Rigidity in Dynamics

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Abstract

In this paper we will review several results of rigidity in one-dimensional dynamics and its relation to renormalization. In many one dimensional dynamical systems the existence of a topological conjugacy between two mappings implies that the restriction of the conjugacy to the attractor extends to a smooth mapping. Thus the combinatorics imposes severe restrictions on the geometry of the attractor.

1 introduction

The phenomenom of rigidity occurs in many situations when a weaker equivalence between certain classes of dynamical systems automatically implies a stronger equivalence. One of the most celebrate results in this direction is Mostow's rigidity theorem stating that if two compact hyperbolic manifolds are homeomorphic (or even if they have the same homotopy type) they are isometric. In one dimensional dynamics the first manifestation of this phenomena appears in the work of M. Herman, [12] where he proved that two smooth circle diffeomorphisms that are topologically conjugate and have a rotation number satisfying a Diophantine condition, the conjugacy is smooth. Also the discovery by Feigenbaum, [10] and Coullet-Tresser, [5] of some universal scaling laws in the transition to chaos in parametrized families of interval maps is related to a similar type of rigidity both in phase space and in the parameter space as we will describe later. Finally the same phenomenon was detected in the critical circle mappings which are in the boundary between circle diffeomorphisms and chaotic circle mappings, [24], [17].

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2 Rigidity

Let M be either the compact interval $[-1,1]$ or the unit circle $\mathbf{S}^1$. The dynamical systems we will discuss here are generated by a C^r map $f\colon M \to M$. The basic examples are given by the one-parameter family of quadratic maps and the two-parameter family of circle maps described below. Let us consider the following parametrized families of maps.

$$q_a\colon [-1,1] \to [-1,1], \qquad q_a(x) = -ax^2 + 1, \quad 0 < a \leq 2$$

$$A_{a,b}\colon \mathbf{S}^1 \to \mathbf{S}^1; \qquad A_{a,b}(x) = x + a + \frac{b}{2\pi}\sin(2\pi x) \pmod 1,$$
$$\text{where } 0 \leq a \leq 2 \quad \text{and } 0 \leq b$$

All the maps in the first family have a unique critical point. If $b < 1$, the mapping $A_{a,b}$ is a circle diffeomorphism; if $b = 1$, it is a critical circle mapping: a smooth homeomorphism with a unique critical point of cubic type and, lastly, for $b > 1$ it is not invertible and the dynamics becomes "chaotic".

An *attractor* for f is a subset $A \subset M$ with the following properties:

- A is closed and f-invariant: $f(A) = A$;
- A is topologically transitive: there exists an orbit in A which is dense in A;
- the basin of attraction of A, $B(A)$, has positive Lebesgue measure. Here $B(A)$ is the set of points in M whose ω-limit set is equal to A.

The simplest type of attractor is an attracting periodic orbit. If f is a circle diffeomorphism of class C^2 without periodic points then, by Denjoy's theorem, the attractor is the whole circle. This is also the case for critical circle maps by a theorem of Yoccoz, [30]. For unimodal maps, like the maps in the quadratic families, we have three types of attractors: periodic orbits, Cantor attractors= closure of the critical orbit, or a cycle of a finite number of intervals that are permuted by f. In the last case, the expanding periodic points are dense in the attractor.

For typical one-parameter families of unimodal maps, the set of parameter values corresponding to maps with a periodic attractor is open and dense by a theorem of Kozlovski, [16], and the set of parameters corresponding to maps whose attractor is a cycle of intervals has positive Lebesgue measure by the theorem of Jakobson, [13]. Finally the set of parameter values corresponding to Cantor attractors contains a Cantor set whose Hausdorff dimension is positive, see [11].

Two maps f and g are topologically conjugate if there exists a homeomorphism $h\colon M \to M$ such that $h \circ f = g \circ h$.

We say that the attractor of f is C^s-rigid if for any map g topologically conjugate to f, the restriction of the topological conjugacy to the attractor extends to a C^s, $1 \leq s$, diffeomorphism of M.

3 Renormalization

A unimodal interval map f is *renormalizable* if there exists an interval J_0 around the critical point such that the first return map to J_0 is an iterate f^k and is again a unimodal map. The family of intervals J_0, $J_1 = f(J_0)$, ..., $J_{k-1} = f(J_{k-2})$ have pairwise disjoint interiors. The smallest such k is called the renormalization period. By rescaling J_0 to the original size, we get the renormalization operator R from the set of renormalizable maps to the set of unimodal maps. The dynamics of this infinite dimension operator plays a fundamental role in the study of rigidity. In order to understand the small scale geometry of the critical orbit of a given map we are led to describe the limit set of the renormalization operator. For typical one parameter families of unimodal maps, as the quadratic family above, the set of parameter values corresponding to maps that are renormalizable with a given combinatorics is usually an interval (or a finite number of disjoint intervals) as in Figure 1. The maps we are interested in are the infinitely renormalizable maps; they belong to the domain of all iterates of the renormalization operator. For such map the closure of the critical orbit is a Cantor set whose geometry we want to understand.

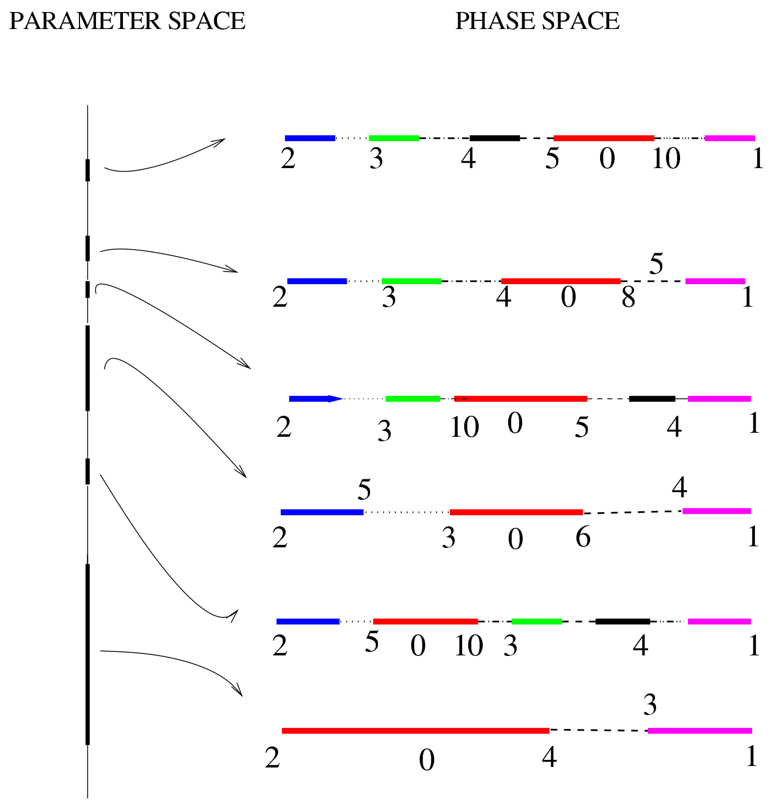

Figure 1: Renormalization

The rigidity of this Cantor set is related to the exponential contraction of the renormalization operator: if f and g are infinitely renormalizable maps with the same combinatorial type then the distance between $R^n(f)$ and $R^n(g)$ converges to zero exponentially fast. That the exponential contraction of the renormalization operator implies the $C^{1+\alpha}$ rigidity of the Cantor attractor was proved in [22] in the case of bounded combinatorial type, i.e., when the renormalization periods are uniformly bounded. To establish the exponential contraction was a much more difficult project that involved many tools from real and complex analysis that were developed by Sullivan,[25],[26], McMullen, [20],[21] and Lyubich, [18], [19]. See [11] for the final result concerning smooth maps with bounded combinatorial type. The first step is to establish the so called real a priori bounds that implies that maps in the limit set of the renormalization operator are real analytic with holomorphic extensions in the Epstein class. The second fundamental step is the so called complex a priori bounds: the high iterates of a map in the Epstein class has a holomorphic extension to a quadratic like map in the sense of Douady-Hubbard, see [7], with a fundamental annulus with conformal modulus bounded from below. The next step came from the rigidity of towers of McMullen that using the theory of geometric limits was able to prove that the quasi-conformal conjugacy between two quadratic like maps of the same bounded combinatorial type is in fact $C^{1+\alpha}$ at the critical point and this implies the exponential convergence. Finally the full hyperbolicity of the renormalization operator in the space of germs of quadratic like maps was obtained by Lyubich even in the case of unbounded combinatorial type.

The combinatorics of circle diffeomorphisms and of critical circle mappings is very simple to describe. For a mapping f without periodic points there is a unique combinatorial invariant: the rotation number $\rho(f)$. If the rotation number is irrational the mapping is semi-conjugate to an irrational rigid rotation and if the map is a C^2 diffeomorphism or a C^3 critical circle mapping it is topologically conjugate to a rotation, see [23]. We say that the rotation number is Diophantine of exponent β if there exists a constant $C > 0$ such that

$$|\rho(f) - \frac{p}{q}| > \frac{C}{q^{2+\beta}}$$

for all rational numbers $\frac{p}{q} \in \mathbf{Q}$. The fundamental rigidity result of M. Herman, [12], states that a smooth diffeomorphism whose rotation number is Diophantine is smoothly conjugate to a rotation. On the other hand, there are examples of circle diffeomorphisms with irrational rotation number such that the conjugacy with an irrational rotation is even not absolutely continuous. The proof of Herman's theorem involves very delicate estimates from real analysis but no complex analysis argument. Recently K. Khanin, found a very simple proof of this theorem that involves only some cross-ratio estimates. In fact he proves that if the diffeomorphism is C^k with $k > \beta + 2$ then the conjugacy to a rotation is $C^{k-1-\beta}$.

The rigidity results for critical circle mappings involve both real analytic estimates and complex dynamics and is related to the behavior of a renormalization operator. To a critical circle mapping f with irrational rotation number we associate a sequence of interval mappings $f_n \colon J_n \to J_n$ which is the first return mapping of f to the interval J_n. The critical point splits the interval J_n in two intervals I_n, I_{n+1} where the interval I_n returns to J_n after q_{n+1} iterates whereas I_{n+1} returns after q_n iterates. Here $\frac{p_n}{q_n}$ are the convergents of the rotation number of f, i.e., the best rational approximations to the rotation number. By rescaling so that the interval I_n becomes the interval $[0, 1]$ we get a sequence of mappings $R^n(f) \colon [-\lambda_n, 1] \to [-\lambda_n, 1]$ where λ_n is the ratio of the lengths of I_{n+1} and I_n. The critical circle mapping has bounded combinatorics if the ratio $\frac{q_{n+1}}{q_n}$ is uniformly bounded. The first step toward rigidity results for critical circle mappings is so called real a priori bounds. This uses tools developed by Yoccoz, Herman, Swiatek that involves the control of the distortion of cross-ratios under iteration, see [8]. The real a priori bounds imply that the sequence λ_n is uniformly bounded and that the sequence of renormalized maps lie in a compact set in the C^0 topology. Furthermore, any convergent subsequence is a commuting pair of real analytic maps that have holomorphic extension belonging to the Epstein class of holomorphic pairs. The next step is again the analog of complex a priori bounds. Finally, using McMullen's tools it was proved in [9] the exponential contraction of renormalization for real analytic maps with the same bounded combinatorial type and the $C^{1+\alpha}$ rigidity of these maps. To study maps with unbounded combinatorial type we first notice that from the real a priori bounds, the renormalized maps of very big renormalization period, i.e., $\frac{q_{n+1}}{q_n}$ big, are very close, in the C^2 topology, to maps that have a parabolic fixed point with bounded second derivative. The next step is to prove the a priori complex bounds for the unbounded case, see [28] and to extend the McMullen's rigidity of towers to include the parabolic towers. Those were the main new tools used in [29] to get the hyperbolicity or the full limit set of the renormalization operator for real analytic critical circle maps.

A different proof of the exponential contraction of the renormalization operator was given in [15] where they prove that the conjugacy between two analytic critical circle maps with the same rotation number is $C^{1+\alpha}$ at the critical point and this implies the exponential contraction of the renormalization operator.

By a careful analysis of the parabolic bifurcation [1] shows the existence of real analytic critical circle maps that are not $C^{1+\alpha}$ rigid for any $\alpha > 0$. However, using a rather precise estimate of the iterates near a saddle-node bifurcation [14] proves that the exponential convergence of the renormalization operator acting on two smooth critical circle maps with the same irrational rotation number implies the existence of a C^1 conjugacy. Hence, combining this with the previous result we get the C^1 rigidity for any real analytic critical circle map with irrational rotation number. This in sharp contrast with the circle diffeomorphism case where, in the case of Liouville rotation number, the conjugacy to a rigid rotation may fail to be even quasi-symmetric.

If the attractor of a unimodal interval map f is a cycle of intervals where the periodic points are dense we should not expect rigidity since, in this case, there exist infinitely many smooth conjugacy invariants: the multipliers of the periodic points. However a remarkable rigidity result was proven in [4] where they proved the existence of a rather big subset X of some Banach space of real analytic maps such that for any two maps in X that are topologically equivalent, the conjugacy, restricted to the attractor is real analytic. The set X is big in the sense that for typical parametrized families of maps, the set of parameters corresponding to maps in X has full Lebesgue measure. In particular X contains maps whose attractor is a cycle of intervals, see [2]. This implies that the holonomy of the dynamical lamination in [2], which is quasi-symmetric, is not absolutely continuous.

4 Open Problems

In this section I will formulate some open problems related to rigidity in dynamics.

- Prove the exponential contraction of renormalization for smooth (C^k, $k \geq 3$) critical circle maps.
- Find all smooth conjugacy invariants of critical circle maps with $n \geq 2$ critical points. Such a map is topologically conjugate to a rigid rotation and so has a unique invariant measure. The ratio of the measures of segments bounded by the critical points are clearly smooth conjugacy invariants.
- A very hard problem: extend the renormalization theory to cover real (non-integer) power law, i.e. for maps of the type $f(x) = \phi(|x|^\alpha)$ where α is a positive real number and ϕ is a smooth interval diffeomorphism. The difficulty here is that we can no longer use the strong tools from complex dynamics.
- Hyperbolicity of renormalization for smooth unimodal maps with unbounded combinatorial type.
- Renormalization of multimodal interval maps.

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