Asymptotic study of planar canard solutions

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Abstract

We are interested in the asymptotic study of canard solutions in real singularly perturbed first order ODE of the form $\varepsilon u' = \Psi(x, u, a, \varepsilon)$, where $\varepsilon > 0$ is a small parameter, and $a \in \mathbb{R}$ is a real control parameter. An operator Ξ_η was defined to prove the existence of canard solutions. This demonstration allows us to conjecture the existence of a generalized asymptotic expansion in fractional powers of ε for those solutions. In this note, we propose an algorithm that computes such an asymptotic expansions for the canard solution. Furthermore, those asymptotic expansions remain uniformly valid.

Introduction

In what follows, we consider equations

$$\varepsilon y' = \Psi(x, y, a, \varepsilon)$$

where Ψ is $\mathcal{C}^\infty$, $x \in I \subset \mathbb{R}$, a is a real control parameter, y is a real function of the variables x and ε, and $\varepsilon \in]0, \varepsilon_0[$ is a small real parameter which is tending to 0.
It is proved [8] (or see [9][7]) that, under assumptions, canard solutions exists in the general equation

$$\eta^{p+1}\dot{u} = (p+1)t^p u + \alpha t^L + S(t, \alpha) + \eta^{p+1} P(t, u, \alpha, \eta) \tag{1}$$

with $t \in [-t_0, t_0]$, $\alpha \in \mathbb{R}$, u is a real function of the variables x and η, $\eta = \varepsilon^{1/(p+1)} \in]0, \eta_0[$, p is odd, $L < p$ is even, and where the functions S and P are $\mathcal{C}^\infty$ in their variables. Furthermore, the function S is such that $S(t, 0) = 0$, and each of its monomial term has a valuation, with pounds 1 for t and $p - L + 1$ for α, strictly

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bigger than $p+1$.

As, for each fixed couple (β, v), the system

$$\begin{cases} \eta^{p+1}\dot{u} = (p+1)t^p u + \alpha t^L + S(t,\alpha) + \eta^{p+1}P(t,v,\beta,\eta) \\ \\ u(-t_0,\eta) = 0 = u(t_0,\eta) \end{cases}$$

has a solution (α, u), we can define an operator $\Xi_\eta : (\beta, v) \mapsto (\alpha, u)$ which was proved [9][7] to be a contraction with Lipschitz constant equals to $\mathcal{O}_{\eta\to 0}(\eta)$.
Consequently, the fixed point theorem implies the existence of a canard solution for this system. By iteration, a sequence $((a^{(n)}, u^{(n)}))_n$ which converges to the expected canard solution (a^*, u^*) [6][2][13] can be defined. Commentaries on this result can be found in the references mentionned below.
This result was already proved in a more general form, by using different methods [10][4][3].
This result brings us to suppose that an asymptotic expansion in the powers of η for the canard solution exists.

In this note, an algorithm that computes an asymptotic expansion in the powers of η for the canard solutions is proposed. We recall that those kind of expansions are series $\sum_k u_k\eta^k$ such that

$$\forall K \in \mathbb{N}, \left\| u^*(.,\eta) - \sum_{k=0}^{K-1} u_k\eta^k \right\| = \mathcal{O}_{\eta\to 0}\left(\eta^K\right)$$

where the coefficients u_k have to be detailed. A formal substitution of this kind of expansion in the studied equation implies that the coefficients u_k cannot simply be functions which are $\mathcal{C}^\infty$ on the variable t. In order to introduce a dependency on η in the coefficient u_k, we fixed a family of functions φ, such that the coefficients u_k are $\mathcal{C}^\infty$ functions of the variables x and φ.

A first application is proposed in the case of an attractive slow curve ($p = 0$), to study the solutions with a boundary layer. We retrieve the so-called Combined Asymptotic Expansions [11][1] for those kind of solutions.
In the case $p = 1$, this method led to the existence of an asymptotic expansion in the powers of η with regular coefficients in t, which is a well-known result [5] too.
When $p > 1$, this method gives the existence of such expansions. Unfortunately question of uniqueness is not solved in the general case. It is due to the interactions between the functions of the family φ that are not sufficiently controlled. This last point will be the subject of a future paper to appear, but remains readable in our PhD work (in french).

1 The formal theorem

This section is dedicated to the demonstration of the formal equivalent of the theorem mentionned in the introduction.

In this section, we give a sequence $((\mathcal{A}_k, \pi_k))_k$ such that, for all $k \in \mathbb{N}$, $\mathcal{A}_k$ is a vector space, and π_k is a linear application defined from $\mathcal{A}_{k+1}$ to $\mathcal{A}_k$ which is surjective. Furthermore, we adopt the notations $\mathcal{A}_{-1} := \{0\}$ and $\pi_{-1} = 0$.

The study of the set

$$\hat{\mathcal{A}} := \{(a_k)_k : \ \forall k \in \mathbb{N}, \ a_k \in \mathcal{A}_k \text{ and } a_k = \pi_k(a_{k+1})\}$$

and of the functions $\hat{\pi}_k$ defined, for all k, by $\hat{\pi}_k((a_i)_{i\in\mathbb{N}}) := a_k$ gives the following theorem.

Theorem 1. $(\hat{\mathcal{A}}, (\hat{\pi}_k)_k)$ *is a projective limit of the system* $((\mathcal{A}_k, \pi_k))_k$

We recall that this property means that for every space E which has a given sequence of functions $(\sigma_k)_k$ which satisfies to

$$\forall k, \ \sigma_k : \ E \to \mathcal{A}_k \ , \text{ and } \pi_k \circ \sigma_{k+1} = \sigma_k$$

there exists a morphism $\chi : \ E \to \hat{\mathcal{A}}$ such that the following diagram is commutative:

$$\begin{tikzcd}
E \arrow[dd, "\chi"'] \arrow[drr, "\sigma_{k+1}"'] \arrow[drrr, "\sigma_k"] & & & & \\
& \cdots \arrow[r] & \mathcal{A}_{k+1} \arrow[r, "\pi_k"] & \mathcal{A}_k \arrow[r] & \cdots \\
\hat{\mathcal{A}} \arrow[urr, "\hat{\pi}_{k+1}"] \arrow[urrr, "\hat{\pi}_k"'] & & & &
\end{tikzcd}$$

We consider a sequence of functions $(\Xi_k)_k$ where, for all $k \in \mathbb{N}$, $\Xi_k : \ \mathcal{A}_k \to \mathcal{A}_k$.

Definition 1. *This sequence is said to be* ***compatible*** *if, for all* $k \in \mathbb{N}$*, the following diagram is commutative:*

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{A}_{k+1} \arrow[r, "\pi_k"] \arrow[d, "\Xi_{k+1}"] & \mathcal{A}_k \arrow[r] \arrow[d, "\Xi_k"] & \cdots \\
\cdots \arrow[r] & \mathcal{A}_{k+1} \arrow[r, "\pi_k"] & \mathcal{A}_k \arrow[r] & \cdots
\end{tikzcd}$$

Proposition 1. *If* $(\Xi_k)_k$ *is compatible, then there exists a function* $\hat{\Xi} : \ \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ *such that, for all* $k \in \mathbb{N}$*, the following diagram is commutative:*

$$\begin{tikzcd}
\hat{\mathcal{A}} \arrow[r, "\hat{\pi}_k"] \arrow[d, "\hat{\Xi}"] & \mathcal{A}_k \arrow[d, "\Xi_k"] \\
\hat{\mathcal{A}} \arrow[r, "\hat{\pi}_k"] & \mathcal{A}_k
\end{tikzcd}$$

The function $\hat{\Xi}$ *is called* ***formal operator****.*

Remark: $\hat{\Xi}$ might be defined by $\hat{\Xi}((a_k)_k) := (\Xi_k(a_k))_k$.

We consider, for all $(\hat{a}, \hat{b}) \in \hat{\mathcal{A}}^2$, the distance

$$d(\hat{a}, \hat{b}) := \inf \left\{ \frac{1}{2^k};\ \hat{\pi}_k(\hat{a}) = \hat{\pi}_k(\hat{b}) \right\}$$

As the set $\{k;\ \hat{\pi}_k(\hat{a}) = \hat{\pi}_k(\hat{b})\}$ is a real interval which contains -1, d is well defined.

The study of the metric space $(\hat{\mathcal{A}}, d)$ gives the following results:

Proposition 2. *If the sequence of functions $(\Xi_k)_k$ is compatible, then the associated formal operator $\hat{\Xi}$ is an 1-Lipschitz function (so is continuous).*
Reciprocally each given operator $\hat{\Xi}:\ \hat{\mathcal{A}} \to \hat{\mathcal{A}}$, which is a 1-Lipschitz function, is the formal operator of some compatible sequence of functions $(\Xi_k)_k$.

By definition of d, this result is a consequence of the equivalence between the continuity of $\hat{\Xi}$ and the property:

$$\forall k \in \mathbb{N},\ \forall (\hat{a}, \hat{b}) \in \hat{\mathcal{A}}^2 :\ \hat{\pi}_k(\hat{a}) = \hat{\pi}_k(\hat{b}) \Rightarrow \hat{\pi}_k \circ \hat{\Xi}(\hat{a}) = \hat{\pi}_k \circ \hat{\Xi}(\hat{b})$$

Furthermore, as we want to apply a fixed point theorem in this complete space, the following definition is needed:

Definition 2. *A sequence of functions $(\Xi_k)_k$ is said to be **formally contractant** if*

$$\forall k \in \mathbb{N},\ \forall (a_k, b_k) \in \mathcal{A}_k^2 :\ \pi_{k-1}(a_k) = \pi_{k-1}(b_k) \Rightarrow \Xi_k(a_k) = \Xi_k(b_k)$$

As a direct consequence of the previous proposition, we have:

Proposition 3. *If $(\Xi_k)_k$ is compatible and formally contractant, then its associated formal operator $\hat{\Xi}:\ \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ is $\frac{1}{2}$-contractant.*
Reciprocally each operator $\hat{\Xi}:\ \hat{\mathcal{A}} \to \hat{\mathcal{A}}$, which is $\frac{1}{2}$-contractant, is the formal operator of some sequence of functions $(\Xi_k)_k$, compatible and formally contractant.

Remark: By construction of the space $\mathcal{A}_{-1}$, the sentence "$(\Xi_k)_k$ is formally contractant" means two different things:

$(k = 0)$ The restriction of Ξ_0 at $\mathcal{A}_0$ is constant.

$(k \geq 1)$ In the following sections, its topological interpretation will be: "*The associated formal operator is a contraction, with Lipschitz constant equals to $\mathcal{O}_{\eta\to 0}(\eta)$*", where η is the relevant small parameter.

Finally, we have the immediate result:

Proposition 4. *The metric space $(\hat{\mathcal{A}}, d)$ is complete.*

As an immediate consequence of the previous propositions, we prove the fundamental theorem of this section.

Theorem 2. *For all sequence of functions $(\Xi_k)_k$, which is compatible and formally contractant, there exists an unique $\hat{a} \in \hat{\mathcal{A}}$ such that*

$$\hat{\Xi}(\hat{a}) = \hat{a}$$

2 Link between the formal theorem and the topological one

In the two following sections, we will denote by τ the small parameter. In practice, τ replaced ε or η, depending on the structure of the studied equation.

This section is a presentation of the general way for linking the formal theorem 2 to the topological theorem mentionned in the introduction.
The main result of this section is the heart of this correspondence.

Let $(\Xi_k)_k$ be a sequence of functions such that, for all integer k, Ξ_k is defined from $\mathcal{A}_k$ to itself, is given. We assume that this sequence is compatible.
The proposition 3 implies that this sequence has a projective limit $\hat{\Xi}$, defined from the complete space $\hat{\mathcal{A}}$ to itself, and that $\hat{\Xi}$ is a contraction if and only if $(\Xi_k)_k$ is formally contractant. Moreover, in this case, the theorem 2 implies that this operator has a fixed point $\hat{a} \in \hat{\mathcal{A}}$.
This fixed point is interpreted as an asymptotic expansion of a topological object.

To construct the formal theory, we need to formalize the property "$\in \circ_{\tau\to 0}(\tau^k)$". So, we consider a sequence of vector spaces $(N_k)_k$ such that

$$\forall k \in \mathbb{N},\ N_{k+1} \subset N_k$$

and, for all $k \in \mathbb{N}$, a function $\tilde{}$ defined from $\mathcal{A}_k$ to N_0, which depends of the integer k. This function is needed to assure a transition between the formal notation and its topological version.

In what follows, we assume that:

$$\underline{\text{Hypothesis (H1):}}\ \forall k \in \mathbb{N},\ \forall a_{k+1} \in \mathcal{A}_{k+1},\ \tilde{a}_{k+1} - \widetilde{\pi_k(a_{k+1})} \in N_k.$$

This hypothesis means that the topological difference between the truncation at order k of the principal term of order $k+1$ of a power series and the principal term of order k of the same series, is negligible in comparison of τ^k, as τ tends to 0.

We denote by $\widetilde{\mathcal{A}_k}$ the space obtained by applying the function $\tilde{}$ to $\mathcal{A}_k$. In particular, for all integer k, $\widetilde{\mathcal{A}_k} \supset N_k$.
We adopt the following terminology.

Definition 3. *A sequence $\hat{a} = (a_k)_k \in \hat{\mathcal{A}}$ is a **semi-asymptotic expansion** of $a \in N_0$ if*

$$\forall k \in \mathbb{N},\ a - \tilde{a}_k \in N_k$$

In this definition we adopt the word "semi" to point out that a problem of uniqueness may occurs. For this reason, we have to assume that:

<u>Hypothesis (H2):</u> For each fixed $k \in \mathbb{N}$, any $a_k \in \mathcal{A}_k$ such that $\tilde{a}_k \in N_k$ is equal to 0.

In practice, (H2) means that, for all $k \in \mathbb{N}$, the family which generates N_k is asymptotically free. Under this hypothesis, for all integer k, N_k is some kind of kernel of $\tilde{} : \mathcal{A}_k \to N_0$.

Proposition 5. *Assuming (H2), every semi-asymptotic expansion of $a \in N_0$ is unique. So, it is called* ***asymptotic expansion*** *of a.*

Before giving the fundamental theorem which is the link between the formal theorem and the topological one, we need a new definition:

Definition 4. *A sequence of functions $(\Xi_k)_k$ is* ***formally equivalent*** *to a topological operator $\Xi : D \subset N_0 \to N_0$ if and only if, for all $(a_k)_k \in \hat{\mathcal{A}}$ and for all integer k:*

- *$\widetilde{a}_k$ belongs to the set on which Ξ is defined.*
- $\Xi(\widetilde{a}_k) - \widetilde{\Xi_k(a_k)} \in N_k$

As a consequence of the propositions proved in the previous section, we have:

Theorem 3. *We consider a sequence $(\Xi_k)_k$, compatible and formally equivalent to a topological operator Ξ defined from N_0 to itself. We suppose that Ξ is N-* ***contractant****, in the sense that*

$$\forall (a,b) \in N_0^2,\ \forall k \in \mathbb{N},\ a - b \in N_{k-1} \Rightarrow \Xi(a) - \Xi(b) \in N_k$$

Then the sequence $(\Xi_k)_k$ is formally contractant.

Proposition 6. *Assuming the hypothesis (H2), every sequence of functions $(\Xi_k)_k$ which is formally equivalent to a topological operator $\hat{\Xi}$, defined from N_0 to itself, is unique.*

Finally, we demonstrate the fundamental theorem of this section. This theorem is the heart of the results that we will present in the section 4.

Theorem 4. *We assume (H1).*
Let $(\Xi_k)_k$ be a sequence of functions such that Ξ_k is defined from $\mathcal{A}_k$ to itself, which is compatible. If this sequence is formally equivalent to a topological operator Ξ, defined from N_0 to itself, which is N-contractant and which has a fixed point a, then the fixed point $\hat{a} = (a_k)_k$ of $\hat{\Xi}$ is a semi-asymptotic expansion of a.
Moreover, if we assume (H2), then $\hat{a}$ is an asymptotic expansion of a.

Demonstration:
By definition, we have to prove that

$$\forall k \in \mathbb{N},\ a - \widetilde{a}_k \in N_k$$

This demonstration consists in a recurrence over the integer $k \in \mathbb{N} \cup \{-1\}$.

The theorem 3 shows that the sequence $(\Xi_k)_k$ is formally contractant. It implies the existence of $\hat{a} = (a_k)_k$, as a direct consequence of the theorem 2.

The initialization property of the recurrence is trivially satisfied at $k = -1$.

So we assume that, for a fixed integer k,

$$a - \widetilde{a}_k \in N_k$$

As $\hat{a}$ (resp. a) is a fixed point of $\hat{\Xi}$ (resp. Ξ), we write

$$a - \widetilde{a_{k+1}} = \Xi(a) - \widetilde{\Xi_{k+1}(a_{k+1})}$$

That is leading us to the egality

$$a - \widetilde{a_{k+1}} = (\Xi(a) - \Xi(\widetilde{a_{k+1}})) + \left(\Xi(\widetilde{a_{k+1}}) - \widetilde{\Xi_{k+1}(a_{k+1})}\right)$$

The assumptions we took earlier implies that $a - \widetilde{a}_k \in N_k$, and the hypothesis (H1) brings to $\widetilde{a}_k - \widetilde{a}_{k+1} = \widetilde{\pi_k(a_{k+1})} - \widetilde{a}_{k+1} \in N_k$. Thus $a - \widetilde{a_{k+1}}$ belongs to the vector space N_k.
Finally, as the operator Ξ is N-contractant, we deduced that

$$\Xi(a) - \Xi(\widetilde{a_{k+1}}) \in N_{k+1}$$

Furthermore, as $(\Xi_k)_k$ is formally equivalent to Ξ, we have

$$\Xi(\widetilde{a_{k+1}}) - \widetilde{\Xi_{k+1}(a_{k+1})} \in N_{k+1}$$

Consequently,

$$a - \widetilde{a}_{k+1} \in N_{k+1}$$

□

3 Computation of the expansion

In this section, we present our choice for the spaces $\mathcal{A}_k$. They link the topological result with the formal theorem 2 for the algorithmic computation of those expansions.
This section begins by a presentation of the relevant spaces $\mathcal{A}_k$ that are used to apply the theorem 4. It is briefly followed by a presentation of an algorithm that computes those expansions.

3.1 Definition of the relevant spaces

To define the spaces $\mathcal{A}_k$ which are relevant in this study, we need a finite family of symbols φ. This family has to be chosen in a contable fixed set to define the function $\tilde{}$, which consists in the substitution of the symbols φ by functions of the variables t and τ which are associated. The choice of such functions is due to the considered equation.

Consequently, the vector spaces $\mathcal{A}_k$ are such that $\widetilde{\mathcal{A}}_k$ is included in the vector space of the functions of the variables t and τ. In practice, for all $k \in \mathbb{N}$, the space $\mathcal{A}_k$ is the vector space which generates the principal term of order k of the expected asymptotic expansion in the powers of τ.
Remind that, by definition, the formal set $\hat{\mathcal{A}}$ is defined as the projective limit of the sequence $(\mathcal{A}_k)_k$. In what follows, we use the alternative definition of $\hat{\mathcal{A}}$, which

consists in regarding its elements as series.

From this point of view, the definition of the spaces $\mathcal{A}_k$ needs the definition of an order ord(.), associated to the asymptotic approximation in the powers of τ to each of the "monomial terms".
Firstly, we choose a set of multi-indices I, such that $I \subset \mathbb{N} \times (\bigcup_n \mathbb{N}^n) \times \mathbb{N}$.

In the studied case, the decompositions carried out show that it is natural for the sets $\mathcal{A}_k$ to be the vector spaces which are generated by terms made up of powers of t, τ, and of the intermediary functions φ.

Terminology: In what follows, we call *monomial* every couple

$$(\tau^n, t^i\varphi^J\tau^l) \text{ such that } n \in \mathbb{N} \text{ , and } (i, J, l) \in I \text{ (where } |J| \text{ is finite)}$$

Definition 5. *We call* ***order****, every application ord :* $I \to \mathbb{N}$ *satisfying to*

- $\forall (i, l) \in \mathbb{N}^2,\ ord(i, 0, l) = l$
- $\forall ((i, J, l), (i', J', l')) \in I \times I$, *such that* $|J|$ *and* $|J'|$ *are finite,*

$$ord(i + i', J + J', l + l') \geq ord(i, J, l) + ord(i', J', l')$$

By convenience, and without ambiguity, we adopt the notation $\mathrm{ord}(t^i\varphi^J\tau^l) := \mathrm{ord}(i, J, l)$.

If the second assumption is natural, the first one is necessary because, whatever the studied framework is, the monomial term $t^i\tau^l$ has always an asymptotic order equals l (i.e. an asymptotic approximation of its associated topological quantity equals $\mathcal{O}_{\tau\to 0}(\tau^l)$).

The studied framework in the continuation being given, we stop working with general concepts. Thus we will assume working hypothesis related to the particularity of the objects on which we will apply the correspondence.
Firstly, we assume that the choice for φ is such that:

> Working hypothesis:
> For all integer k, the set $\{(i, J, l) \in I;\ |J| \geq 1 \text{ and } \mathrm{ord}(i, J, l) = k\}$ is finite.

So, for all integer k, we can define

$$\mathcal{B}_k := \{t^i\varphi^J\tau^l :\ |J| \geq 1,\ \mathrm{ord}(i, J, l) = k\} \text{ , and } \mathcal{C}_k := \mathbb{R}.\tau^k \times \left(\mathcal{C}^\infty([-t_0, t_0]).\tau^k \oplus \mathrm{Vect}\ \mathcal{B}_k\right)$$

Remark: *The goal of this working hypothesis is to give sense to the topological equivalent (i.e the image by $\tilde{}$) of elements of those sets (i.e. the sets obtained by the substitution of the symbols φ by a fixed family of functions).*

Finally, for all integer k, we define

$$\mathcal{A}_k := \mathcal{A}_{k-1} \oplus \mathcal{C}_k = \oplus_{l\leq k}\mathcal{C}_l \text{ (where we denote } \mathcal{A}_{-1} := \mathcal{C}_{-1} := \{(0, 0)\}\text{)}$$

By definition of $\tilde{}$, it is natural to define N_k as

$$N_k := \left\{(\alpha, u) \in \bigcup_l \mathcal{A}_l;\ \tilde{\alpha} = \mathcal{O}_{\tau\to 0}(\tau^{k+1}),\ ||\tilde{u}(., \tau)|| = \mathcal{O}_{\tau\to 0}(\tau^{k+1})\right\}$$

Proposition 7. *For the sets defined above, the assumption (H1) is satisfied. Moreover:*
Every topological operator Ξ_τ which is a contraction with Lipschitz constant equals to $\mathcal{O}_{\tau\to 0}(\tau)$, is N-contractant and has a fixed point.

Finally we denote, for all $k \in \mathbb{N}$, by π_k the natural projection from $\mathcal{A}_{k+1}$ to $\mathcal{A}_k$.

Remark:
The choice of intermediary functions φ which are such that the order ord is satisfying, for all monomials u and v, the inverse inequality

$$\mathrm{ord}(uv) \leq \mathrm{ord}(u) + \mathrm{ord}(v)$$

is not alleviating. It it is strongly related to the fact that the family $(\mathcal{A}_k)_k$ defined in the preceding part is asymptotically free.

3.2 Computation of the expected asymptotic expansions

This part is a presentation of the way the method described above is applied in practice.

For the particular spaces defined in the previous part, we have the definition:

Definition 6. *A couple (α, u), where $\alpha \in \mathbb{R}$ and $u : \ (t,\tau) \to u(t,\tau)$ is a $\mathcal{C}^\infty$ function, has the couple $(\sum_k \alpha_k, \sum_k u_k) \in \hat{\mathcal{A}}$ for* ***semi-asymptotic expansion*** *if, for all $n \in \mathbb{N}$, $(\alpha_n, u_n) \in \mathcal{C}_n$, and*

$$\forall k \in \mathbb{N}, \ \left|\alpha - \sum_{n=0}^{k} \widetilde{\alpha}_n\right| = \mathcal{O}_{\tau\to 0}(\tau^{k+1}), \quad \sup_{t\in[-t_0,t_0]} \left\{\left|u(t,\tau) - \sum_{n=0}^{k} \widetilde{u}_n(t, \widetilde{\varphi}(t,\tau), \tau)\right|\right\} = \mathcal{O}_{\tau\to 0}(\tau^{k+1})$$

We say that (α, u) has an ***asymptotic expansion*** *if it has an unique semi-asymptotic expansion.*

The proof of the existence of an asymptotic expansion for (α^*, u^*) needs the definition of a family φ of intermediary functions. This choice gives a sequence of spaces $(\mathcal{A}_k)_k$. So we define, from the topological operator Ξ_τ given in the introduction, a sequence of functions $(\Xi_k)_k$ which is compatible and formally equivalent to Ξ_τ.

As the uniqueness of such expansions is equivalent to the assumption (H2), it is a consequence of:

Proposition 8. *If the family $(\mathcal{A}_k)_k$ is* ***asymptotically free****, i.e. satisfied to*

$$\forall k \in \mathbb{N}, \ \forall a_k \in \mathcal{A}_k, \ \widetilde{a}_k = \mathcal{O}_{\tau\to 0}(\tau^{k+1}) \ \Rightarrow a_k = 0$$

then every semi-asymptotic expansion of $\widetilde{u}$ is an asymptotic expansion.

Remark: Proving the property "$(\mathcal{A}_k)_k$ is asymptotically free" is equivalent to prove that, in $\hat{\mathcal{A}}$, every semi-asymptotic expansion of 0 is the series 0.

All the results that have been presented gives a decomposition of the computation of the asymptotic expansion in two steps:

- *Proving that the choice of intermediary functions φ (and so the construction of the sets $\mathcal{A}_k$) allows us to define, from the contraction Ξ_τ, a sequence of functions $(\Xi_k)_k$, where Ξ_k is defined from $\mathcal{A}_k$ to itself, which is compatible and formally equivalent to Ξ_τ.*
  In practice, it will be a direct consequence of the construction of $(\Xi_k)_k$.
  Consequently, the theorem 4 implies the **existence** of a semi-asymptotic expansion for the canard solution.

- *Proving that the choice of intermediary functions φ is such that the family $(\mathcal{A}_k)_k$ is asymptotically free.*
  Consequently, the last proposition implies the **uniqueness** of the computed semi-asymptotic expansion obtained in the first step.
  In this case, we say that we have defined an *asymptotic scale* $\{t^i\varphi^J\tau^l\}$.

Remark:
In order to give a formal sense to the definition with integrals of Ξ_τ, the application of this correspondence consists in the substitution of the monomial terms of Ξ_τ by some formal objects which can be written as a finite linear combination of elements of $\bigcup_k \mathcal{A}_k$. This decomposition give the definition of the sequence $(\Xi_k)_k$.

In practice we substitute, in the integral form of Ξ_τ, the functions P and γ by their respective Taylor expansions that are truncated at order k. Then, all of the terms are replaced by their associated formal terms in $\mathcal{A}_k$.

In this construction, functions f satisfying to the property

$$<< \text{ When } \tau \to 0,\ \exists (c, C) \in \mathbb{R}_+^2,\ \ \forall t \in [-t_0, t_0],\ |\widetilde{f}(t,\tau)| < Ce^{-ct/\tau} >>$$

appeared. Such terms are called *exponentially decreasing*. As those functions have for asymptotic expansions in the powers of τ the series 0, they will systematically be put out of our study.

Finally, in order to avoid problems of boundary layer in the neighborhood of the initial condition, which do not concern the framework of our study, we apply this formal framework to values of $t \in [-t_1, t_1] \subset]-t_0, t_0[$ (because we are interested by the study of an asymptotic expansion of the solution in an appreciable neighborhood of 0).

In the last section, we present the application of this correpondence to two well-known cases, and then we conclude this note by giving a few words about the application in the degenerate case.

- Firstly, using the trivial asymptotic scale (which is the one which has no intermediary functions), we prove that such structures have sense. It is proved that, in the non-degenerate case ($p = 1$), this correspondence implies the existence and the uniqueness of an asymptotic expansion in the powers of η, with regular coefficients in t, for the canard solutions.

- In the case $p = 0$, which is not consisting in a study of canard solutions, we study the solutions which have a limit layer.

More particulary, we retrieve the existence and the uniqueness of a combined asymptotic expansion of solutions living near an attractive curve, after a limit layer.
This last study can be generalized, in the general case, to compute an asymptotic expansion of a canard solution which has a limit layer (i.e. which is not satisfying at $u(\pm t_0, \eta) = u_0(\pm t_0, \eta)$).

4 Main results

In this last section we present two applications of the correspondence given in the previous sections. The last part consists in a rapid survey of the application in the degenerate case, whose study will be presented in a future paper.

4.1 Application to the non-degenerate case

In this part, we assume that $p = 1$. Thus, the hypothesis assumed on the general equations bring us to consider the equations

$$\eta^2 \dot{u} = 2tu + \alpha(1 + \gamma(t, \alpha)) + \eta^2 P(t, u, \alpha, \eta)$$

where $t \in [-t_1, t_1]$, u is a real function of the variables t and η, $\alpha \in \mathbb{R}$, $\eta \in]0, \eta_0[$, γ is a $\mathcal{C}^\infty$ function in t and α such that $\gamma(0, 0) = 0$, and P is a $\mathcal{C}^\infty$ function.
In this case, the application of the correspondence does not need intermediary functions. Consequently, the uniqueness of the expansions that are algorithmically computed is trivial.

So, we are only concerned with the study of the existence of such expansions, which is resumed in the following result:

Theorem 5. *The canard solution (α^*, u^*) has an unique asymptotic expansion $\hat{\alpha}^* \sim \sum_l a_l \eta^l$ and $\hat{u}^* \sim \sum_l u_l(t)\eta^l$, where the functions $\tilde{u}_l$ are $\mathcal{C}^\infty$ in t.*

This result is a well-known result in asymptotics (see [5], for example).

To conclude this part, we give a sketch of the demonstration for the existence of such an expansions (i.e. the existence of a sequence $(\Xi_k)_k$ which is compatible and formally equivalent to Ξ_η).

To define each function Ξ_k, we fixed an integer k and a couple $(\beta_k, v_k) \in \widetilde{\mathcal{A}}_k$.
By definition of $\widetilde{\mathcal{A}}_k$, it is associated to $(\hat{\beta}_k, \hat{v}_k) \in \mathcal{A}_k$. Finally, we note

$$(\alpha_k, u_k) := \Xi_\eta(\beta_k, v_k)$$

Construction of $\hat{\alpha}_k$:

The definition of Ξ_η shows that the parameter α_k is a solution of the equation

$$0 = \alpha_k \left(\int_{-t_0}^{t_0} e^{-(\xi/\eta)^2} d\xi + \int_{-t_0}^{t_0} \gamma(\xi, \alpha_k) e^{-(\xi/\eta)^2} d\xi \right) + \eta^2 \int_{-t_0}^{t_0} P(\xi, v_k(\xi, \eta), \beta_k, \eta) e^{-(\xi/\eta)^2} d\xi$$

We substitute to γ and P, which are $\mathcal{C}^\infty$ functions, their Taylor expansions truncated at order k, and to β_k (resp. v_k) the topological version of $\hat{\beta}_k$ (resp. $\hat{v}_k$). Then, an asymptotic approximation of the integrals gives an equation of the form

$$E(\alpha_k, \eta) = 0$$

where E is a $\mathcal{C}^\infty$ function of its variables.
Moreover, as all of the monomial terms which generated E are asymptotically dominated by the term $\alpha_k \int_{-t_0}^{t_0} e^{-(\xi/\eta)^2} d\xi$, which is one of them, the implicit function theorem implies that α_k is a $\mathcal{C}^\infty$ function of η.
In conclusion, the expected term $\hat{\alpha}_k$ is defined as the formal truncature at order k of the Taylor expansion of α_k, as a function of η.

Construction of $\hat{u}_k$:

By definition of Ξ_η we write, for all t and η, the function u_k as

$$u_k(t,\eta) = \frac{1}{\eta^2} e^{(t/\eta)^2} \int_{-t_0}^{t} \Big(\alpha_k \left(1+\gamma(\xi,\alpha_k)\right) + \eta^2 P(\xi, v_k(\xi,\eta), \beta_k, \eta)\Big) e^{-(\xi/\eta)^2} d\xi$$

Considering the equation satisfied by α_k, we arrive at

$$\alpha_k = -\alpha_k \frac{\int_{-t_0}^{t_0} \gamma(\xi,\alpha_k) e^{-(\xi/\eta)^2} d\xi}{\int_{-t_0}^{t_0} e^{-(\xi/\eta)^2} d\xi} - \eta^2 \frac{\int_{-t_0}^{t_0} P(\xi, v_k(\xi,\eta), \beta_k, \eta) e^{-(\xi/\eta)^2}}{\int_{-t_0}^{t_0} e^{-(\xi/\eta)^2} d\xi}$$

which gives the formula

$$u_k(t,\eta) = \frac{1}{\eta^2} e^{(t/\eta)^2} \int_{-t_0}^{t} \alpha_k \left(\gamma(\xi,\alpha_k) - \frac{\int_{-t_0}^{t_0} \gamma(\xi,\alpha_k) e^{-(\xi/\eta)^2} d\xi}{\int_{-t_0}^{t_0} e^{-(s/\eta)^2} ds}\right) e^{-(\xi/\eta)^2} d\xi + \dots$$

$$+ e^{(t/\eta)^2} \int_{-t_0}^{t} \left(P(\xi, v_k(\xi,\eta), \beta_k, \eta) - \frac{\int_{-t_0}^{t_0} P(s, v_k(s,\eta), \beta_k, \eta) e^{-(s/\eta)^2} ds}{\int_{-t_0}^{t_0} e^{-(s/\eta)^2} ds}\right) e^{-(\xi/\eta)^2} d\xi$$

We substitute to the functions γ and P their respectives Taylor expansions truncated at order k, and to β_k (resp. α_k, v_k) the topological version of $\hat{\beta}_k$ (resp. $\hat{\alpha}_k$, $\hat{v}_k$). Then we reduce this study to a linear combination of functions like

$$(t,\eta) \mapsto e^{(t/\eta)^2} \int_{-t_0}^{t} \eta \left(w_n(\xi)\eta^n - \frac{\int_{-t_0}^{t_0} w_n(s)\eta^n e^{-(s/\eta)^2} ds}{\int_{-t_0}^{t_0} e^{-(s/\eta)^2} ds}\right) e^{-(\xi/\eta)^2} d\xi$$

where $w_n \in \mathcal{C}^\infty([-t_1, t_1])$.
Such functions can be properly studied as solutions of

$$\eta^2 \dot{u} = 2tu + \eta^2 \lambda_w + \eta^2 w \text{ , where } \lim_{t\to\pm\infty} u(t,\eta)$$

It can be proved that those solutions, denoted by $\mathcal{I}_\eta(w)$, have a formal equivalent in $\hat{\mathcal{A}}$. This gives the definition of $\hat{u}_k$.

Consequently, we are able to define an operator Ξ_k, leaving from $\mathcal{A}_k$ to itself, by

$$\Xi_k(\hat{\beta}_k, \hat{v}_k) := (\hat{\alpha}_k, \hat{u}_k)$$

And, by construction, the sequence $(\Xi_k)_k$ is compatible and is formally equivalent to Ξ_η.

4.2 Application to the combined asymptotic expansion

In this part, we assume that $p = 0$. The hypothesis assumed on the general equations leads us to consider the equations

$$\varepsilon \dot{u} = -u + \varepsilon P(t, u, \varepsilon)$$

where $t \in [0, t_0]$, u is a real function of the variables t and ε, $\varepsilon \in]0, \varepsilon_0[$, and P is a $\mathcal{C}^\infty$ function.
As $p = 0$, it is not a study of canard solution. We assume that $u(0, \varepsilon) := u_0$ is far away 0 and that we are studying the asymptotic behavior of this function, which has a boundary layer at $t = 0$, before living in the neighborhood of the attractive slow curve $u = 0$. So, we restrict our study to $t \in [0, t_0]$.

In this case, the relevant intermediary function is $\varphi(t, \varepsilon) = e^{-t/\varepsilon}$. So the order ord(.) is well-defined by:

$$\mathrm{ord}(t^i \varphi^j \varepsilon^l) := \begin{cases} i + l \text{ if } j > 0 \\ \\ l \text{ if } j = 0 \end{cases}$$

Furthermore, the relevant spaces for our study are

$$\mathcal{C}_k = \left\{ f(t)\varepsilon^k + h\left(\frac{t}{\varepsilon}, \varphi\right)\varphi\varepsilon^k \ : \ f \in \mathcal{C}^\infty([0, t_0]),\ h \text{ is polynomial in } T \text{ and } \mathcal{C}^\infty \text{ in } \varphi \right\}$$

$$\mathcal{A}_k := \mathcal{A}_{k-1} \oplus \mathcal{C}_k = \oplus_{l \le k} \mathcal{C}_l \text{ (where } \mathcal{A}_{-1} := \mathcal{C}_{-1} := \{(0,0)\} \text{)}$$

For this choice of spaces, the demonstration of the existence of a semi-asymptotic expansion for this solution is similar to the one presented in the previous part, so we won't be interested by it.

On the other hand, it remains to prove that the family $(\mathcal{A}_k)_k$ is asymptotically free, which will imply the theorem:

Theorem 6. *Each solution has an unique asymptotic expansion of the form*

$$\sum_k \left(f_k(t) + h_k\left(\frac{t}{\varepsilon}, e^{-t/\varepsilon}\right) e^{-t/\varepsilon} \right) \varepsilon^k$$

where, for all $k \in \mathbb{N}$, f_k (resp. h_k) is $\mathcal{C}^\infty$ in t (resp. polynomial in T and $\mathcal{C}^\infty$ in φ).

This expansion is a particular form of the combined asymptotic expansion [11][1], which are expansions that can be written

$$\sum_n \left(f_n(t) + g_n\left(\frac{t}{\varepsilon}\right) \right) \varepsilon^n$$

where, for all $k \in \mathbb{N}$, f_k (resp. h_k) is $\mathcal{C}^\infty$ in t (resp. $\mathcal{C}^\infty$ and exponentially decreasing in $T \in \mathbb{R}$).

In conclusion, it remains to prove the following result:

Proposition 9. *The family $(\mathcal{A}_k)_k$ is asymptotically free, which means that:*

For each fixed $K \in \mathbb{N}$, any $u_K \in \mathcal{A}_K$ such that $||\widetilde{u}_K(., \varepsilon)|| = \mathcal{O}_{\varepsilon \to 0}(\varepsilon^{K+1})$ is equal to 0

Proof:
We have to prove that if, for all $K \in \mathbb{N}$,

$$\sup_{t\in[0,t_0]} \left\{ \left| \sum_{k=0}^{K} \left(f_k(t) + h_k\left(\frac{t}{\varepsilon}, \widetilde{\varphi}(t,\varepsilon)\right) \widetilde{\varphi}(t,\varepsilon) \right) \varepsilon^k \right| \right\} = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{K+1})$$

then all of the functions f_k and h_k are 0.
The demonstration of these result consists in a recurrence over the integer K.

So, we assume that this is satisfied for a fixed integer $K-1$. As we have

$$\left(f_K(t) + h_K\left(\frac{t}{\varepsilon}, \varphi\right) \varphi \right) \varepsilon^K \in \mathcal{C}_K$$

The definition of $\mathcal{C}_K$ implies that its topological version is equal to $\mathcal{O}_{\varepsilon\to 0}(\varepsilon^K)$.
So, the hypothesis implies that, for all $l \leq K-1$, the terms f_l and $h_{i,l}$ are 0, because the term $\sum_{k=0}^{K-1} \left(f_k(t) + h_k\left(\frac{t}{\varepsilon}, \varphi(t,\varepsilon)\right) \varphi(t,\varepsilon) \right) \varepsilon^k$ belongs to $\mathcal{A}_{K-1}$, and is such that

$$\sup_{t\in[0,t_0]} \left\{ \left| \sum_{k=0}^{K-1} \left(f_k(t) + h_k\left(\frac{t}{\varepsilon}, \widetilde{\varphi}(t,\varepsilon)\right) \widetilde{\varphi}(t,\varepsilon) \right) \varepsilon^k \right| \right\} = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{K})$$

Consequently, we have

$$\sup_{t\in[0,t_0]} \left\{ \left| f_K(t) + h_K\left(\frac{t}{\varepsilon}, \widetilde{\varphi}(t,\varepsilon)\right) \widetilde{\varphi}(t,\varepsilon) \right| \right\} = \mathcal{O}_{\varepsilon\to 0}(\varepsilon)$$

As the term φ is factorized in the second term, a study of the case $t \neq \mathcal{O}_{\varepsilon\to 0}(\varepsilon)$ implies that this term is exponentially small. It gives that $f_K(t) = \mathcal{O}_{\varepsilon\to 0}(\varepsilon)$. As the term $f_K(t)$ do not depends of ε, we conclude that it is 0. So, we have

$$\sup_{t\in[0,t_0]} \left\{ \left| h_K\left(\frac{t}{\varepsilon}, \widetilde{\varphi}(t,\varepsilon)\right) \widetilde{\varphi}(t,\varepsilon) \right| \right\} = \mathcal{O}_{\varepsilon\to 0}(\varepsilon)$$

The change of variable $t = \varepsilon\xi$ gives

$$\sup_{\xi\in\left[0,\frac{t_0}{\varepsilon}\right]} \left\{ \left| h_K(\xi, e^{-\xi}) e^{-\xi} \right| \right\} = \mathcal{O}_{\varepsilon\to 0}(\varepsilon)$$

Assuming that ξ is fixed and considering that ε is tending to 0, we extend this formula from $\left[0, \frac{t_0}{\varepsilon}\right]$ to $[0, +\infty[$.
Moreover, as all of the terms do not depend of ε, we conclude that

$$\forall \xi \in [0, +\infty[, h_K(\xi, e^{-\xi}) e^{-\xi} = 0$$

By the change of variable $\phi = e^{-\xi}$, a Taylor expansion of the function h_K with respect to its first variable brings

$$\forall \phi \in]0,1], \quad \sum_{i=0}^{\deg(h_K(.,\phi))} h_{i,K}(\phi).(-\ln \phi)^i \phi = 0$$

which gives that all of the functions $h_{i,K}$ are 0. As this linear comination is finite, it implies that the function h_K is 0.

So, we conclude that $(\mathcal{A}_k)_k$ is asymptotically free.

□

4.3 A remark about the application in the degenerate case

The application of the correspondence in the case $p \geq 3$ will be detailed in a future paper. Its demonstration, and many commentaries, can be found in our PhD work [8] (in french).

It is proved that our relevant choice for the family φ allows us to prove the existence and the uniqueness of the expected asymptotic expansion for the canard solution, when the considered equation is linear in u.

In the general case, the correspondence gives the existence of a semi-asymptotic expansion for the canard solutions, and the uniqueness of such expansions is conjectured but not proved. This problem is due to difficulties generated by the interactions between our intermediary functions.
Nevertheless, a theorem which states the existence of a general asymptotic expansion in the powers of η for the canard solution is proved. But this is at the price of the loss of a formal method which computes such expansions.

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