Multidimensional formal Takens normal form*

Ewa Stróżyna Henryk Żołądek

Dedicated to Freddy Dumortier

Abstract

We present a multidimensional analogue of the classical Takens normal form for a nilpotent singularity of a vector field.

Recall the result of F. Takens.

Theorem 1. ([13]) *Given an analytic germ of planar vector field of the form* $V = x_2\partial_{x_1} + h.o.t.$ *there exists a formal change of the coordinates* x_1, x_2 *reducing it to the form*

$$V^{Takens} = (x_2 + a(x_1))\partial_{x_1} + b(x_1)\partial_{x_2}$$

where $a(x_1) = a_2x_1^2 + \ldots$ *and* $b(x_1) = b_2x_1^2 + \ldots$ *are formal power series.*

The Takens normal form is obtained by solving the homological equation

$$[x_2\partial_{x_1}, Z] = W$$

which is a linear approximation to the condition

$$\left(g_Z^1\right)^* V = V^{Takens},$$

where g_Z^t is the formal flow generated by a formal vector field Z and $V = V^{Takens} + W$. It means that the space $x_1^2\mathbb{C}[[x_1]]\partial_{x_1} + x_1^2\mathbb{C}_2[[x_1]]\partial_{x_2}$ is complementary to the space $\mathrm{ad}_{x_2\partial_{x_1}}\{\mathbb{C}[[x_1,x_2]]_{\geq 2}\partial_{x_1} + \mathbb{C}[[x_1,x_2]]_{\geq 2}\partial_{x_2}\}$, where $\mathbb{C}[[x_1]]_{\geq 2}$ is the space of series with second order zero at $x_1 = x_2 = 0$. This is the definition of the Takens normal form.

Remark 1. The Takens normal form is not complete. A. Baider and J. Sanders [2], A. Algaba, E. Freire and E. Gamaro [1] and H. Kokubu, H. Oka and D. Wang

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[8] showed that some terms in the power series $a(x_1)$ and $b(x_1)$ can be cancelled. In some cases a complete normal form was obtained, but many cases still remain unsolved.

Consider now germs of analytic vector fields in $(\mathbb{C}^n, 0)$ with nilpotent linear part at the singular point $x = 0$. Assume firstly that there is only one Jordan cell. Therefore we take

$$V = X + h.o.t. \tag{0.1}$$

where

$$X = (n-1)x_2\partial_{x_1} + (n-2)x_3\partial_{x_2} + \ldots + x_n\partial_{x_{n-1}}. \tag{0.2}$$

(The coefficients before $x_{i+1}\partial_{x_i}$ can be chosen arbitrarily). Define the following additional vector fields

$$\begin{array}{rcl} Y & = & x_1\partial_{x_2} + 2x_2\partial_{x_3} + \ldots + (n-1)x_{n-1}\partial_{x_n}, \\ H & = & -(n-1)x_1\partial_{x_1} - (n-3)x_2\partial_{x_2} + \ldots + (n-1)x_n\partial_{x_n}. \end{array} \tag{0.3}$$

Lemma 1. *The vector fields X, Y, H define an irreducible representation σ of the Lie algebra $sl\,(2,\mathbb{C})$ such that*

$$\sigma(A) = X, \ \ \sigma(B) = Y, \ \ \sigma(C) = H,$$

where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *and* $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *generate* $sl(2,\mathbb{C})$.

Proof. See the book of J.-P. Serre [11] and the papers [5], [6]. ■

The vector field Y, treated as a differentiation of the ring $\mathbb{C}\,[x] = \mathbb{C}[x_1, \ldots, x_n]$, is a so-called *locally nilpotent derivation* (see [7]). It means that for any polynomial $f\,(x) \in \mathbb{C}[x]$ we have

$$Y^N(f) \equiv 0$$

for some $N > 0$. (Of course, X is also a locally nilpotent derivation). With any locally nilpotent derivation one associates its ring of constants, i.e.

$$\mathbb{C}\,[x]^Y = \{g \in \mathbb{C}\,[x] : Yg = 0\}\,.$$

Lemma 2. *We have*

$$\mathbb{C}\,[x]^Y = \mathbb{C}\,[G_1, G_2, \ldots, G_{n-1}]\,[x_1^{-1}] \cap \mathbb{C}[x]$$

where $G_1 = C_1 = x_1$ and G_j are homogeneous polynomial of degree j defined inductively by

$$\begin{array}{rcl} G_j & = & C_j \cdot {x_1}^{j-1}, \\ C_j & = & x_{j+1} - \binom{j}{1} C_{j-1} \left(\dfrac{x_2}{x_1}\right)^1 - \ldots - \binom{j}{j-2} C_2 \left(\dfrac{x_2}{x_1}\right)^{j-2} \\ & & - \binom{j}{j} C_1 \left(\dfrac{x_2}{x_1}\right)^j. \end{array}$$

Proof. The system of equations defining the vector field Y is following

$$\dot{x}_1 = 0, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = 2x_2, \ldots$$

Since $x_1(t) \equiv C_1 = \text{const}$ and since we can shift the time t, we can assume that $x_2(t) = x_1 t$, or

$$t = x_2/x_1.$$

The other equations are solved in the form

$$x_{j+1}(t) = C_j + j\int_0^t x_j(s)ds.$$

From this the formulas from the lemma follow. Also the homogeneity of the polynomials G_j from this follows.

On the other hand, the space of solutions is parametrized by the constants of motion C_j. Each C_j, $j \geq 2$, depends linearly on x_{j+1}, with coefficient being a power of x_1; the same is true for G_j, $j \geq 2$. Since any polynomial first integral depends polynomially on $x_3, \ldots, x_n$, we can replace the latter variables by functions of $G_2, \ldots, G_{n-1}$ and of x_1 and x_2; moreover, the dependence on x_2 is polynomial. Thus our first integral becomes a polynomial in x_2 with coefficients depending on elementary first integrals $G_1, \ldots, G_{n-1}$.

As the latter polynomial represents a first integral of Y, it cannot contain positive powers of x_2. ∎

Remark 2. For $n = 2$ we get $\mathbb{C}[x]^Y = \mathbb{C}[x_1]$. It is easy to prove that for $n = 3$ we have $\mathbb{C}[x]^Y = \mathbb{C}[G_1, G_2]$.

But for $n = 4$ the ring of constants of the derivation Y is not equal the polynomial ring of our three polynomials. We have $G_2 = x_1x_3 - x_2^2$, $G_3 = x_1^2x_4 - 3x_1x_2x_3 + 2x_2^3$. However the following first integral $\widetilde{G}_4 = 3x_2^2x_3^2 - 4x_2^3x_4 + 6x_1x_2x_3x_4 - 4x_1x_3^3 - x_1^2x_4^2$ cannot be expressed as a polynomial in G_1, G_2, G_3. In fact, for $n = 4$ the ring $\mathbb{C}[x]^Y$ is a ring of regular functions on the algebraic hypersurface in $\mathbb{C}^4$ defined by $8x^2u - y^3 + 8z^2 = 0$ (see [10]). Also for greater dimensions the ring $\mathbb{C}[x]^Y$ is not equal $\mathbb{C}[\mathbb{C}^{n-1}]$.

By a theorem of Weitzenböck [14] the ring $\mathbb{C}[x]^Y$ is finitely generated, but its structure for general n is not known. There exist examples of locally nilpotent derivations such that their rings of constants are not finitely generated.

For more informations we refer the reader to the habilitation thesis of A. Nowicki [10] and to the book of Freudenburg [7].

Among the first integrals for the vector field Y we distinguish those which are also first integrals for the vector field X. It is easy to see that they are altogether first integrals for the vector field H.

From the examples in Remark 2 we find that $G_2 = x_1x_3 - x_2^2$ is also first integral for X when $n = 3$; it is invariant with respect to the change $(x_1, x_2, x_3) \to (x_3, x_2, x_1)$. Similarly, the integral $\widetilde{G}_4$ is a first integral for $sl(2, \mathbb{C})$ when $n = 4$.

The vector field H defines a quasi-homogeneous gradation $\deg_H$ in the ring $\mathbb{C}[x]$, such that

$$\deg_H x_j = 2j - n - 1.$$

It follows that the first integrals F for Y which are first integrals for $sl(2,\mathbb{C})$ can be characterized by the property

$$\deg_H F = 0,$$

i.e. that they contain only monomials of quasi-homogeneous degree 0.

Note that the first integrals G_j defined in Lemma 2 have $\deg_H G_j < 0$. Generally any first integral of Y contains only terms of $\deg_H \leq 0$. Denote by $\mathbb{C}\,[x]^{Y,0} = \ker Y \cap \ker X$, respectively by $\mathbb{C}\,[x]^{Y,<0} = \ker Y \ominus \ker X$, the ring of polynomial first integrals for $sl(2,\mathbb{C})$, respectively the ring of polynomial first integrals for Y which contain only terms of nonzero quasi-homogeneous degree $\deg_H$.

Remark 3. The three vector fields X, Y, H define a distribution $\mathcal{D} \subset T\mathbb{C}^n$, i.e. a (singular) subbundle such that the fiber $\mathcal{D}_x$ at a point x is spanned by the vectors $X(x), Y(x), H(x)$. If $n \geq 4$ then at a general point the dimension of the space $\mathcal{D}_x$ equals 3, but at some points this dimension falls down. If $n = 2, 3$ then typically $\dim \mathcal{D}_x = 2$.

Since the vector fields generate a Lie algebra, the distribution is integrable. By the Frobenius theorem there exists a foliation $\mathcal{F}$ with typical leaves L of dimension 3 (for $n \geq 4$) or of dimension 2 ($n = 3$). In fact, the leaves are orbits of the action of the Lie group $SL(2,\mathbb{C})$. Since the phase flows g_X^t and g_Y^t are polynomial (as X and Y are locally nilpotent derivations) and since $(g_H^t)^* x_j = e^{t\cdot \deg_H x_j} x_j$ arises from an algebraic action of $\mathbb{C}^*$, the leaves L are algebraic varietes. So there should exist algebraic first integrals for the foliation $\mathcal{F}$.

Existence of polynomial first integrals for $\mathcal{F}$ follows also from the Clebsch–Gordan formula.

We can now formulate the main result of this work. Denote by $\mathbb{C}[x]_k$ and $\mathbb{C}[[x]]_{\geq k}$ (respectively $\mathbb{C}[x]_k^Y$, $\mathbb{C}[[x]]_{\geq k}^Y$, $\mathbb{C}[x]_k^{Y,<0}$, $\mathbb{C}[[x]]_{\geq k}^{Y,<0}$) the subspaces of $\mathbb{C}[[x]]$ (respectively of $\mathbb{C}[[x]]^Y$, $\mathbb{C}[[x]]^{Y,<0}$) consisting of homogeneous polynomials of degree k and of series which have zero of order $\geq k$ at the origin.

Theorem 2. *Any germ of the form (0.1) can be reduced by means of a formal change of variables $x_1, ..., x_n$ to the following*

$$V^{Takens} = X + F_1(G)\,\partial_{x_1} + \ldots + F_n(G)\,\partial_{x_n}, \tag{0.4}$$

where $F_j(G) = F_j(G_1, \ldots, G_{n-1})$ are formal power series in $G_2, \ldots, G_{n-1}$ with coefficients being Laurent polynomials in $G_1 = x_1$ and such that $F_j \circ G(x) \in \mathbb{C}[[x]]_{\geq 2}$ and $F_j \in \mathbb{C}[[x]]^{Y,<0}$ for $j = 1, \ldots, n-1$. Moreover, the form (0.4) is unique in a sense that the space

$$\mathbb{C}[[x]]_{\geq 2}^{Y,<0} \cdot \partial_{x_1} + \ldots + \mathbb{C}[[x]]_{\geq 2}^{Y,<0} \cdot \partial_{x_{n-1}} + \mathbb{C}[[x]]_{\geq 2}^{Y} \cdot \partial_{x_n}$$

is complementary to the space

$$\mathrm{ad}_X \left\{\mathbb{C}[[x]]_{\geq 2} \cdot \partial_{x_1} + \ldots + \mathbb{C}[[x]]_{\geq 2} \cdot \partial_{x_n}\right\}.$$

Example 1. For $n = 3$ the Takens normal form is following

$$\dot{x}_1 = 2x_2 + x_1\Phi_1(x_1, G_2), \quad \dot{x}_2 = x_3 + x_1\Phi_2(x_1, G_2), \quad \dot{x}_3 = \Phi_3(x_1, G_2).$$

For $n = 4$ we have

$$F_j = \sum_{a,b,c,d \geq 0} f^{(j)}_{a,b,c,d} G_1^a G_2^b G_3^c \tilde{G}_4^d,$$

where $a + 2b + 3c + 4d \geq 2$, $a = 0, 1$ if $d > 0$, and $3a + 2b + 3c > 0$ for $j = 1, 2, 3$.

Proof of Theorem 2. Let $Z = Z_1(x)\,\partial_{x_1} + \ldots + Z_n(x)\,\partial_{x_n}$ be a homogenous vector field of degree k. We have

$$\begin{aligned} \mathrm{ad}_X Z &= X(Z_n)\,\partial_{x_n} \\ &\quad + (X(Z_{n-1}) - (n-1)Z_n)\,\partial_{x_{n-1}} \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + (X(Z_1) - Z_2)\,\partial_{x_1}. \end{aligned}$$

Theorem 2 follows from the following two lemmas. ■

Lemma 3. *In the space* $\mathbb{C}[x]_k$ *of homogeneous polynomials we have*

$$\begin{aligned} \ker Y \oplus \operatorname{Im} X &= \mathbb{C}[x]_k, \\ \ker X \ominus \ker Y &\subset \operatorname{Im} X, \end{aligned}$$

where $\ker X \ominus \ker Y = \mathbb{C}[x]_k^{X,>0}$ *denotes the space of first integrals for* X *which contain only terms with nonzero quasi-homogeneous degree* $\deg_H$.

Proof. The vector fields X, Y, H define a representation of the Lie algebra $sl(2, \mathbb{C})$ in the space $\mathbb{C}[x]_k$ of homogeneous polynomials. It is known that any finite dimensional representation is split into irreducible representations, so-called *highest weight representations* (see [11]). Therefore

$$\mathbb{C}[x]_k = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_m,$$

and any $\mathcal{H}_j$ has a basis $\{e_1, \ldots, e_d\}$ such that

$$Xe_1 = 0,\ Xe_2 = (d-1)e_1, \ldots, Xe_d = e_{d-1},$$

$$Ye_1 = e_2, \ldots, Ye_{d-1} = (d-1)e_d,\ Ye_d = 0,$$

$$H(e_j) = (2j - d - 1)e_j.$$

We see that $\operatorname{Im} X = \operatorname{span}(e_1, \ldots, e_{d-1})$, $\ker X = \operatorname{span}(e_1)$, $\ker Y = \operatorname{span}(e_d)$. Hence $\ker Y \oplus \operatorname{Im} X = \mathcal{H}_j$.

If $d > 1$ then we see that $\ker X \subset \operatorname{Im} X$. If $d = 1$ then $X = Y = H = 0$ and $\ker X \ominus \ker Y = 0 \subset \operatorname{Im} X$.

Now the equalities from Lemma 3 hold when restricted to any subspace $\mathcal{H}_j$. Therefore they hold also in $\mathbb{C}[x]_k$. ■

Lemma 4. *The space* $\ker Y \ominus \ker X \cdot \partial_{x_1} + \ldots + \ker Y \ominus \ker X \cdot \partial_{x_{n-1}} + \ker Y \cdot \partial_{x_n}$ *is complementary to the space* $\mathrm{ad}_X \mathcal{X}_k$ *in the space* $\mathcal{X}_k$ *of homogeneous vector fields of degree* k.

Proof. From Lemma 3 we see that the last component of the action of ad_X on Z equals $X(Z_n)$, i.e. lies in the image of X in $\mathbb{C}[x]_k$. So the n–th component of the normal form should be the kernel of $Y|_{\mathbb{C}[x]_k}$. Note that the Z_n is not unique, when killing a suitable part in ∂_{x_n}; we can add some $\tilde{Z}_n \in \ker X$ to Z_n.

The $(n-1)$–th component of the action ad_X equals $X(Z_{n-1}) - \lambda_{n-1} Z_n$. So all polynomials from $\mathrm{Im}\, X$ can be killed.

We can hope to make an additional cancellation using $\tilde{Z}_n$ from $\ker X$. Lemma 3 says that we can write $\tilde{Z}_n = \tilde{Z}_n^{<0} + \tilde{Z}_n^0$, where

— $\tilde{Z}_n^{<0}$ lies in $\mathrm{Im}\, X$ (and we gain nothing);

— $\tilde{Z}_n^0$ belongs to $\ker Y \cap \ker X$ (here we cancel terms from $\mathbb{C}[x]_k^{Y,0}$).

So, the $(n-1)$–th component in the normal form is in $\ker Y \ominus \ker X$.

Analogously we consider successively other components. ∎

Remark 4. We can generalize Theorem 2 to the case when X, the linear part of V, has several nilpotent Jordan cells. For example, when X is given by the matrix

$$\begin{pmatrix} \begin{bmatrix} 0 & n-1 & \ldots & 0 \\ & 0 & \ldots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & m-1 & \ldots & 0 \\ & 0 & \ldots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \end{pmatrix}$$

Then X and the vector field Y, which is given by the matrix

$$\begin{pmatrix} \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ldots & 0 & \\ & & n-1 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ldots & 0 & \\ & & m-1 & 0 \end{bmatrix} \end{pmatrix},$$

define a representation of the Lie algebra $sl(2, \mathbb{C})$. The normal form is

$$V^{Takens} = X + \sum_{j=1}^{m+n} F_j(G)\, \partial_{x_j}$$

where $F_j(G_1, \ldots, G_{n-1}, G'_1, \ldots, G'_{m-1})$ are formal series of polynomials $G_2, \ldots, G_{n-1}$, $G'_2, \ldots, G'_{m-1}$ with coefficients being Laurent polynomials in $G_1 = x_1$ and $G'_1 = x_{n+1}$. The polynomials $G'_1, G'_2, \ldots, G'_{m-1}$ generate the field of constants of the part of Y associated with the variables $x_{n+1}, \ldots, x_{n+m}$. The polynomials F_j, $\neq n, n+m$, do not contain terms with zero quasi-homogeneous degree.

Remark 5. Another question is whether the Takens form is analytic (provided that the initial vector field is analytic near the origin). In the two-dimensional case the analyticity was proved in [12] and [9]. Some partial results in this direction were obtained also by V. Basov [3, 4].

We began to study this problem for $n \geq 3$, but it looks very difficult. We think that when $n \geq 3$ the above normal form is not analytic in general. We plan to continue investigations.

Remark 6. R. Cushman and J. Sanders [5, 6] also studied the normal form for the nilpotent singularities and also used the representation theory of the Lie algebra $sl(2, \mathbb{C})$. However their normal form is more complicated than ours. In fact, they applied the representation of this Lie algebra directly in the space $\mathcal{X}_k$ of homogeneous vector fields using the operator ad_X, ad_Y and ad_H, while we are working in the space $\mathbb{C}[x]_k$ of homogeneous polynomials. Moreover, they seem not to explore the property $\ker X \ominus \ker Y \subset \mathrm{Im}\, X$ from Lemma 3.

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Faculty of Mathematics and Information Science,
Warsaw University of Technology,
pl. Politechniki 1, 00-661 Warsaw, Poland
email:strozyna@mini.pw.edu.pl

Institute of Mathematics, University of Warsaw,
ul. Banacha 2, 02-097 Warsaw, Poland
email:zoladek@mimuw.edu.pl