Large dimensional classical groups
and linear spaces

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Abstract

Suppose that a group G has socle L a simple large-rank classical group. Suppose furthermore that G acts transitively on the set of lines of a linear space $\mathcal{S}$. We prove that, provided L has dimension at least 25, then G acts transitively on the set of flags of $\mathcal{S}$ and hence the action is known. For particular families of classical groups our results hold for dimension smaller than 25.

The group theoretic methods used to prove the result (described in Section 3) are robust and general and are likely to have wider application in the study of almost simple groups acting on finite linear spaces.

1   Introduction

A *linear space* $\mathcal{S}$ is an incidence structure consisting of a set of points Π and a set of lines Λ in the power set of Π such that any two points are incident with exactly one line. The linear space is called *non-trivial* if every line contains at least three points and there are at least two lines. The linear space is called *finite* provided Π and Λ are finite. All linear spaces which we consider are finite; we write $v = |\Pi|$ and $b = |\Lambda|$.

This paper is part of a sequence attempting to classify those groups which can act line-transitively on a finite linear space. In [CP01, CP93] it was shown that

L	Conditions	N_1	N_2	N_3
$\Omega_n(q)$	n odd, q odd	7	13	21
$P\Omega_n^\epsilon(q)$	n even, q odd	18	26	26
$\Omega_n^\epsilon(q)$	n even, q even	16	26	26
$PSU_n(q)$	nq odd	11	15	15
$PSU_n(q)$	n even, q odd	16	22	22
$PSU_n(q)$	q even	11	13	13
$PSp_n(q)$	n even, q odd	12	22	22
$Sp_n(q)$	n even, q even	8	14	14
$PSL_n(q)$	nq odd	7	17	17
$PSL_n(q)$	n even, q odd	12	22	22
$PSL_n(q)$	q even	8	17	17

Table 1: Values for (N_1, N_2, N_3)

such a group takes one of three forms: Either it contains a normal subgroup which acts intransitively on the set of points, or it contains an elementary-abelian normal subgroup acting regularly on the set of points, or it is almost simple. This last case is equivalent to the group having *simple socle* (recall that the socle of a finite group G is the product of the minimal normal subgroups of G).

The results in [CP01, CP93] have inspired the study of almost simple groups acting line-transitively. In particular studies have been made when the socle is a sporadic group ([CS00]), an alternating group ([CNP03]) or a member of particular families of low rank groups of Lie type ([Gila, Liu03c, LLM01, Liu01, Liu03a, Liu03b, Liu03d, LZLF04, LLG06, ZLL00, Zho05, Zho02]). We continue this investigation by considering the situation when the socle of a line-transitive automorphism group G is L a simple large-rank classical group.

Our major result is that, in this situation, any line-transitive action is in fact flag-transitive. (Here a *flag* is an incident point-line pair.) The flag-transitive actions of almost simple groups were fully classified in [BDD+90] (with proofs in [BDD, Dav87, Del01, Del86, Kle90, Sax02, Spi97]) and are well-known. When the socle is a large-rank classical group there is one infinite family of flag-transitive actions:

Example 1. *We have $\mathcal{S} = PG(n-1, q)$, projective space of dimension $n - 1 \geq 2$ over a field of size q where q is a prime power. Any group G with $PSL_n(q) \leq G \leq P\Gamma L_n(q)$ acts flag-transitively on $\mathcal{S}$.*

In order to state our theorem we need to define a triple $(N_1, N_2, N_3) \in \mathbb{Z}^3$, $N_1 \leq N_2 \leq N_3$, dependent on the socle L. Values for this triple are given in Table 1.

Theorem A. *Let G be a group which acts transitively on the set of lines of a linear space $\mathcal{S}$. Suppose that G has socle L a simple classical group of dimension n. The following statements hold:*

- *If $n \geq N_1$ then L acts transitively on the set of lines of $\mathcal{S}$.*

- *If $n \geq N_2$ and L acts primitively on the set of points of $\mathcal{S}$ then we have Example 1.*

- *If $n \geq N_3$ then we have Example 1.*

Corollary 2. *Let G be a group which acts transitively on the set of lines of a linear space $\mathcal{S}$. Suppose that G has socle L a simple classical group of dimension $n \geq 25$. Then G acts transitively on the set of flags of $\mathcal{S}$ and we have Example 1.*

The rest of the paper will be occupied with proving Theorem A. In Section 2 we will outline a number of general background lemmas concerning groups acting line-transitively on finite linear spaces; in particular we record several lemmas which first appeared in [CNP03] and which will be crucial to our proof of Theorem A. In Section 3 we outline a method for applying these results to the case where G is almost-simple. Then, in the remaining sections, we apply the method of Section 3 to the different families of almost simple groups listed in Theorem A. Note that in our final section we are able to state a stronger result than Corollary 2 by referring to some cases not covered by Theorem A.

We write α for a point of $\mathcal{S}$, $\mathfrak{L}$ for a line of $\mathcal{S}$ and G_α, $G_\mathfrak{L}$ the respective stabilizers in G.

2   Some background lemmas

We list here some well-known lemmas which we will use later. Suppose throughout this section that G acts line-transitively on a linear space $\mathcal{S}$. Block [Blo67] proved that line-transitivity implies point-transitivity. Thus, if a linear space $\mathcal{S}$ is line-transitive then every line has the same number, k, of points and every point lies on the same number, r, of lines. Such a linear space is called *regular*; all linear spaces which we consider from here on will be assumed to be regular. The first lemma is proved easily by counting and holds for all regular finite linear spaces.

Lemma 3.   *1. $b = \frac{v(v-1)}{k(k-1)} \geq v$ (Fisher's inequality);*

2. $r = \frac{v-1}{k-1} \geq k$;

Lemma 4. *[Dav87, CS89] If $g \in G$ then g fixes at most $r + k - 3$ points.*

Lemma 5. *[CS89, Lemma 2] If $g \in G$ is an involution and g fixes no points then G acts flag-transitively on $\mathcal{S}$.*

If $\mathcal{S}$ is not a projective plane then by Fisher's inequality $b > v$ and, since $b = v(v-1)/(k(k-1))$, there must be some prime t that divides both $v-1$ and b. We shall refer to such a prime as a *significant* prime of $\mathcal{S}$.

Lemma 6. *[CNP03, Lemma 6.1] Suppose that $\mathcal{S}$ is not a projective plane and let t be a significant prime of $\mathcal{S}$. Let S be a Sylow t-subgroup of G_α. Then S is a Sylow t-subgroup of G and G_α contains the normalizer $N_G(S)$.*

Lemma 7. $N_G(G_\alpha) = G_\alpha$.

Proof. Let $S \in Syl_t G_\alpha$ and suppose that $N_G(G_\alpha) > G_\alpha$. Then $N_G(S) > N_{G_\alpha}(S)$ hence G_α does not contain the normalizer of a Sylow t-subgroup for any prime t. This contradicts Lemma 6. ∎

Finally we present a series of results which first appeared in [CNP03] and which will be central to our analysis of the almost simple case.

Lemma 8. *[CNP03, Lemma 2.2] Let $g \in G$ fix f points. Then $k < 2v/f$.*

Lemma 9. *[CNP03, Corollary 2.4] Let $g \in G_\alpha$. Assume that g has w conjugates in G and that a of them lie in G_α. Then*

$$k < \frac{2w}{a}.$$

Lemma 10. *[CNP03, Proposition 2.7] Let $h, g \in G$ with $g \in G_\alpha$ and $h \notin G_\alpha$. Define*

$$w = |G : C_G(g)|, \ c = |G : C_G(h)| \geq |G_\alpha : C_{G_\alpha}(h)|.$$

and write a for the number of conjugates of g in G_α. Then $r \leq kc$, $k < \frac{2w}{a}$ and

$$v = r(k-1) + 1 \leq ck(k-1) + 1 < \frac{2wc}{a}(\frac{2w}{a} - 1) + 1 \leq \frac{4w^2c}{a^2}.$$

Proof. Look at $K = G_\alpha \cap G_{h\alpha}$. This subgroup has to lie in the stabilizer $G_{\mathfrak{L}}$ of some line $\mathfrak{L}$. Then

$$|G : K| = |G : G_{\mathfrak{L}}| \cdot |G_{\mathfrak{L}} : K| = |G : G_\alpha| \cdot |G_\alpha : K|.$$

It follows that

$$|G_\alpha : K| = \frac{|G : G_{\mathfrak{L}}| \cdot |G_{\mathfrak{L}} : K|}{|G : G_\alpha|} = \frac{b|G_{\mathfrak{L}} : K|}{v} \text{ so } r \leq k|G_\alpha : K|.$$

We need to obtain an upper bound for the right hand side. Let $C = C_G(h)$ and $L = G_\alpha \cap C$. Then $[h, L] = 1$ so $hLh^{-1} = L \subset G_{h\alpha}$ so $L \subset G_\alpha \cap G_{h\alpha} = K$. Hence $|G_\alpha : K| \leq |G_\alpha : L|$. Set $c = |G : C|$ and observe that

$$|G_\alpha : L| = |G_\alpha : G_\alpha \cap C| \leq |\langle G_\alpha, C \rangle : C| \leq |G : C| = c.$$

This implies the first inequality in the lemma. The second follows from Lemma 9:

$$v = r(k-1) + 1 \leq |G_\alpha : K| \cdot ck(k-1) + 1 < \frac{2w}{a}\left(\frac{2w}{a} - 1\right) + 1.$$

∎

We can slightly strengthen this result in the following ways:

Corollary 11.   • *If h is an involution then $r \leq \frac{kc}{2}$ and*

$$v < \frac{wc}{a}(\frac{2w}{a} - 1) + 1 < \frac{2w^2c}{a^2}.$$

- *In general $|G : C_{G_\alpha}(g)| \leq 4w^2c$.*

Proof. To prove the first statement suppose that h is an involution. Then, in the notation of Lemma 10, $h \in G_{\mathfrak{L}} \backslash K$. This implies that

$$|G_\alpha : K| = \frac{|G : G_{\mathfrak{L}}| \cdot |G_{\mathfrak{L}} : K|}{|G : G_\alpha|} \geq \frac{2b}{v} \text{ so } r \leq \frac{1}{2}k|G_\alpha : K|.$$

To prove the second statement observe that, for $g \in G_\alpha$, it is clear that $a \geq |G_\alpha : C_{G_\alpha}(g)|$. This implies that

$$|G : G_\alpha| = v < \frac{4w^2c}{a^2} \leq \frac{4w^2c}{|G_\alpha : C_{G_\alpha}(g)|^2}$$

$$\implies |G : C_{G_\alpha}(g)| \leq \frac{4w^2c}{|G_\alpha : C_{G_\alpha}(g)|} \leq 4w^2c. \qquad \blacksquare$$

3 The almost simple case

Lemma 10 will be our primary tool in the analysis of an almost simple group G acting transitively on the set of lines of a linear space $\mathcal{S}$. We first outline our notation for the rest of the paper.

3.1 Notation

Let L be the socle of G. We consider L in the different families of classical groups; thus L is a simple group of dimension n defined over a field of size q where $q = p^a$ for some prime p. Let $\mathcal{C}_i, i = 1, \ldots, 8$ be the Aschbacher families of maximal subgroups of G as described in [KL90]. The collection of all maximal subgroups in these families will be written $\mathcal{C}(G)$. Let L^* be the linear group which covers L. For $H < L$ (resp. $g \in L$) let H^* (resp. g^*) be the pre-image of H (resp. g) in L^* under the natural homomorphism.

Take V to be the n-dimensional vector space over the field of size q on which L^* acts naturally. Let κ be the non-degenerate form on V which is preserved by L^*, i.e. $L^* \trianglelefteq I(V, \kappa)$, the set of isometries of κ in $GL(V)$. We take $S(V, \kappa)$ to be the set of isometries of κ in $SL(V)$ while $\Omega(V, \kappa)$ coincides with $S(V, \kappa)$ except in the orthogonal case when it is of index 2 in $S(V, \kappa)$. We will often write simply $I(V), S(V)$ or $\Omega(V)$ when the form of κ is clear.

We will reserve lower case Greek letters to represent the symbols $+$ and $-$; in particular $\Omega_n^\zeta(q)$ is used to represent one of the groups $\Omega_n^+(q)$ or $\Omega_n^-(q)$.

For a group G take $P(G)$ to be the degree of the minimum permutation representation of G. We represent cyclic groups of order c by the integer c, while soluble groups of order c will be represented by $[c]$. An extension of a group G by a group H will be written $G.H$; if the extension is split we write $G : H$. We will sometimes precede the structure of a subgroup H of a projective group with ˆ which means that we are giving the structure of the pre-image in the corresponding universal group (i.e. H^*). The notation $\frac{1}{2}G$ refers to a normal subgroup in G of index 2; $O_p(G)$ refers to the largest normal p-group in G. We write $|H|_p$ for the highest divisor of $|H|$ which is a power of a prime p; similarly $|H|_{p'} = \frac{|H|}{|H|_p}$.

3.2 Our Method

The remainder of the paper will be concerned with proving Theorem A. We operate from now on under the following hypothesis.

Hypothesis. *Let $L \trianglelefteq G \leq AutL$ where L is a finite simple classical group defined over a finite field with $q = p^a$ elements, for a prime p. Suppose that G acts line-transitively but not flag-transitively on a linear space $\mathcal{S}$ with parameters (b, v, r, k). Suppose that $L_\alpha \leq M < L$.*

Let n be the dimension of the classical geometry for L. For each family of finite simple groups which we consider we will define three integers $N_1 \leq N_2 \leq N_3$.

- *For Steps 1 and 2 we assume that $n \geq N_1$.*

- *From Step 3 (Irreducibles) on we assume that $n \geq N_2$ and that $G = L$.*

- *For Step 6 (Imprimitivity) we assume that $n \geq N_3$.*

We will prove in Steps 1 and 2 that, for $n \geq N_1$, L acts line-transitively on $\mathcal{S}$. Then we will prove that, for $n \geq N_2$ and G acting point-primitively, no actions exist; then in Step 6 (Imprimitivity) we prove that, for $n \geq N_3$, no actions exist.

Two particular situations can be excluded immediately.

First of all [Gilb, Theorem A] implies that $\mathcal{S}$ is not a projective plane. In particular we must have $b > v$ and so $\mathcal{S}$ has a significant prime.

Secondly [Gil06, Theorem 1] implies that the characteristic prime, p, of L is not significant.

We now describe the steps that we will use to prove Theorem A in each case. In the course of our description we prove a number of lemmas which can be seen to hold for a much weaker hypothesis; in particular the lower bounds on dimension are often not used.

- **Step 1 (Bound)**: Find small values for w and c (we use [KL90, GLS94]). Note that all involutions must fix a point so we can take g to be an involution (Lemma 5). Use Lemma 10 to get a rough bound for k and v.

- **Step 2 (Simplicity)**: We apply the principle of "exceptionality": Let B be a normal subgroup in a group G which acts upon a set Π. Then (G, B, Π) is called *exceptional* if the only common orbital of B and G in their action upon Π is the diagonal (see [GMS03]). We refer to the following lemma:

  Lemma 12. *[Gila, Lemma 26] Suppose a group G acts line-transitively on a linear space $\mathcal{S}$. Let B be normal in G such that $|G : B|$ is a prime. If B is not line-transitive on $\mathcal{S}$ then either $\mathcal{S}$ is a projective plane or (G, B, Π) is exceptional.*

  We know that $\mathcal{S}$ is not a projective plane. Suppose that L is not line-transitive on $\mathcal{S}$. Then there exist groups G_1, G_2 such that $L \trianglelefteq G_1 \triangleleft G_2 \leq G \leq AutL$ where $|G_2 : G_1|$ is a prime and G_1 is not line-transitive on $\mathcal{S}$ while G_2 is. By Lemma 12, (G_2, G_1, Π) is an exceptional triple.

Now let M_2 be a maximal subgroup of G_2 which contains a point-stabilizer in G_2. If (G_2, G_1, Π) is exceptional then $(G_2, G_1, G_2/M_2)$ is also exceptional and [GMS03, Theorem 1.5] implies that $M_2 \cap L \leq M$, a maximal subgroup of L from family $\mathcal{C}_5$. In fact in all cases we know that $|M| \leq |\hat{\ }G(q^{\frac{1}{5}})|$ where $G(q^{\frac{1}{5}})$ is the group of similarities of the same classical geometry over a field of size $q^{\frac{1}{5}}$. We will use this to yield a contradiction for $n \geq N_1$.

For the remaining steps we will assume that $G = L$, so G is simple. This will often enable us to refine our estimate of w and c to improve the strength of our upper bound on v.

- **Step 3 (Irreducibles)**: We prove that, for $n \geq N_2$, L_α must lie in a reducible subgroup. Our analysis primarily makes use of the theorems provided by Liebeck in [Lie85]. Liebeck lists the irreducible subgroups M with minimal index; it is enough for us to show that $|L : M|$ is greater than the upper bound for v.

  When $L = PSp_n(q)$ with q even the results of [Lie85] are not quite strong enough and we will also need to make use of the following lemma:

  Lemma 13. *[KL90, Theorem 5.2.4] Let L be a classical simple group with associated geometry of dimension n over a field of size q. Also let $L \trianglelefteq G \leq Aut(L)$, and let H be any subgroup of G not containing L. Then either H is contained in a member of $\mathcal{C}(G)$ or one of the following holds:*

  1. *H is A_m or S_m with $n + 1 \leq m \leq n + 2$;*
  2. *$|H| < q^{3n}$.*

- **Step 4 (Reducibles)**: We (usually) have two possibilities for maximal reducible subgroups - those which are parabolic and those which stabilize a non-degenerate subspace. We are able to rule out the first case in almost all situations as follows.

  Lemma 14. *Suppose that L is a finite simple group isomorphic to one of $PSp_n(q)$ ($n \geq 4$), $PSU_n(q)$ ($n > 2$), $\Omega_n(q)$ (n odd, $n \geq 7$), $P\Omega_{2s}^\epsilon(q)$ ($s \geq 4$). If L_α lies in a parabolic subgroup, P, of L then $L = P\Omega_{2s}^+(q)$ with s odd. Furthermore $P = P_s$, the stabilizer of an s-dimensional totally singular subspace.*

  Proof. Suppose that L_α does lie in a parabolic subgroup P of L. Let P_L be a Levi-complement of P. Now P^* fixes W, a totally singular (or totally isotropic) subspace of V. We can take a basis for W, $\{e_1, \dots, e_m\}$, such that there exists $f_1, \dots f_m$ in V with (e_i, f_i) orthogonal hyperbolic pairs. Furthermore we can choose $f_1, \dots f_m$ such that P_L stabilizes $\langle f_1, \dots f_m \rangle$.

  We first show that there exists an element $g \in L^*$ such that $\langle e_i \rangle g = \langle f_i \rangle$ and $\langle f_i \rangle g = \langle e_i \rangle$ for $i = 1, \dots, m$.

    - If (V, κ) is symplectic or unitary, then we can define g to be such that $e_i g = f_i$ and $f_i g = -e_i$ for $i = 1, \dots, m$, with the trivial action on the orthogonal complement of $\langle e_1, \dots e_m, f_1 \dots f_m \rangle$.

– Let (V, κ) be orthogonal and take X to be the orthogonal complement of all e_i, f_i. Let σ be the projection of $I(V, \kappa)$ into $I(V, \kappa)/L^*$ which is a group of exponent 2 and of order $2(2, q-1)$. Define $g_i \in I(V, \kappa)$ as follows: $e_i g_i = f_i$, $f_i g_i = e_i$ and $v g_i = v$ for all $v \in (e_i^\perp \cap f_i^\perp)$. Set $g = g_1 \cdots g_m$.

By Witt's theorem, all g_i are conjugate in $I(V, \kappa)$, hence $\sigma(g_i) = \sigma(g_j)$ for $1 \le i, j \le m$. If m is even then $\sigma(g) = \sigma(g_1) \cdots \sigma(g_m) = 1$ as $I(V, \kappa)/L^*$ is of exponent 2. So $g \in L^*$.

Suppose m is odd. If $\dim X > 1$ then there exists an element $y \in I(V, \kappa)$ fixing all e_i, f_i such that $\sigma(y) = \sigma(g_m)$. Then $y \in P^*$ and $gy \in L^*$. So the claim follows by replacing g with gy.

The case $\dim X = 0$ corresponds to the possibility given in the theorem, as $X = 0$ means that $I(V, \kappa) = O^+(V)$.

Finally let $\dim X = 1$; then $n = 2m+1$ and q is odd. Set $T = \langle e_m, f_m, X \rangle$. Let $\mu : I(T, \kappa|_T) \to I(V, \kappa)$ be the natural embedding. It is well known that $\mu(\Omega(T)) \subset \Omega(V)$.

Now $PSL(2, q) \cong \Omega(T)$ and this isomorphism is obtained from the adjoint action of $SL(2, q)$ on the space S of (2×2)-matrices over F with zero trace on which the symmetric bilinear form is defined as $\text{trace}(ab)$ for $a, b \in S$. Consider the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $e, f, t \in S$ and $heh^{-1} = f$, $hfh^{-1} = e$ and $hth^{-1} = -t$. Let g'_m be the image of h in $\Omega(T)$. Then we replace g by $g' = g_1 \cdots g_{m-1} \cdot \mu(g'_m)$. The images of e, f in T can be taken for e_m, f_m. It is easy to observe that then $g' \in P^*$ acts as g on e_i, f_i and $g' \in \Omega(V)$.

Thus we have our element g in all cases. Let h be the projection of g in L. Then h normalizes P_L but h does not lie in P_L. Let $S \in Syl_t P_L$ for some prime t. Then $N_L(S) > N_P(S)$ hence the only t for which P contains the normalizer of a Sylow t-subgroup is for t the characteristic prime. But this means that p is a significant prime for $\mathcal{S}$ and this possibility is excluded by [Gil06, Theorem 1]. ∎

For Step 4 (Reducibles) we deal with those subgroups not excluded by Lemma 14. We apply the rough bound to those maximal subgroups M in $\mathcal{C}_1$ to give bounds on the dimension of subspaces stabilized by M.

- **Step 5 (Primitivity)**: Although we have labelled this step "Primitivity", the situations we consider are in fact more general than the primitive case. Nonetheless, at the completion of Step 5 (Primitivity), Theorem A will be proved for the situation where G acts primitively on the set of points of $\mathcal{S}$. There are two situations which we need to consider separately:

    – $L \ne PSL_n(q)$: In this case the primary remaining case is when L_α preserves a non-degenerate subspace of V. We will typically take M to be the projective image of M^*, the stabilizer of a non-degenerate m-dimensional

subspace W of V. Then $M^* = (X_m(q) \times Y_{n-m}(q)).[s]$ where $[s]$ is a small soluble group,

$$(X, Y) \in \{(Sp, Sp), (SU, SU), (\Omega^\zeta, \Omega^\eta), (\Omega, \Omega^\eta), (\Omega^\eta, \Omega)\}$$

and $m \leq \frac{n}{2}$. We set $U = W^\perp$ and write $I(W)$ for the group of isometries of $(W, \kappa|_W)$, similarly for $I(U)$. Formally,

$$I(W) = \{g \oplus 1_U : g \in I(W, \kappa|_W)\}.$$

In this step we add one supposition to our hypothesis as follows: **Suppose that $L_\alpha^* \geq \Omega(U)$, the quasi-simple subgroup normal in $I(U)$.** We prove that, with this extra supposition, our hypothesis leads to a contradiction. Note that $\Omega(U) \cong Y_{n-m}(q)$. We will make use of the following lemma:

Lemma 15. *Suppose q is odd and (V, κ) is symplectic, orthogonal or unitary. Take $\Omega(U) \leq L_\alpha^* \leq M^*$. Then there exists an involution g in L_α^* which is not central in $\Omega(U)$. Furthermore we can choose g so that it acts as the identity on a 1 (resp. 2) dimensional subspace, X, of V for n odd (resp. n even) while taking all vectors in $X^\perp$ to their negative.*

Proof. Take n odd. There exists a non-singular vector, u, in U. Define g to be the involution in $I(V)$ which fixes u and which takes all vectors in $\langle u \rangle^\perp$ to their negative. In the unitary case all such g must lie in $\Omega(V)$. In the orthogonal case this is not necessarily true however if we choose u carefully we can ensure that g lies in $\Omega(V)$.

Clearly g normalizes $\Omega(U)$ and centralizes $I(W)$. Thus g normalizes L_α^* and hence, by Lemma 7, lies in L_α^*. It is clear that g is not centralized by $\Omega(U)$.

When n is even we do the same but instead of using a non-singular vector in U we use a hyperbolic pair or, in the orthogonal case, we may take U to be non-degenerate and anisotropic. ∎

This covers most of the *point-primitive* situations. There are some occasional other possibilities (in even characteristic, or thrown up by Lemma 14) which we also consider in this step.

- $L = PSL_n(q)$: In this case Lemma 14 does not apply. Hence we need to consider the situation where $L_\alpha \leq P_m$ a parabolic subgroup associated with an m-dimensional subspace of V.

  In this step we add one supposition to our hypothesis as follows: **Suppose that $H \leq L_\alpha \leq P_m$ where $H \cong \hat{\ }SL_{n-m}(q)$ and H is normal in a Levi-complement of P_m.** We prove that, with the addition of this supposition, our hypothesis implies that we have Example 1. This also excludes all point-primitive situations.

  Note that a similar line of argument to that given in Lemma 14 allows us to conclude that $m < \frac{n}{2}$ (i.e. $m \neq \frac{n}{2}$).

- **Step 6 (Imprimitivity)**: We are left with the *point-imprimitive* situation. In this case we have the following result of Delandtsheer and Doyen:

Proposition 16. *[DD89] Suppose that L acts transitively on the set of lines of a linear space $\mathcal{S}$ and acts imprimitively on the set of points of $\mathcal{S}$. Then, if L_α is the stabilizer of a point, we have that*

$$|L : M| < \binom{k}{2}, \ \ |M : L_\alpha| < \binom{k}{2}$$

where M is any group such that $L_\alpha < M < L$.

When using this result we will most often take M to be a maximal subgroup of L. Again we distinguish between when $L = PSL_n(q)$ and when $L \neq PSL_n(q)$. The former case is reasonably straightforward so, for now, we consider only the latter case.

We suppose that L_α stabilizes a non-degenerate subspace W of dimension $m \leq \frac{n}{2}$. As before write $U = W^\perp$, $\Omega(W) \cong X_m(q)$ etc. The following lemmas will be useful:

Lemma 17. *Suppose $L_\alpha < M = \hat{\ }(X_m(q) \times Y_{n-m}(q)).[s]$ and $L_\alpha^* \cap \Omega(U)$ lies inside a maximal parabolic subgroup, P^*, of $\Omega(U) \cong Y_{n-m}(q)$. Then $m = 1, Y_{n-m} = \Omega_{n-1}^+, P = P_{\frac{n-1}{2}}$ and $q(\frac{n-1}{2})$ is odd.*

Proof. Our proof is very similar to the proof of Lemma 14. [KL90, Lemma 4.1.1] implies that

$$L_\alpha^* \leq \Omega(W)(P^*.[x])$$

where $[x]$ is a subgroup of $[s]$ and $P^*.[x]$ is isomorphic to a parabolic subgroup of some automorphism group of $\Omega(U)$.

We know that, except in the case listed, there exists

$$h \in \Omega(U) \backslash P^*$$

which normalizes the Levi-complement, P_L^*, of P^*. In fact it is not hard to see that it must normalize $\Omega(W)(P_L^*.[x])$.

Now, as in Lemma 14, let $S \in Syl_t(P_L.[x])$. Then $N_L(S) > N_P(S)$ hence the only t for which P contains the normalizer of a Sylow t-subgroup is for t the characteristic prime. But this means that p is a significant prime for $\mathcal{S}$ and this possibility is excluded by [Gil06, Theorem 1]. ∎

Lemma 18. *Take $H < L^*$ and suppose that $H < M^* = (\Omega(W)\Omega(U)).[s]$ and $H \cap \Omega(U) \leq (\Omega(U_a)\Omega(U_b)).[t]$ where $U = U_a \perp U_b$ is a decomposition into non-degenerate subspaces. Then H is a subgroup of $M_1^* = (\Omega(W \perp U_a)\Omega(U_b)).[u]$ where M_1^* is the stabilizer of the decomposition $V = (W \perp U_a) \perp U_b$.*

Proof. Again, by [KL90, Lemma 4.1.1], if $g \in M^*$ then $g = g_1 g_2$ where $g_1 \in I(W), g_2 \in I(U)$. Furthermore, if $g \in H$ then $g_2 = g_{2a}g_{2b}$ where $g_{2a} \in I(U_a), g_{2b} \in I(U_b)$ and $U = U_a \perp U_b$ is a decomposition of W into non-degenerate subspaces of dimension r and $n - m - r$ respectively. Thus if $g \in H$ then $g = (g_1 g_{2a})g_{2b}$ and we have the required decomposition. ∎

Lemma 19. *Suppose L_α is a subgroup of maximal subgroups isomorphic to $\hat{\ }(X_{m_i}(q) \times Y_{n-m_i}(q)).[s]$ with $m_i \leq m^\dagger \leq \frac{n}{2}$ where $m^\dagger$ is a constant. Suppose $L_\alpha^* \cap \Omega(U_i)$ (where $\Omega(U_i) \cong X_{m_i}(q)$) is not irreducible in $\Omega(U)$ then one of the following situations holds:*

1. *There exists j such that $L_\alpha^* \geq \Omega(U_j)$ and L_α^* preserves a non-degenerate decomposition, $V = W_j \perp U_j$, where W_j has dimension at most $m^\dagger$;*

2. *(V, κ) is orthogonal, n is odd, $L_\alpha^* \leq \Omega_{n-1}^+(q).2$, $L_\alpha^* \cap \Omega_{n-1}^+(q)$ is a subgroup of a parabolic $P_{\frac{n-1}{2}}^*$ and $q^{\frac{n-1}{2}}$ is odd;*

3. *(V, κ) is orthogonal, n is even, $Y_{n-m} = \Omega_{n-m}^\epsilon$, $L_\alpha \cap \Omega(U) \leq Sp_{n-m-2}(q)$ and q is even;*

Proof. Suppose that the second and third possibilities do not occur. By Lemma 17, $L_\alpha^* \cap \Omega(U)$ does not lie in a parabolic subgroup of $\Omega(U)$. Thus $L_\alpha^* \cap \Omega(U)$ preserves a non-degenerate subspace of U and so, by Lemma 18, either $L_\alpha^* \geq \Omega(U)$ (and the first situation holds) or $L_\alpha^* \leq I(W_0)I(U_0)$ where $V = W_0 \perp U_0$ and W_0 has dimension $m + r$ for some $r > 0$.

We repeat our analysis using U_0 instead of U. Since $m + r \leq m^\dagger$ this process must eventually terminate with $L_\alpha^* \geq \Omega(U_j)$ as required. ∎

Note that the first possibility of Lemma 19 has already been analyzed in Step 5 (Primitivity). Thus, to complete our analysis of the situation where L_α preserves a non-degenerate subspace of W, we need only consider the irreducible subgroups of $\Omega(U)$ and the exceptions listed.

4 The First Steps

4.1 Steps 1 and 2

Steps 1 and 2 and of our method can be completed very easily. We refer to Table 2 and Table 3 which give information about our elements g and h for different socles L with $n \geq N_1$. We choose our elements g and h as follows:

- Suppose nq is odd. When $L = PSL_n(q)$, we choose g and h to be involutions such that g^* and h^* act as the identity on a 1-dimensional non-degenerate subspace X while taking all vectors in Y to their negative, where Y is some subspace such that $V = X \oplus Y$. When $L \neq PSL_n(q)$ we make the same choice but we require that X is non-degenerate and $Y = X^\perp$.

- When q is odd and n is even, we choose g to be as above but this time X is 2-dimensional. If L is not orthogonal then h is taken to be a transvection. If L is orthogonal then no transvections exist and h is taken to be of the same type as g.

- When q is even, we take g and h to be unipotent elements (transvections in the non-orthogonal case and root elements in the orthogonal case).

L	Conditions	$\lvert L : C_L(h)\rvert$
$PSU_n(q)$	n even, q odd	$\frac{(q^n-1)(q^{n-1}+1)}{q+1}$
$PSL_n(q)$	n even, q odd	$\frac{(q^n-1)(q^{n-1}-1)}{q-1}$

Table 2: Values for c when h is not conjugate to g, $n \geq N_1$

L	Conditions	$\lvert L : C_L(g)\rvert$	$\log_q(v_{as})$	$\log_q(v_s)$
$\Omega_n(q)$	n odd, q odd	$\frac{1}{2}q^{\frac{n-1}{2}}(q^{\frac{n-1}{2}}+\epsilon)$	3n-3	3n-3
$P\Omega_n^\epsilon(q)$	n even, q odd	$\frac{q^{n-2}(q^{\frac{n-2}{2}}+\zeta)(q^{\frac{n}{2}}-\epsilon)}{2(q-\eta)}$	6n-5	6n-11
$\Omega_n^\epsilon(q)$	n even, q even	$\frac{(q^{n-2}-1)(q^{\frac{n}{2}}-\epsilon)(q^{\frac{n-4}{2}}+\epsilon)}{q^2-1}$	6n-5	6n-14
$PSU_n(q)$	nq odd	$\frac{q^{n-1}(q^n+1)}{q+1}$	6n-5	6n-5
$PSU_n(q)$	n even, q odd	$\frac{q^{2n-4}(q^n-1)(q^{n-1}+1)}{(q^2-1)(q+1)}$	10n-5	10n-6
$PSU_n(q)$	q even	$\frac{(q^n-(-1)^n)(q^{n-1}-(-1)^{n-1})}{q+1}$	6n	6n-5
$PSp_n(q)$	n even, q odd	$\frac{q^{n-2}(q^n-1)}{q^2-1}$	5n-1	5n-4
$Sp_n(q)$	n even, q even	q^n-1	3n+1	3n+1
$PSL_n(q)$	nq odd	$\frac{q^{n-1}(q^n-1)}{q-1}$	6n-2	6n-4
$PSL_n(q)$	n even, q odd	$\frac{q^{2n-4}(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}$	10n-13	10n-14
$PSL_n(q)$	q even	$\frac{(q^n-1)(q^{n-1}-1)}{q-1}$	6n+1	6n-2

Table 3: Upper bounds for v, $n \geq N_1$

Note that, for later steps, we may change our choice of g and h. Now Table 3 gives values, when $n \geq N_1$, for:

- $v_{as} = 2w^2c$: This is an upper bound for v obtained using Lemma 10. This bound applies when G is almost simple.

- $v_s = 2w^2c$: Again this is an upper bound for v obtained using Lemma 10. However this bound applies when G is simple (thus, from Step 3 (Irreducibles) onwards.

In order to complete Steps 1 and 2 it is enough to observe that $\lvert L : \hat{\ }G(q^{\frac{1}{5}})\rvert > v_{as}$ for $n \geq N_1$. Recall that $G(q^{\frac{1}{5}})$ is the group of similarities, of the same classical geometry as L, over a field of size $q^{\frac{1}{5}}$. In certain borderline cases ($n = N_1$, say) it may be necessary to refine our value for v_{as} using precise values for w and c.

Thus, from now on, we take $G = L$, $n \geq N_2$ and observe that $v < v_s$ in all cases.

4.2 Steps 3 and 4

For Steps 3 and 4 we refer to Table 4. We define

- M_{Irr} to be the largest irreducible subgroup in L. Then $|L : M_{Irr}|$ is a lower bound for the index of a proper irreducible subgroup in L; values are obtained using [Lie85] and are valid for $n \geq \frac{N_3}{2}$. There is one exception to this (see * in the table): When $L = Sp_n(q)$, for q even, there is an irreducible subgroup $O_n^\epsilon(q)$ which is the largest proper irreducible subgroup of $Sp_n(q)$. The value given at * in Table 4 is a lower bound for the index of all other proper irreducible subgroups in L.

- M_{Red} is a reducible subgroup of L. When $L = PSL_n(q)$, $M_{Red} = P_m$, a parabolic subgroup preserving an m-dimensional subspace where $m \leq \frac{n}{2}$. When $L \neq PSL_n(q)$, M_{Red} is of type $X_m \perp Y_{n-m}$, and so preserves a decomposition into non-degenerate subspaces. Again this value is valid for $n \geq \frac{N_3}{2}$.

- $m^\dagger$ is the largest value of m such that $|L : M_{Red}| \leq v_s$. This calculation holds for $n \geq \max\{N_1, \frac{N_3}{2}\}$.

In almost all cases to complete Step 3 (Irreducibles) we simply observe that $|L : M_{Irr}| > v_s$ for $n \geq N_2$. Again, for small values of n ($n = N_2$, say) it may be necessary to refine our value for v_{as} and $|L : M_{Irr}|$. Then to complete Step 4 (Reducibles) we read off the value for $m^\dagger$ (again, with refinement if necessary). When $L = PSL_n(q)$, no value for $m^\dagger$ is given. Instead Step 4 (Reducibles) is subsumed into Steps 5 and 6 in this case.

There is one special situation in which extra work is needed in order to complete Step 3 (Irreducibles). This occurs when $L = Sp_n(q)$, with q even, and $L_\alpha \leq M < L$ for $M = O_n^\epsilon(q) \in \mathcal{C}_8$. Then M consists of the subgroup of elements which preserve a quadratic form κ over our vector space V such that κ polarises to a symplectic form f, for which L is a set of linear isometries.

We need to prove that, for $n \geq 14$ our hypothesis leads to a contradiction. Then there are two possibilities:

- $L_\alpha = M = O_n^\epsilon(q)$: Then $v = \frac{1}{2}q^{\frac{n}{2}}(q^{\frac{n}{2}} + \epsilon)$. When $q = 2$, L acts transitively on the conjugates of L_α ([CKS76]; hence linear-space action will be flag-transitive ([BDD88]). If we consult the list of flag-transitive actions ([BDD+90]) we find that no such actions exist and so we assume that $q > 2$.

  Now $O_n^\epsilon(q) > Sp_{n-2}(q)$ and so $a \geq q^{n-2} - 1$. Hence $k \leq \frac{w}{a} \leq 2(q^2 + 1)$ and $b > \frac{v^2}{k^2} > \frac{1}{32}q^{2n-4}$.

  Examining intersections of conjugates of $O_n^\epsilon(q)$ in $Sp_n(q)$ we find that there exist distinct conjugates which both contain $U_P : O_{n-2}^\epsilon(q)$ where $U_P = O_p(P_1)$. Here P_1 is a parabolic subgroup of $O_n^\epsilon(q)$ which preserves a 1-dimensional non-singular subspace of the n-dimensional orthogonal space. Then b divides

$$(v(v-1), |Sp_n(q) : (U_P : O_{n-2}(q))|)$$
$$= \frac{1}{2}q^{\frac{n}{2}}(q^{\frac{n}{2}} + \epsilon)(q^{\frac{n}{2}} - \epsilon)(\frac{1}{2}(q^{\frac{n}{2}} + 2\epsilon), q^{\frac{n-2}{2}} + \epsilon).$$

L	Conditions	$\log_q |L : M_{Irr}|$	$\log_q |L : M_{Red}|$	$m^\dagger$
$\Omega_n(q)$	n odd, q odd	$\frac{1}{4}n^2 - \frac{1}{4}n - 1$	$mn - m^2 - n + m - 2$	3
$P\Omega_n^\epsilon(q)$	n even, q odd	$\frac{1}{4}n^2 - \frac{1}{2}n - \frac{5}{2}$	$mn - m^2 - n - m - 3$	14
$\Omega_n^\epsilon(q)$	n even, q even	$\frac{1}{4}n^2 - \frac{1}{2}n - 4$	$mn - m^2 - n - m - 3$	14
$PSU_n(q)$	nq odd	$\frac{1}{2}n^2 + \frac{1}{2}n - 3$	$2mn - 2m^2 - n + m$	4
$PSU_n(q)$	n even, q odd	$\frac{1}{2}n^2 - \frac{1}{2}n - 3$	$2mn - 2m^2 - n + m$	7
$PSU_n(q)$	q even	$\frac{1}{3}n^2$	$2mn - 2m^2 - n + m$	5
$PSp_n(q)$	n even, q odd	$\frac{1}{4}n^2 - \frac{1}{4}n - 1$	$mn - m^2$	6
$Sp_n(q)$	n even, q even	$* \ \frac{1}{4}n^2 - \frac{1}{4}n - 1$	$mn - m^2$	6
$PSL_n(q)$	-	$\frac{1}{2}n^2 - n$	$mn - m^2$	-

<div align="center">Table 4: Index lower bounds for $n \geq \frac{N_3}{2}$</div>

For $q > 2$, $(\frac{1}{2}(q^{\frac{n}{2}} + 2\epsilon), q^{\frac{n-2}{2}} + \epsilon) < \frac{1}{2}q$ and $b < \frac{1}{4}q^{\frac{3n}{2}+1}$. But then $\frac{1}{32}q^{2n-4} < \frac{1}{4}q^{\frac{3n}{2}+1}$ implies that $n < 14$ as required.

- $L_\alpha < M$: If L_α lies in a reducible subgroup of M then L_α lies in a reducible subgroup of L and this is dealt with in Section 12. If L_α lies in an irreducible subgroup of M then we can apply bounds given by [Lie85, Theorems 5.4 and 5.5] to conclude that $|M : L_\alpha| > q^{\frac{1}{4}(n^2-2n-16)}$. Since $|L : M| > q^{n-2}$ we know that $|L : L_\alpha| > \frac{1}{4}(n^2 + 2n - 24)$. Since $v < q^{3n+1}$ we find that $n \leq 12$ as required.

We now proceed with Steps 5 and 6 for L in various families of simple classical groups.

5 $L = \Omega_n(q)$, nq **odd**

In this section we set $(N_1, N_2, N_3) = (7, 13, 21)$. We will prove that $n \geq 13$ leads to a contradiction for G point-primitive, while $n \geq 21$ leads to a contradiction in all cases.

5.1   Step 5 (Primitivity)

Here $M = (\Omega(U)\Omega(W)).[s]$. Since $m^\dagger = 3$ we have three cases here corresponding to $m = 1, 2$ or 3. Assume that $\Omega(U) \leq L_\alpha$. By Lemma 15 we know that $a \geq P(\Omega(U))$ (recall that a is the number of conjugates of g which lie in L_α).

If $m = 3$ then $M \cong (\Omega_{n-3}^\zeta(q) \times \Omega_3(q)).[4]$ and $|L : M| > q^{3n-9}$. Now $a \geq P(\Omega_{n-3}^\zeta(q)) > q^{n-5}$ ([KL90, Table 5.2.A]) and so $v < q^{n+7}$ which is a contradiction for $n \geq 13$.

If $m = 2$ then $M \cong (\Omega_{n-2}(q) \times \Omega_2^\zeta(q)).[4]$ and $|L : M| > q^{2n-5}$. Now $a \geq P(\Omega_{n-2}(q)) > q^{n-4}$ ([KL90, Table 5.2.A]) and so $v < \frac{q^{3n-3}}{a^2} \leq q^{n+5}$ which is a contradiction for $n \geq 13$.

If $m = 1$ then $M \cong \Omega_{n-1}^{\zeta}(q).2$. Then $a \geq P(\Omega_{n-1}^{\zeta}(q)) > q^{n-3} + q^{\frac{n-5}{2}}$ ([KL90, Table 5.2.A]); this implies that $k < \frac{2w}{a} < q^2 + 1$ and so $b > \frac{v^2}{k^2} > q^{2n-8}$.

Clearly there exists a distinct conjugate of L_α, L_β say, such that $L_\alpha \cap L_\beta \geq \Omega_{n-2}(q)$. Thus $\Omega_{n-2}(q)$ fixes the line between α and β. Hence b divides $|\Omega_n(q) : \Omega_{n-2}(q)| = q^{n-2}(q^{n-1} - 1)$.

Since $b = \frac{v(v-1)}{k(k-1)}$ divides $q^{n-2}(q^{n-1} - 1)$ and $|v|_p = q^{\frac{n-1}{2}}$ we deduce that b divides $q^{\frac{n-1}{2}}(q^{n-1} - 1)$. This is a contradiction for $n \geq 13$.

5.2 Step 6 (Imprimitivity)

Here we consider the imprimitive situation, $L_\alpha < M$. Since $k < q^{\frac{n-1}{2}}(q^{\frac{n-1}{2}} + 1)$ we can apply the bounds of [DD89] to get $|L : M| < q^{2n-2}$.

Since $|L : M| < q^{2n-2}$, M must stabilize a non-degenerate subspace W of dimension at most 2.

Suppose that W has dimension 2. Then $M \cong (\Omega_2^{\zeta}(q) \times \Omega_{n-2}(q)).[4]$, $|L : M| > q^{2n-5}$ and $|M : L_\alpha| < q^{n+2}$. By Lemma 19 $L_\alpha \cap \Omega(U)$ is irreducible. But, by [Lie85, Theorem 5.6], this implies that $|M : L_\alpha| > q^{n+2}$.

Suppose that W has dimension 1. Then $M \cong \Omega_{n-1}^{\zeta}(q).2$, $|L : M| = \frac{1}{2}q^{\frac{n-1}{2}}(q^{\frac{n-1}{2}} + \zeta)$ and $|M : L_\alpha| < q^{2n-1}$. By Lemma 19 there are two possibilities:

- $L_\alpha \cap \Omega(U)$ is irreducible. But, by [Lie85, Theorem 5.6], this implies that $|M : L_\alpha| > q^{2n-1}$ for $n \geq 21$.

- $\epsilon = +$ and $L_\alpha \cap \Omega(U)$ lies in a parabolic subgroup $P_{\frac{n-1}{2}}$ (see Lemma 19). But then $|M : L_\alpha| > q^{\frac{1}{8}(n-1)(n-3)} > q^{2n-1}$ for $n \geq 21$.

6 $\quad L = P\Omega_n^{\epsilon}(q)$, n **even**, q **odd**

In this section we set $(N_1, N_2, N_3) = (18, 26, 26)$. We will prove that $n \geq 26$ leads to a contradiction.

6.1 Step 5 (Primitivity)

Suppose first of all that L_α lies inside a maximal subgroup of type $O_m \perp O_{n-m}$ with $m \leq 14$. Then, by assumption, $L_\alpha > \hat{\Omega}_{n-14}^{\zeta}(q)$ and L_α contains at least

$$\frac{q^{n-16}(q^{\frac{n-14}{2}} - 1)(q^{\frac{n-16}{2}} - 1)}{4(q+1)}$$

conjugates of g. Then, by Lemma 10, we have $k < \frac{2w}{a} \leq 2q^{31}$ and $v < q^{2n+59}$. This implies that

$$mn - n - m^2 - m - 3 < 2n + 59.$$

Since $2m \leq n$ and $n \geq 26$ we conclude that $m \leq 9$. But this implies that $L_\alpha > \Omega_{n-10}^{\zeta}(q)$ and we repeat the process to find that $v < q^{2n+43}$. But this implies that $L_\alpha > \Omega_{n-8}^{\zeta}(q)$, $v < q^{2n+35}$ and $m \leq 6$. We continue to find that $v < q^{2n+31}$ and $m \leq 2$.

When $m = 2$ we find that $q^{2n-6} < v$ and $|v|_p = q^{n-2}$. Clearly there exists a distinct conjugate of L_α, L_β say, such that $L_\alpha \cap L_\beta \geq \Omega_{n-4}^\eta(q)$. This must lie in a line-stabilizer hence we conclude that

$$b < q^{n-2}|\Omega_n^\epsilon(q) : \Omega_{n-4}^\eta(q)|_{p'} < q^{3n-5}.$$

Now, since $L_\alpha > \Omega_{n-2}^\zeta(q)$, $k < \frac{2w}{a} < 2q^3$. Hence $b > \frac{v^2}{k^2} > q^{4n-20}$. But this gives a contradiction for $n \geq 26$.

When $m = 1$, $L_\alpha \cong \Omega_{n-1}(q)$ and $v = \frac{1}{2}q^{\frac{n-2}{2}}(q^{\frac{n}{2}} - 1) > q^{n-3}$. Then, as before, $b > \frac{v^2}{k^2} > \frac{1}{4}q^{2n-12}$. Now, similarly to before, we know that a line-stabilizer must contain $\Omega_{n-2}^\epsilon(q)$ and so

$$b < q^{\frac{n-2}{2}}|\Omega_n^\epsilon(q) : \Omega_{n-2}^\epsilon(q)| \leq q^{\frac{n-2}{2}}(q^{\frac{n}{2}} + 1)(q^{\frac{n-2}{2}} + 1).$$

Once again this yields a contradiction for $n \geq 26$.

6.1.1   Other primitive possibilities

Suppose that $\epsilon = +$ and L_α stabilizes a totally singular subspace of dimension $\frac{n}{2}$. So $L_\alpha \cong {}^\wedge[q^{\frac{1}{8}n(n-2)}] : \frac{1}{2}GL_{\frac{n}{2}}(q)$. Then

$$a \geq \frac{q^{\frac{1}{8}n(n-2)} \times \frac{1}{2} \times |GL_{\frac{n}{2}}(q)|}{q^{\frac{1}{8}(n-2)(n-4)} \times \frac{1}{2} \times |GL_{\frac{n-2}{2}}(q)| \times 4 \times |\Omega_2^-(q)|}.$$

This implies that $k \leq \frac{2w}{a} < q^{\frac{n}{2}+1}$ and $v < q^{3n}$. But $v > q^{\frac{1}{8}n(n-2)}$ and we have a contradiction for $n \geq 26$.

6.2   Step 6 (Imprimitivity)

Here we consider the imprimitive situation, $L_\alpha < M$. We know that $k < 2q^{2n-4}$ and we can apply the bounds of [DD89] to get $|L : M| < \frac{1}{2}k^2 < q^{4n-7}$.

Suppose that M is a maximal subgroup of type $O_m \perp O_{n-m}$. This implies that

$$mn - n - m^2 - m - 3 < 4n - 7.$$

Since $2m \leq n$ and $n \geq 26$, we conclude that $m \leq 6$. By Lemma 19, $L_\alpha^* \cap \Omega(U)$ is irreducible. Then Theorems 5.4 and 5.5 of [Lie85] imply that $|\Omega(U) : L_\alpha^* \cap \Omega(U)| > q^{\frac{1}{4}((n-m)^2 - 2(n-m) - 10)}$ and $v > q^{6n-11}$ for $n \geq 32$. Checking the indices of maximal irreducible subgroups for $n = 26, 28$ and 30 we are able to exclude these also.

6.2.1   Other imprimitive possibilities

Suppose that $\epsilon = +$. If L_α lies in a parabolic subgroup $P_{\frac{n}{2}}$ then

$$\frac{1}{8}n(n-2) < 4n - 7.$$

This implies that $n \leq 32$. We need only consider the situation where $\frac{n}{2}$ is odd hence we are left with $n = 30$ and $n = 26$.

Let $U = O_p(P_{\frac{n}{2}})$. We consider two situations:

- **Suppose that** $L_\alpha \cap U < U$. Then we can apply Proposition 16 to U : $(L_\alpha \cap \hat{}SL_{\frac{n}{2}}(q))$. Thus this subgroup must have index at most q^{4n-7}. When $n = 30$ this implies that $L_\alpha \geq \hat{}SL_{15}(q)$. But then $a > q^{15}$ and $\frac{1}{8}n(n-2) > 4n - 37$ which is a contradiction. When $n = 26$ we know that $L_\alpha \cap \hat{}SL_{13}(q))$ must lie inside a parabolic subgroup, P_1. Iterating the procedure we find that $L_\alpha \geq SL_{12}(q)$ and $a > q^{12}$ which yields a contradiction.

- **Suppose that** $L_\alpha \geq U$. Then let g and h be root elements and observe that U contains more than q^{2n} root elements. Then $w = c = \frac{(q^{n-2}-1)(q^{\frac{n}{2}}-1)(q^{\frac{n-4}{2}}+1)}{q^2-1}$ and, by Lemma 10, $v < q^{2n-10}$. This gives a contradiction for $n = 26$ and $n = 30$.

7 $L = \Omega_n^\epsilon(q)$, n **even**, q **even**

In this section we set $(N_1, N_2, N_3) = (16, 26, 26)$. We will prove that $n \geq 26$ leads to a contradiction.

7.1 Step 5 (Primitivity)

Suppose first of all that L_α is a subgroup of a maximal subgroup of type $O_m \perp O_{n-m}$ with m even. Now, by assumption, $L_\alpha > \Omega_{n-m}^\zeta(q)$. Then $a > q^{2n-2m-7}$ and so $v < q^{2n+4m}$. This implies that

$$mn - n - m^2 - m - 3 < 2n + 4m.$$

Since $2m \leq n$ and $n \geq 26$ we conclude that $m \leq 4$.

When $m = 4$ we find that $v < q^{2n+16}$ and, examining $|L : L_\alpha|$, we obtain a contradiction for $n \geq 26$. When $m = 2$ we find that $q^{2n-6} < v$ and $|v|_p \leq q^{n-2}$. As in the odd characteristic case $b < q^{3n-5}$. Since $L_\alpha > \Omega_{n-2}^\zeta(q)$ we have $k < \frac{2w}{a} < q^8$ and $b > \frac{v^2}{k^2} > q^{4n-28}$. But this gives a contradiction for $n \geq 26$.

7.1.1 Other primitive possibilities

Suppose that $\epsilon = +$ and L_α stabilizes a totally singular subspace of dimension $\frac{n}{2}$. So $L_\alpha \cong [q^{\frac{1}{8}n(n-2)}] : GL_{\frac{n}{2}}(q)$ and it is easy to see that $a \geq q^{\frac{3n}{2}-6}$. By Lemma 10, we have that $v < q^{3n-2}$. But $v > q^{\frac{1}{8}n(n-2)}$ and we have a contradiction for $n \geq 26$.

Finally consider the possibility that $L_\alpha = Sp_{n-2}(q)$ and $v = q^{\frac{n-2}{2}}(q^{\frac{n}{2}} - \epsilon)$. Now L_α is the set of elements in L which stabilizes a 1-dimensional non-singular subspace of V. Consider $H = \Omega_2^\zeta(q)) \times \Omega_{n-2}^\epsilon(q)$ stabilizing a 2-dimensional non-degenerate subspace of V. Then it is easy to see that $\Omega_{n-2}^\epsilon(q)$, normal in H, stabilizes 2 distinct 1-dimensional non-singular subspaces of V. We conclude that $\Omega_{n-2}^\epsilon(q)$ stabilizes a line and so b divides

$$|v|_p |\Omega_n^\epsilon(q) : \Omega_{n-2}^\epsilon(q)|_{p'} = q^{\frac{n-2}{2}}(q^{\frac{n}{2}} - \epsilon)(q^{\frac{n-2}{2}} + \epsilon) < q^{\frac{3n}{2}-1}.$$

Now observe that, since $\Omega_{n-2}^{\epsilon}(q)$ stabilizes a point,

$$w \leq \frac{(q^{n-2}-1)(q^{\frac{n}{2}}-\epsilon)(q^{\frac{n-4}{2}}+\epsilon)}{q^2-1}, \quad a \geq \frac{(q^{n-4}-1)(q^{\frac{n-2}{2}}-\epsilon)(q^{\frac{n-6}{2}}+\epsilon)}{q^2-1}.$$

Applying our bound we find that

$$k < \frac{2w}{a} \leq \frac{2(q^{\frac{n-2}{2}}+\epsilon)(q^{\frac{n}{2}}-\epsilon)}{(q^{\frac{n-4}{2}}-\epsilon)(q^{\frac{n-6}{2}}+\epsilon)} \leq q^6$$

and so $b > \frac{v^2}{k^2} > q^{2n-13}$. But then $2n - 13 < \frac{3n}{2} - 1$ implies that $n < 26$ as required.

7.2   Step 6 (Imprimitivity)

Here we consider the imprimitive situation, $L_\alpha < M$. We know that $k < 2q^{2n-4}$ and so we can apply the bounds of [DD89] to get $|L : M| < 2w^2 < q^{4n-9}$.

Suppose that M is a maximal subgroup of type $O_m \perp O_{n-m}$ with m even. This implies that

$$mn - n - m^2 - m - 3 < 4n - 9.$$

Since $2m \leq n$ and $n \geq 26$, we conclude that $m \leq 6$. In what follows we take m to be the largest integer such that L_α is contained in a maximal subgroup of type $O_m \perp O_{n-m}$; thus $m = 2, 4$ or 6. We refer to Lemma 19 and go through all possibilities:

- $L_\alpha \cap \Omega(U)$ is irreducible. In this case $|M : L_\alpha| \geq q^{\frac{1}{4}((n-m)^2-2(n-m)-16)}$. But then $|L : L_\alpha| > q^{6n-14}$ for $n \geq 26$.

- $L_\alpha \cap \Omega(U) \leq Sp_{n-m-2}(q)$. In this case

$$L_\alpha \leq M_1 = (\Omega(W) \times Sp_{n-m-2}(q)).2$$

  and $|L : M_1| > q^{mn+n-m^2-m-4}$. Once again we have a number of possibilities:

  – $L_\alpha \cap Sp_{n-m-2}(q)$ is irreducible:

    * $L_\alpha \cap Sp_{n-m-2}(q) \geq \Omega_{n-m-2}^{\epsilon}(q)$. Then, similarly to before, $v < q^{2n+4(m+2)}$ which is a contradiction for $n \geq 26$.
    * $L_\alpha \cap Sp_{n-m-2}(q) \leq O_{n-m-2}^{\epsilon}(q)$. Then $L_\alpha < (\Omega_{m+2}^{\eta}(q) \times \Omega_{n-m-2}^{\epsilon}(q)).2$ which is a contradiction.
    * $L_\alpha \cap Sp_{n-m-2}(q)$ lies in any other irreducible subgroup. Then, referring to [KL90, Table 3.5.C], we find that $|M_1 : L_\alpha| > q^{\frac{1}{4}(n-m-2)^2-\frac{1}{4}(n-m-2)-1}$ and $|L : L_\alpha| > q^{6n-14}$ for $n \geq 26$.

  – $L_\alpha \cap Sp_{n-m-2}(q)$ lies inside P a parabolic subgroup of $Sp_{n-m-2}(q)$. This possibility can be excluded by an argument similar to the proof for Lemma 17.

$$- L_\alpha \cap Sp_{n-m-2}(q) \leq Sp_r(q) \times Sp_{n-m-2-r}(q). \text{ Examining}$$

$$|L : (\Omega(W) \times Sp_r(q) \times Sp_{n-m-2-r}(q)).2|$$

we conclude that $m + r \leq 8$. In a similar way to Lemma 19 we need only examine the possibility that $L_\alpha \cap Sp_{n-m-2-r}(q)$ is irreducible. There are a number of possibilities:

* $L_\alpha \geq Sp_{n-m-2-r}(q) \geq \Omega_{n-m-2-r}^\epsilon(q)$. Then, as before, $v < q^{2n+4(m+r+2)}$ which is a contradiction for $n \geq 26$.

* $L_\alpha \cap Sp_{n-m-2-r}(q) \leq O_{n-m-2-r}(q)$.
  Then $L_\alpha < (\Omega_{m+2+r}^\eta(q) \times \Omega_{n-m-2-r}^\zeta(q)).2$ which is a contradiction.

* $L_\alpha \cap Sp_{n-m-2-r}(q)$ lies in any other irreducible subgroup. Then, referring to [KL90, Table3.5.C], we find that

$$|Sp_{n-m-2-r}(q) : L_\alpha \cap Sp_{n-m-2-r}(q)| > q^{\frac{1}{4}(n-m-r-2)^2 - \frac{1}{4}(n-m-2-r)-1}$$

and $|L : L_\alpha| > q^{6n-14}$ for $n \geq 26$.

7.2.1 Other imprimitive possibilities

Suppose first of all that $\epsilon = +$ and L_α lies in a parabolic subgroup $P_{\frac{n}{2}}$. Then

$$\frac{1}{8}n(n-2) < 4n - 9$$

and so $n \leq 32$. We need only consider the situation where $\frac{n}{2}$ is odd hence we are left with $n = 30$ and $n = 26$. These cases are ruled out very similarly to when q is odd.

Now suppose that $L_\alpha < M = Sp_{n-2}(q)$. Then $|L : M| > q^{n-2}$ and so $|M : L_\alpha| < q^{5n-12}$. If $L_\alpha < O_{n-2}(q)$ then L_α preserves a non-degenerate subspace of dimension 2 and this is covered above. If L_α lies in any other irreducible subgroup of $Sp_{n-2}(q)$ then $|M : L_\alpha| > q^{5n-12}$ which is a contradiction for $n \geq 26$.

By a similar argument to Lemma 14 we can conclude that L_α does not lie in a parabolic subgroup of $Sp_{n-2}(q)$. Thus L_α must lie in a maximal subgroup M_1 of type $Sp_m \perp Sp_{n-m-2}$. Since $|M : L_\alpha| < q^{5n-12}$ we conclude that $m \leq 6$.

In a similar way to Lemma 19 we need only examine the possibility that $L_\alpha \cap Sp_{n-m-2}(q)$ is irreducible. There are a number of possibilities:

- $L_\alpha \geq Sp_{n-m-2}(q) \geq \Omega_{n-m-2}^\epsilon(q)$. Once more this implies that $v < q^{2n+4(m+2)}$ which is a contradiction for $n \geq 26$.

- $L_\alpha \cap Sp_{n-m-2}(q) \leq O_{n-m-2}(q)$. Then $L_\alpha < (\Omega_{m+2}^\eta(q) \times \Omega_{n-m-2}^\zeta(q)).2$ which is covered above.

- $L_\alpha \cap Sp_{n-m-2}(q)$ lies in any other irreducible subgroup. Then, referring to [KL90, Table3.5.C], we find that $|Sp_{n-m-2}(q) : L_\alpha \cap Sp_{n-m-2}(q)| > q^{\frac{1}{4}(n-m-2)^2 - \frac{1}{4}(n-m-2)-1}$ and $|L : L_\alpha| > q^{6n-14}$ for $n \geq 26$.

8 $L = PSU_n(q)$, nq **odd**

In this section we set $(N_1, N_2, N_3) = (11, 15, 15)$. We will prove that $n \geq 15$ leads to a contradiction.

8.1   Step 5 (Primitivity)

Here $L_\alpha \leq \hat{}((SU_m(q) \times SU_{n-m}(q)).(q+1))$ for some $m \leq 4$. Observe that in all cases $L_\alpha > \hat{}SU_{n-6}(q)$ and so $a \geq q^{n-5}\frac{q^{n-4}+1}{q+1}$. Thus $k \leq \frac{2w}{a} \leq q^9$ and so $v < q^{2n+16}$. This implies that

$$2mn - 2m^2 - n + m < 2n + 16.$$

Since $2m < n$ and $n \geq 15$ we conclude that $m \leq 2$. But this implies that $L_\alpha > \hat{}SU_{n-2}(q)$ and so $a \geq q^{n-3}\frac{q^{n-2}+1}{q+1}$ and $k < q^5$. Repeating the process we find that $v < q^{2n+8}$ and $m = 1$.

We have $\hat{}SU_{n-1}(q) \leq L_\alpha \leq \hat{}GU_{n-1}(q)$. Now, by Lemma 7, $L_\alpha = \hat{}GU_{n-1}(q)$, $k < q^5$ and $v = q^{n-1}\frac{q^n+1}{q+1}$. Then $b > \frac{v^2}{k^2} > q^{4n-16}$. Clearly there exists a distinct conjugate of L_α, L_β say, such that $L_\alpha \cap L_\beta \geq \hat{}SU_{n-2}(q)$. This must lie in a line-stabilizer hence we conclude that $b | q^{2n-3}(q^n + 1)(q^{n-1} - 1)$. Since $v = q^{n-1}\frac{q^n+1}{q+1}$ we must have $b | q^{n-1}(q^n + 1)(q^{n-1} - 1)$. But then $q^{4n-16} < b < q^{3n-3}$ which is a contradiction.

8.2   Step 6 (Imprimitivity)

Here we consider the imprimitive situation, $L_\alpha < M$. We know that $k < 2q^{2n-1}$ and we can apply the bounds of [DD89] to get $|L : M| < q^{4n-1}$. This implies that

$$2mn - 2m^2 - n + m < 4n - 1.$$

Once again, since $2m < n$ and $n \geq 15$, we conclude that $m \leq 2$.

If $m = 2$ then $|L : M| > q^{4n-9}$ and so $|M : L_\alpha| < q^{2n+4}$. We examine $L_\alpha \cap \Omega(U)$; by Lemma 19 this must be an irreducible subgroup of $\Omega(U)$. But then, by [Lie85, Theorem 5.3], $|\Omega(U) : L_\alpha \cap \Omega(U)| > q^{2n+4}$ which is a contradiction.

If $m = 1$ then $|L : M| > q^{2n-3}$ and so $|M : L_\alpha| < q^{4n-2}$. Once again $L_\alpha \cap \Omega(U)$ must be an irreducible subgroup of $\Omega(U)$. But then, by [Lie85, Theorem 5.3], $|\Omega(U) : L_\alpha \cap \Omega(U)| > q^{4n-2}$ which is a contradiction.

9 $L = PSU_n(q)$, n **even**, q **odd**

In this section we set $(N_1, N_2, N_3) = (16, 22, 22)$. We will prove that $n \geq 22$ leads to a contradiction.

9.1   Step 5 (Primitivity)

Here $L_\alpha \leq \hat{}((SU_m(q) \times SU_{n-m}(q)).(q+1))$ for some $m \leq 7$. Observe that in all cases $L_\alpha > \hat{}SU_{n-8}(q)$ and so

$$a \geq q^{2n-20}(q^{n-10} + \cdots + q^2 + 1)(q^{n-10} - \cdots - q + 1).$$

Thus $k \leq \frac{2w}{a} \leq 2q^{26}(q^8 + 1)$ and so $v < q^{2n+64}$. This implies that

$$2mn - 2m^2 - n + m < 2n + 64.$$

Since $2m < n$ and $n \geq 22$ we conclude that $m \leq 4$. But this implies that $L_\alpha > \hat{\ }SU_{n-4}(q)$. We repeat the process to find that $v < q^{2n+32}$ and $m \leq 2$. But this implies that $L_\alpha > \hat{\ }SU_{n-2}(q)$. Again we repeat the process to find that $v < q^{2n+16}$ and $m = 1$.

Thus $\hat{\ }SU_{n-1}(q) \leq L_\alpha \leq \hat{\ }GU_{n-1}(q)$. By Lemma 7 $L_\alpha = \hat{\ }GU_{n-1}(q)$, $k < q^5$ and $v = q^{n-1}\frac{q^n-1}{q+1}$. Then $b > \frac{v^2}{k^2} > q^{4n-18}$. Clearly there exists a distinct conjugate of L_α, L_β say, such that $L_\alpha \cap L_\beta \geq \hat{\ }SU_{n-2}(q)$. This must lie in a line-stabilizer hence we conclude that $b \big| q^{2n-3}(q^n - 1)(q^{n-1} + 1)$. Since $v = q^{n-1}\frac{q^n+1}{q+1}$ we must have $b \big| q^{n-1}(q^n - 1)(q^{n-1} + 1)$. But then $q^{4n-18} < b < q^{3n-3}$ which is a contradiction.

9.2   Step 6 (Imprimitivity)

Here we consider the imprimitive situation, $L_\alpha < M$. We know that $k < 2q^{3n-6}(q^{n-2} + \ldots q^2 + 1)$ and we can apply the bounds of [DD89] to get $|L : M| < \frac{1}{2}k^2 < q^{8n-15}$. This implies that

$$2mn - 2m^2 - n + m < 8n - 15.$$

Since $2m < n$ and $n \geq 22$, we conclude that $m \leq 4$ or $m = 5$ and $n \leq 30$.

Now $|L : M| > q^{2mn-2m^2-n-m}$. We examine $L_\alpha \cap \Omega(U)$; by Lemma 19 this must be an irreducible subgroup of $\Omega(U)$. But then, by [Lie85, Theorem 5.3], $|\Omega(U) : L_\alpha \cap \Omega(U)| > q^{\frac{1}{2}(n-m)^2 - \frac{1}{2}(n-m)-3}$ which is a contradiction for $n \geq 26$. A closer examination of bounds for $n = 20$ and 22 also yields a contradiction as required.

10 $L = PSU_n(q)$, q **even**

In this section we set $(N_1, N_2, N_3) = (11, 13, 13)$. We will prove that $n \geq 13$ leads to a contradiction.

10.1   Step 5 (Primitivity)

Here $L_\alpha \leq \hat{\ }((SU_m(q) \times SU_{n-m}(q)).(q+1))$ for some $m \leq 5$. Observe that in all cases $L_\alpha > \hat{\ }SU_{n-5}(q)$ and so $a \geq \frac{(q^{n-5}-(-1)^{n-5})(q^{n-6}-(-1)^{n-6})}{q+1}$. Thus $k \leq \frac{2w}{a} \leq 2q^5(q^5 + 1)$ and so $v < q^{2n+20}$. This implies that

$$2mn - 2m^2 - n + m < 2n + 20.$$

Since $2m < n$ and $n \geq 13$ we conclude that $m \leq 2$. But this implies that $L_\alpha > \hat{\ }SU_{n-2}(q)$. We repeat the process to find that $v < q^{2n+7}$ and $m = 1$ or $(m, n, q) = (2, 13, 2)$. This last situation is easily excluded.

Thus $\hat{\ }SU_{n-1}(q) \leq L_\alpha \leq \hat{\ }GU_{n-1}(q)$. By Lemma 7 $L_\alpha = \hat{\ }GU_{n-1}(q)$, $k < q^3$ and $v = q^{n-1}\frac{q^n\pm1}{q+1}$. Then $b > \frac{v^2}{k^2} > q^{4n-12}$. Clearly there exists a distinct conjugate of L_α, L_β say, such that $L_\alpha \cap L_\beta \geq \hat{\ }SU_{n-2}(q)$. This must lie in a line-stabilizer hence we conclude that $b \big| q^{2n-3}(q^n \pm 1)(q^{n-1} \mp 1)$. Since $v = q^{n-1}\frac{q^n\pm1}{q+1}$ we must have $b \big| q^{n-1}(q^n \pm 1)(q^{n-1} \mp 1)$. But then $q^{4n-12} < b < q^{3n-3}$ which is a contradiction.

10.2   Step 6 (Imprimitivity)

Here we consider the imprimitive situation, $L_\alpha < M$. We know that $k < 2q^{2n-2}$ and we can apply the bounds of [DD89]to get $|L : M| < \frac{1}{2}k^2 < q^{4n-3}$. This implies that

$$2mn - 2m^2 - n + m < 4n - 3.$$

Since $2m < n$ and $n \geq 14$, we conclude that $m \leq 2$.

Now $|L : M| > q^{2mn-2m^2-n-m}$. We examine $L_\alpha \cap \Omega(U)$; by Lemma 19 this must be an irreducible subgroup of $\Omega(U)$. But then, by [Lie85, Theorem 5.3], $|\Omega(U) : L_\alpha \cap \Omega(U)| > q^{\frac{1}{3}(n-m)^2}$ which is a contradiction for $n \geq 20$. A closer examination of bounds for $13 \leq n \leq 19$ also yields a contradiction as required.

11 $L = PSp_n(q)$, n **even**, q **odd**

In this section we set $(N_1, N_2, N_3) = (12, 22, 22)$. We will prove that $n \geq 22$ leads to a contradiction.

11.1   Step 5 (Primitivity)

Here $L_\alpha \leq \hat{\ }(Sp_m(q) \circ Sp_{n-m}(q))$ for some even $m \leq 6$. Observe that, by supposition, in all cases $L_\alpha > Sp_{n-6}(q)$ and so $a \geq q^{n-8}(q^{n-8} + \cdots + q^2 + 1)$. Thus $v < q^{n+26}$. This implies that

$$mn - m^2 < n + 26.$$

Since $2m < n$ and $n \geq 22$ we conclude that $m = 2$. But this implies that $L_\alpha > Sp_{n-2}(q)$. We repeat the process to find that $v < q^{n+10}$ which is a contradiction.

11.2   Step 6 (Imprimitivity)

Here we consider the imprimitive situation, $L_\alpha < M$. We know that $k < 2q^{n-2}(q^{n-2} + \ldots q^2 + 1)$ and we can apply the bounds of [DD89] to get $|L : M| < \frac{1}{2}k^2 < q^{4n-7}$. This implies that

$$mn - m^2 < 4n - 7.$$

Since $2m < n$ and $n \geq 22$, we conclude that $m \leq 4$.

Now $|L : M| > q^{mn-m^2}$. We examine $L_\alpha \cap \Omega(U)$; by Lemma 19 this must be an irreducible subgroup of $\Omega(U)$. But then, by [Lie85, Theorem 5.3], $|\Omega(U) : L_\alpha \cap \Omega(U)| > q^{\frac{1}{4}(n-m)^2 - \frac{1}{4}(n-m)-1}$ which is a contradiction for $n \geq 22$.

12 $L = Sp_n(q)$, n **even**, q **even**

In this section we set $(N_1, N_2, N_3) = (8, 14, 14)$. We will prove that $n \geq 14$ leads to a contradiction.

12.1 Step 5 (Primitivity)

Here $L_\alpha \leq \hat{}(Sp_m(q) \circ Sp_{n-m}(q))$ for some even $m \leq 6$. Observe that in all cases $L_\alpha > Sp_{n-6}(q)$ and so $a \geq q^{n-6} - 1$. Thus $v < q^{n+16}$. This implies that

$$mn - m^2 < n + 16.$$

Since $2m < n$ and $n \geq 14$ we conclude that $m = 2$. But this implies that $L_\alpha > Sp_{n-2}(q)$. We repeat the process to find that $v < q^{n+10}$ which is a contradiction.

12.2 Step 6 (Imprimitivity)

Here we consider the imprimitive situation, $L_\alpha < M$. We know that $k < 2q^n$ and we can apply the bounds of [DD89] to get $|L : M| < \frac{1}{2}k^2 < q^{2n+1}$. This implies that

$$mn - m^2 < 2n + 1.$$

Since $2m < n$ and $n \geq 14$, we conclude that $m = 2$.

Now $|L : M| > q^{2n-4}$. Examining [KL90, Table 5.2.A] we find that $P(Sp_{n-2}(q)) > q^5$ in all cases. Thus $|L : (Sp_2(q) \times M_1)| > q^{2n+1}$ where M_1 is maximal in $Sp_{n-2}(q)$. By Proposition 16, this implies that $L_\alpha \cap \Omega(U) = M_1$ and so is and so is maximal in $\Omega(U)$.

Observe now that $|M : L_\alpha| < q^{n+5}$. We examine $L_\alpha \cap \Omega(U)$; by Lemma 19 this must be an irreducible subgroup of $\Omega(U)$. There are two possibilities:

- $L_\alpha = O_{n-2}^\epsilon(q) > Sp_{n-4}(q)$. Then $a \geq q^{n-4} - 1$ and $v < q^{n+11}$ which is a contradiction.

- $L_\alpha \cap \Omega(U)$ lies in any other irreducible subgroup of $Sp_{n-2}(q)$. But in this case $|L : L_\alpha| > q^{3n+1}$ which is a contradiction.

13 $L = PSL_n(q)$

In this section L_α lies in a parabolic subgroup of L. We will prove that, if $n \geq N_2$, then only Example 1 occurs.

13.1 Step 5 (Primitivity)

Suppose that $H \leq L_\alpha \leq P_m$ where $H \cong \hat{}SL_{n-m}(q)$ and H is normal in a Levi-complement of P_m. Clearly, in all cases, L_α contains transvections. Thus we take g and h to be transvections and observe that $a \geq (q^{n-m-1} - 1)(q^{n-m-1} + \cdots + q + 1)$. By Lemma 10 we have $v < q^{2n+4m+2}$. Since $v > q^{m(n-m)}$ we conclude that $m \leq 3$.

If $L_\alpha < P_m$ then we can apply the bounds of Delandtsheer and Doyen. Thus $|L : P_m| < \frac{2w^2}{a^2} < 2(q^3 + 1)^4$. This is a contradiction for $n \geq 17$.

If $L_\alpha = P_1$ then this action is 2-transitive, hence flag-transitive ([BDD88]) and so corresponds to the known action on $PG(n-1, q)$.

If $L_\alpha = P_m$, for $m = 2$ or 3, then there exists a conjugate P_m^x not equal to P_m such that $\hat{}SL_{n-m-1}(q) < P_m \cap P_m^x$. Hence b divides $|SL_n(q) : SL_{n-m-1}(q)|$. We

know that p is not significant ([Gil06]) and p does not divide v. Hence b divides $|SL_n(q) : SL_{n-m-1}(q)|_{p'} < q^{(m+1)(n-\frac{1}{2}m)}$. Now $v > q^{m(n-m)}$ and $k < q^{2m+1}$ hence $b > \frac{v^2}{k^2} > q^{2mn-2m^2-4m-4}$. This gives a contradiction for $n \geq 17$.

13.2 Step 6 (Imprimitivity)

We are left with the possibility that $L_\alpha < P_m$ and L_α does not contain H. Our upper bound for v varies in this case. The worst case scenario is when n is even and q is odd in which case $v < q^{10n-14}$ and, by Proposition 16, $|L : P_m| < q^{8n-14}$.

Let $U = O_p(P_m)$. We have two situations:

- **Suppose that L_α does not contain U.** Then, by Proposition 16 we can apply the Delandtsheer-Doyen bound to the group $M_1 = UL_\alpha$. Then we have $|L : P_m| \cdot |H : L_\alpha \cap H| < q^{8n-14}$. We suppose that $L_\alpha \cap H$ is an irreducible subgroup of H. Referring to [Lie85, Theorem 5.1] we see that this implies that

$$|H : L_\alpha \cap H| \geq q^{\frac{1}{2}(n-m)^2-(n-m)}.$$

  But then the Delandtsheer-Doyen bound is violated for $n \geq 20$.

  For q even or nq odd the Delandtsheer-Doyen bound implies that $|L : P_m| \cdot |H : L_\alpha \cap H| < q^{4n}$. This implies that, for $n \geq 17$, $L_\alpha \cap H$ is not an irreducible subgroup of H.

  Thus $L_\alpha \cap H$ lies in a parabolic subgroup of H. We can continue to iterate in this way by considering $L_\alpha \cap \hat{\ }SL_{n-s}(q)$ for some s. This will either terminate with $L_\alpha \geq \hat{\ }SL_{n-s}(q)$ for some s or it will increase the cumulative index by at least q^s and increase s by at least 1. It is easy to see that this must terminate with a final value for $s \leq n - 2$.

  Thus $L_\alpha > \hat{\ }SL_2(q)$ and so L_α must contain transvections and, once more, we have $|L : M_1| < q^{4n}$. Repeating our analysis with this new bound we find that $s \leq 10$. In fact, for $L_\alpha > \hat{\ }SL_{n-s}(q)$, we have $|L : M_1| < q^{4s+4}$. This implies that $s \geq n - 4$ which is a contradiction for $n \geq 17$.

- **Suppose that $L_\alpha \geq U$.** Then L_α contains transvections and we can take $w = c = (q^{n-1}-1)(q^{n-1}+\cdots+q+1)$. Then $a > q^{n-1}-1$ and so $v < q^{4n}$. This implies that $m \leq 6$. Once again we can exclude the possibility that $L_\alpha \cap H$ is an irreducible subgroup of H.

  Thus $H \cap L_\alpha \leq P_{m_1}$, a parabolic subgroup of H. Since $v < q^{4n}$ we must have $m + m_1 \leq 6$. If $H \cap L_\alpha = P_{m_1}$ then observe that $L_\alpha \geq \hat{\ }SL_{n-6}(q)$.

  $H \cap L_\alpha < P_{m_1}$ then we can apply Proposition 16 to $U : P_{m_1}$. This implies that $q^{n(m+m_1)-(m+m_1)^2} < |L : (U : P_{m_1})| < q^{2n+1}$. Then $m + m_1 = 2$ and $m = m_1 = 1$. We can iterate this procedure once again by examining $H_1 \cap L_\alpha$ where H_1 is normal in a Levi-complement of P_{m_1}. Clearly $H_1 \cap L_\alpha$ must lie in a parabolic subgroup of H_1. In fact, by considering Proposition 16, it must equal such a parabolic subgroup and once more $L_\alpha \geq \hat{\ }SL_{n-6}(q)$.

  Now $a \geq (q^{n-7} - 1)(q^{n-7} + \cdots + q + 1)$ and $v < q^{2n+28}$. Then L_α^* must stabilize a subspace of dimension at most 4 and $L_\alpha > \hat{\ }SL_{n-4}(q)$. Repeating

the process we conclude that $L_\alpha > \hat{\ }SL_{n-3}(q)$ and $|L : M| < 2(q^3 + 1)^4$ which is a contradiction for $n \geq 17$.

14 Concluding remarks

We have also investigated groups with socle $P\Omega^\epsilon_{22}(q)$ or $P\Omega^\epsilon_{24}(q)$ using the techniques described in Section 3. We found that such groups do not act line-transitively on a finite linear space and we are therefore able to record the following result:

Corollary 20. *Let G be a group which acts transitively on the set of lines of a linear space $\mathcal{S}$. Suppose that G has socle L a simple classical group of dimension $n > 20$. Then G acts transitively on the set of flags of $\mathcal{S}$ and we have Example 1.*

No doubt the techniques of Section 3 can be applied to further reduce the lower bound in Corollary 20.

References

[BDD]   F. Buekenhout, A. Delandtsheer, and J. Doyen, *Finite linear spaces with flag-transitive sporadic groups*, Unpublished notes.

[BDD88]   _____, *Finite linear spaces with flag-transitive groups*, J. Combin. Theory, Series A **49** (1988), 268–293.

[BDD+90] F. Buekenhout, A. Delandtsheer, J. Doyen, P. Kleidman, M. Liebeck, and J. Saxl, *Linear spaces with flag-transitive automorphism groups*, Geom. Ded. **36** (1990), 89–94.

[Blo67]   R.E. Block, *On the orbits of collineation groups*, Math. Zeitschrift **96** (1967), 33–49.

[CKS76]   Charles W. Curtis, William M. Kantor, and Gary M. Seitz, *The 2-transitive permutation representations of the finite Chevalley groups*, Trans. Amer. Math. Soc. **218** (1976), 1–59.

[CNP03]   A. Camina, P. Neumann, and C. Praeger, *Alternating groups acting on linear spaces*, Proc. London Math. Soc.(3) **87** (2003), no. 1, 29–53.

[CP93]   Alan R. Camina and Cheryl E. Praeger, *Line-transitive automorphism groups of linear spaces*, Bull. London Math. Soc. **25** (1993), 309–315.

[CP01]   A. Camina and C. Praeger, *Line-transitive, point quasiprimitive automorphism groups of finite linear spaces are affine or almost simple*, Aequationes Math. **61** (2001), 221–232.

[CS89]   A. Camina and J. Siemons, *Block transitive automorphism groups of 2-$(v, k, 1)$ block designs*, J. Combin. Theory, Series A **51** (1989), 268–276.

[CS00]   A. Camina and F. Spiezia, *Sporadic groups and automorphisms of linear spaces*, J. Combin. Designs **8** (2000), 353–362.

[Dav87]   D.H. Davies, *Automorphism of designs*, Ph.D. thesis, University of East Anglia, 1987.

[DD89]   A. Delandtsheer and J. Doyen, *Most block-transitive t-designs are point-primitive*, Geom. Ded. **29** (1989), 307–310.

[Del86]   A. Delandtsheer, *Flag-transitive finite simple groups*, Arch. Math. **47** (1986), 395–400.

[Del01]   ———, *Finite flag-transitive linear spaces with alternating socle*, Algebraic combinatorics and applications (Gößweinstein, 1999), Springer, Berlin, 2001, pp. 79–88.

[Gila]   Nick Gill, $PSL(3, q)$ *and line-transitive linear spaces*, Beiträge zur Algebra und Geometrie, To appear.

[Gilb]   ———, *Transitive projective planes*, Advances in Geometry, To appear.

[Gil06]   ———, *Linear spaces with significant characteristic prime*, Innovations in Incidence Geometry **3** (2006), 109–119.

[GLS94]   Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The classification of the finite simple groups, number 3*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, 1994.

[GMS03]   Robert M. Guralnick, Peter Müller, and Jan Saxl, *The rational function analogue of a question of Schur and exceptionality of permutation representations*, Mem. Amer. Math. Soc. **162** (2003), no. 773, 1–79.

[KL90]   P. Kleidman and M. Liebeck, *The subgroup structure of the finite classical groups*, London Mathematical Society Lecture Note Series, vol. 129, Cambridge University Press, Cambridge, 1990.

[Kle90]   Peter B. Kleidman, *The finite flag-transitive linear spaces with an exceptional automorphism group*, Finite geometries and combinatorial designs (Lincoln, NE, 1987), Contemp. Math., vol. 111, Amer. Math. Soc., Providence, RI, 1990, pp. 117–136.

[Lie85]   Martin W. Liebeck, *On the orders of maximal subgroups of the finite classical groups*, Proc. London Math. Soc. (3) **50** (1985), no. 3, 426–446.

[Liu01]   Weijun Liu, *The Chevalley groups $G_2(2^n)$ and 2-$(v, k, 1)$ designs*, Algebra Colloq. **8** (2001), no. 4, 471–480.

[Liu03a]   ———, *The Chevalley groups $G_2(q)$ with q odd and 2-$(v, k, 1)$ designs*, European J. Combin. **24** (2003), no. 3, 331–346.

[Liu03b]   ———, *Finite linear spaces admitting a projective group* PSU(3, q) *with q even*, Linear Algebra Appl. **374** (2003), 291–305.

[Liu03c]   ———, *Finite linear spaces admitting a two-dimensional projective linear group*, J. Combin. Theory Ser. A **103** (2003), no. 2, 209–222.

[Liu03d]   _____, *Steinberg triality groups acting on 2-(v, k, 1) designs*, Sci. China Ser. A **46** (2003), no. 6, 872–883.

[LLG06]   Weijun Liu, Shangzhao Li, and Luozheng Gong, *Almost simple groups with socle* Ree(q) *acting on finite linear spaces*, European J. Combin. **27** (2006), no. 6, 788–800.

[LLM01]   Weijun Liu, Huiling Li, and Chungui Ma, *Suzuki groups and* 2-(v, k, 1) *designs*, European J. Combin. **22** (2001), no. 4, 513–519.

[LZLF04]   Weijun Liu, Shenglin Zhou, Huiling Li, and Xingui Fang, *Finite linear spaces admitting a Ree simple group*, European J. Combin. **25** (2004), no. 3, 311–325.

[Sax02]   Jan Saxl, *On finite linear spaces with almost simple flag-transitive automorphism groups*, J. Combin. Theory A **100** (2002), 322–348.

[Spi97]   F. Spiezia, *Simple groups and automorphisms of linear spaces*, PhD thesis, University of East Anglia, 1997.

[Zho02]   Shenglin Zhou, *Block primitive* 2-(v, k, 1) *designs admitting a Ree simple group*, European J. Combin. **23** (2002), no. 8, 1085–1090.

[Zho05]   _____, *Block primitive 2-(v, k, 1) designs admitting a Ree group of characteristic two*, Des. Codes Cryptogr. **36** (2005), no. 2, 159–169.

[ZLL00]   Shenglin Zhou, Huiling Li, and Weijun Liu, *The Ree groups* $^2G_2(q)$ *and* 2 − (v, k, 1) *block designs*, Discrete Math. **224** (2000), no. 1-3, 251–258.

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