vector fields, Bulletin des Sciences Mathématiques, 129/6 (2005), 457–485.

[20] D. Žubrinić, V. Županović, Fractal analysis of spiral trajectories of some vector fields in $\mathbb{R}^3$, C. R. Acad. Sci. Paris, Série I, Vol. 342, 12 (2006), 959-963.

[21] D. Žubrinić, V. Županović, Box dimension of spiral trajectories of some vector fields in $\mathbb{R}^3$, Qualitative Theory of Dynamical Systems, Vol. 6, No. 2 (2005), 203-222.

[22] V. Županović, D. Žubrinić, Recent results on fractal analysis of trajectories of some dynamical systems, in *Proceedings of Conference in Functional Analysis IX in Dubrovnik 2005*, Aarhus 48, G. Muić, J. Hoffmann-Jörgensen (eds.), Aarhus, Denmark, 2007, 126-140.

[23] V. Županović, D. Žubrinić, Fractal dimensions in dynamics, in *Encyclopedia of Mathematical Physics*, eds. J.-P. Françoise, G.L. Naber and Tsou S.T. Oxford: Elsevier, (2006), vol 2, 394–402.

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