Span of Dold Manifolds

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Abstract

We study span and stable span of Dold manifolds $P(m,n)$. We present several lower and upper bounds for these invariants, and for some pairs (m,n), we determine the exact values of span or stable span. In the case $P(m,1)$, $m \not\equiv 15 \pmod{16}$, we show how the maximum number of everywhere linearly independent vector fields can be constructed.

1 Introduction

Given a smooth manifold M, its span is the maximum number of everywhere linearly independent vector fields on M, denoted by $\operatorname{span} M$. Its stable span is the number $\operatorname{span}(M \times S^1) - 1$, denoted by $\operatorname{stabspan} M$. Span M and stabspan M are important characteristics of M. In many situations, some information about stable span can be obtained more easily than about span, thus the stable span can play a relevant role in finding the span. But the stable span in its own right is significant in various applications, for instance, it is related to the existence of fold maps from M to $\mathbb{R}^p$ (see O. Saeki [15]). Further information on span and stable span (and related questions) can be found, for example, in [20], [6], [7], [9].

In this paper, we derive new results on span and stable span of the Dold manifolds $P(m,n) = (S^m \times \mathbb{C}P^n)/\sim$, with $(x,z) \sim (-x,\bar{z})$, where $\mathbb{C}P^n$ is the complex projective space of (complex) dimension n. In particular, $P(m,0) = \mathbb{R}P^m$ and $P(0,n) = \mathbb{C}P^n$ are nothing but real and complex projective spaces.

We recall that the Dold manifolds were first introduced by A. Dold in [2], to describe the generators in the unoriented cobordism ring $\mathfrak{N}$. Later on, many other authors studied various properties of Dold manifolds. For instance, immersions

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and embeddings of these manifolds were studied by J. J. Ucci [22], W.-L. Ting [21], T. Kobayashi [5]. The existence question for almost complex structures was analyzed by Z. Tang [19]. R. Stong [18] describes possible Stiefel-Whitney classes of vector bundles over Dold manifolds and a partial smooth classification of manifolds homotopy equivalent to a Dold manifold was given by H. K. Mukerjee [12]. But up to now not much was known about the values of span $P(m,n)$ and stabspan $P(m,n)$, except for the particular cases $P(m,0) = \mathbb{R}P^m$ and $P(0,n) = \mathbb{C}P^n$. All that we are aware of are the following papers. J. H. Kwak's [11] attacks the question of which manifolds $P(m,n)$ are parallelizable (that is, such that span $P(m,n)$ is as big as $\dim P(m,n) = m + 2n := D$) or stably parallelizable (that is, such that stabspan $P(m,n) = D$). In addition to this, in B. Junod and U. Suter [4] and M.-Y. Sohn [16], some upper bounds for the span of Dold manifolds or products of Dold manifolds can be found.

This paper is organized as follows. In Sec. 2, after recalling some basic facts, we derive several bounds for (stab)span $P(m,n)$, and in some cases we find the value of span $P(m,n)$.

In Sec. 3, we find more bounds for the stable span of $P(m,n)$ and we present further results on span $P(m,n)$ (in particular, on span $P(1,n)$). In this section, we also compare the results of [4] and [16] with our upper bounds obtained in Sec. 2 and show that, in most cases, our results are better.

Finally, in Sec. 4 we show how for the Dold manifolds $P(m,1)$ with $m \not\equiv 15 \pmod{16}$ the maximum number of everywhere linearly independent vector fields can be constructed.

2 Bounds for (stab)span $P(m,n)$ implied by Stiefel-Whitney classes

The upper bounds coming from Stiefel-Whitney class calculations apply to stable span, so they also apply to span since stable span is greater than or equal to span.

We recall ([2], [22]) that the canonical map $S^m \times \mathbb{C}P^n \to P(m,n)$ induces a double covering,

$$\mathbb{Z}_2 \longrightarrow S^m \times \mathbb{C}P^n \longrightarrow P(m,n). \tag{2.1}$$

The map $p\colon P(m,n) \to \mathbb{R}P^m$ induced by the projection $S^m \times \mathbb{C}P^n \to S^m$ defines a smooth fibre bundle,

$$\mathbb{C}P^n \longrightarrow P(m,n) \longrightarrow \mathbb{R}P^m. \tag{2.2}$$

The cohomology ring with coefficients in $\mathbb{Z}_2$ is given by

$$H^*(P(m,n),\mathbb{Z}_2) = \mathbb{Z}_2[c,d]/(c^{m+1} = d^{n+1} = 0), \text{ where } c \in H^1,\ d \in H^2.$$

For the total Stiefel-Whitney class we have

$$w(P(m,n)) = (1+c)^m(1+c+d)^{n+1}. \tag{2.3}$$

Since $\chi(\mathbb{C}P^n) = n+1$ and $\chi(S^m) = 1 + (-1)^m$, and we have the covering (2.1), for the Euler characteristic of the Dold manifold we readily derive that

$$\chi(P(m,n)) = \tfrac{1}{2}\chi(S^m)\chi(\mathbb{C}P^n) = \begin{cases} n+1 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases} \tag{2.4}$$

From (2.3) we obtain $w_1(P(m,n)) = (m+n+1)c$, so $P(m,n)$ is orientable if and only if $m+n$ is odd or $m=0$ (in the latter case, $c=0$).

As an initial piece of information on $\operatorname{span} P(m,n)$, we now obtain the following.

Proposition 2.5. *We always have*

$$\operatorname{span} P(m,n) \geq \operatorname{span} S^m.$$

Proof. This is readily seen, using the fibre bundle (2.2), [6, 3.1.6(1)] (if $F \to E \to B$ is a smooth fibre bundle, then $\operatorname{stabspan} E \geq \operatorname{stabspan} B$ and $\operatorname{span} E \geq \operatorname{span} B$), and recalling that $\operatorname{span} S^m = \operatorname{span} \mathbb{R}P^m$. ■

By (2.4), if m is even, then there is no everywhere nonzero vector field on $P(m,n)$. On the other hand, we have $\operatorname{span} P(m,n) \geq 1$ whenever m is odd (this also follows from Proposition 2.5). Hence, when interested in the span of $P(m,n)$, we shall concentrate our attention on the Dold manifolds $P(m,n)$ with m odd. For even values of m, the question about stable span can have meaning.

Now we present some upper bounds for (stab)span $P(m,n)$ using Stiefel-Whitney characteristic classes.

From (2.3) we obtain

$$w(P(m,n)) = \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{n+1}{j}\binom{m+n+1-j}{i} c^i d^j. \tag{2.6}$$

For the k-th Stiefel-Whitney class (for $k = 0,1,\ldots,D$), we have then the formula

$$w_k(P(m,n)) = \sum_{j=\max\{0,\lfloor (k-m)/2\rfloor\}}^{\min\{n,\lfloor k/2\rfloor\}} \binom{n+1}{j}\binom{m+n+1-j}{k-2j} c^{k-2j} d^j.$$

We recall one standard fact from number theory, which we shall use in the sequel.

Proposition 2.7 (Lucas' theorem)**.** *Let p be a prime and a, b be nonnegative integers with base p expansions*

$$a = a_0 + a_1 p + a_2 p^2 + \ldots,$$
$$b = b_0 + b_1 p + b_2 p^2 + \ldots \qquad (\textit{of course, } 0 \leq a_i < p,\ 0 \leq b_i < p).$$

Then

$$\binom{a}{b} \equiv \prod_{i=0}^{\infty} \binom{a_i}{b_i} \pmod p,$$

where $\binom{a_i}{b_i} = 0$ whenever $a_i < b_i$.

In particular, when $p=2$, then the number $\binom{a}{b}$ is even if and only if the binary expansion of b has the digit 1 at a position where the binary expansion of a has the digit 0. It will also be useful to denote, for a positive integer t, by $2^{\nu(t)}$ the highest power of 2 dividing t.

Using the obvious implication

$$w_{D-k}(P(m,n)) \neq 0 \quad \Longrightarrow \quad (\text{stab})\operatorname{span} P(m,n) \leq k, \tag{2.8}$$

we derive the following upper bound for the (stab)span of $P(m,n)$.

Theorem 2.9. *Given a pair (m,n) of nonnegative integers, write m in the form*

$$m = 2^{\nu(n+1)} \cdot k + l, \qquad \text{with } 0 \le l < 2^{\nu(n+1)}.$$

Then we have

$$\text{(stab)span}\, P(m,n) \le 2^{\nu(n+1)}(2^{\nu(k+1)} + 1) - 2.$$

Proof. The coefficient of $c^{m-2^{\nu(n+1)}(2^{\nu(k+1)}-1)} d^{n-(2^{\nu(n+1)}-1)}$ in the Stiefel-Whitney class

$$w_{m-2^{\nu(n+1)}(2^{\nu(k+1)}-1)+2(n-(2^{\nu(n+1)}-1))}(P(m,n))$$

is by (2.6) equal to

$$\binom{n+1}{n+1-2^{\nu(n+1)}}\binom{m+n+1-n-1+2^{\nu(n+1)}}{m-2^{\nu(n+1)}(2^{\nu(k+1)}-1)} = \binom{n+1}{2^{\nu(n+1)}}\binom{m+2^{\nu(n+1)}}{2^{\nu(n+1)+\nu(k+1)}}.$$

The first binomial coefficient is odd by Lucas' theorem. The second one can be transformed into

$$\binom{m+2^{\nu(n+1)}}{2^{\nu(n+1)+\nu(k+1)}} = \binom{2^{\nu(n+1)}\cdot(k+1)+l}{2^{\nu(n+1)+\nu(k+1)}} = \binom{2^{\nu(n+1)+\nu(k+1)}\cdot r + l}{2^{\nu(n+1)+\nu(k+1)}}$$

for some odd r, which also is odd by Lucas' theorem. Thus the Stiefel-Whitney class under consideration is nonzero, and by (2.8) we obtain

$$\begin{aligned}\text{(stab)span}\, P(m,n) &\le 2^{\nu(n+1)}(2^{\nu(k+1)} - 1) + 2(2^{\nu(n+1)} - 1) = \\ &= 2^{\nu(n+1)}(2^{\nu(k+1)} + 1) - 2.\end{aligned}$$

■

This upper bound leads us to the following observations.

Proposition 2.10. *If n is even, then*

$$\text{(stab)span}\, P(m,n) \le 2^{\nu(m+1)} - 1.$$

Proof. Using the notation from Theorem 2.9, we have $\nu(n+1) = 0$ and $k = m$, hence

$$\text{(stab)span}\, P(m,n) \le 2^0(2^{\nu(m+1)} + 1) - 2 = 2^{\nu(m+1)} - 1.$$

■

It is well known (J. F. Adams [1]) that span $S^m = \rho(m+1) - 1$, where $\rho(m+1)$ is the Hurwitz-Radon number defined by the formula $\rho(m+1) = 2^c + 8d$ for $\nu(m+1) = c + 4d$, $c, d \ge 0$, $c \le 3$. Therefore Propositions 2.10 and 2.5 imply the following.

Corollary 2.11. *If n is even and $\nu(m+1) \in \{1,2,3\}$, then*

$$\text{span}\, P(m,n) = \text{span}\, S^m = 2^{\nu(m+1)} - 1.$$

Proposition 2.12. *If $n \equiv 1 \pmod 4$ and m is odd (i. e., $\nu(m+1) > 0$), then*

$$\text{(stab)span}\, P(m,n) \le 2^{\nu(m+1)}.$$

Proof. Using the notation from Theorem 2.9, we have $\nu(n+1) = 1$, $k = \frac{m-1}{2}$ and $\nu(k+1) = \nu(m+1) - 1$, hence

$$\text{(stab)span}\, P(m,n) \le 2^1(2^{\nu(m+1)-1} + 1) - 2 = 2^{\nu(m+1)}.$$

■

In the same way as in 2.11, we obtain the following estimate, with upper and lower bounds differing by just one.

Corollary 2.13. *If* $n \equiv 1 \pmod 4$ *and* $\nu(m+1) \in \{1,2,3\}$, *then*

$$2^{\nu(m+1)} - 1 \le \operatorname{span} P(m,n) \le 2^{\nu(m+1)}.$$

For $n = 1$, we shall improve this result in Sec. 4 and prove that

$$\operatorname{span} P(m,1) = 2^{\nu(m+1)} \qquad \text{if } \nu(m+1) \in \{1,2,3\}.$$

Except for the cases described in 2.11 and the cases when m is even, it is impossible to derive $\operatorname{span} P(m,n) = \operatorname{span} S^m$, if true, using only Proposition 2.5 and the implication (2.8). Namely, the class $w_{D-\operatorname{span} S^m}(P(m,n))$ vanishes by the following theorem (recall that D denotes the dimension of $P(m,n)$).

Theorem 2.14. *Let* $\nu(m+1) > 0$ (*i.e.,* m *is odd*). *The Stiefel-Whitney classes* $w_{D-i}(P(m,n))$, $i = 0,1,\ldots,2^{\nu(m+1)}-2$, *are all zero. If* n *is odd, then also the class* $w_{D-(2^{\nu(m+1)}-1)}(P(m,n))$ *is zero.*

Proof. In $w_{D-i}(P(m,n))$, only the summands

$$c^{m-i}d^n, c^{m-i+2}d^{n-1}, c^{m-i+4}d^{n-2}, \ldots$$

can occur, i.e., the terms of the form $c^{m-i+2j}d^{n-j}$ with $j \ge 0$, $j \le n$, $2j \le i$. So we wish to show (see (2.6)) that the coefficients $\binom{n+1}{n-j}\binom{m+1+j}{m-i+2j}$ are even for $i = 0,1,\ldots,2^{\nu(m+1)}-2$ (or for i up to $2^{\nu(m+1)}-1$ for n odd) and for $j = 0,1,\ldots,\min\{n,\lfloor i/2\rfloor\}$. We proceed by induction on j. More precisely, we show that the second factor of the product of the binomial coefficients under consideration is even (except for the case when n is odd, $i = 2^{\nu(m+1)}-1$ and $j=0$, but then obviously the first factor is even).

For $j = 0$, the number we are interested in,

$$\binom{m+1}{m-i} = \binom{q\cdot 2^{\nu(m+1)}}{i+1} \qquad (q \text{ odd}),$$

is even for every $i = 0,1,\ldots,2^{\nu(m+1)}-2$ by Lucas' theorem.

For $j \ge 1$, $j \le \min\{n,\lfloor i/2\rfloor\}$ we have

$$\begin{aligned}\binom{m+1+j}{m-i+2j} &= \binom{m+1+(j-1)}{m-i+2j} + \binom{m+1+(j-1)}{m-i+2j-1} = \\ &= \binom{m+1+(j-1)}{m-(i-2)+2(j-1)} + \binom{m+1+(j-1)}{m-(i-1)+2(j-1)}.\end{aligned}$$

By the induction hypothesis, the latter two binomial coefficients are even. ■

Further, we recall the following two criteria for the existence of two or three everywhere independent vector fields.

Fact 2.15 ([13], [10]). *Let M be a smooth closed connected nonorientable manifold of dimension D with $D \equiv 3 \pmod 4$. If $D = 3$, then* $\operatorname{span} M \geq 2$ *if and only if* $w_1^2(M) = 0$. *If $D \geq 7$, then* $\operatorname{span} M \geq 2$ *if and only if* $w_1^2(M) = 0$ *or if* $w_1^2(M) \neq 0$ *and* $w_{D-1}(M) = 0$.

Fact 2.16 ([14]). *Let M be a smooth closed connected nonorientable manifold of dimension D with $D \equiv 3 \pmod 4$. If $D \geq 7$, then* $\operatorname{span} M \geq 3$ *if and only if* $\beta^* w_{n-3}(M) = 0$ (*where β^* is the Bockstein homomorphism*).

By applying 2.15 and 2.16, we obtain the following two results on $\operatorname{span} P(m,n)$.

Proposition 2.17. *If* $m \equiv n \equiv 1 \pmod 4$, *then*

$$\operatorname{span} P(m,n) = 2.$$

Proof. For $n \equiv 1 \pmod 4$ we can use Corollary 2.13. If $m \equiv 1 \pmod 4$, then we have $\nu(m+1) = 1$, hence $\operatorname{span} P(m,n) \leq 2$. For the dimension we have $D = 2n + m \equiv 3 \pmod 4$. Since $m+n$ is even and $m \neq 0$, $P(m,n)$ is nonorientable. By Theorem 2.14 we see that $w_{D-1} = 0$ (and in particular for $P(1,1)$, of dimension 3, we have $w_1^2 = 0$). Applying 2.15, we obtain $\operatorname{span} P(m,n) \geq 2$. ∎

Proposition 2.18. *If* $m \equiv 1 \pmod 4$ *and* $n \equiv 3 \pmod 4$, *then*

$$\operatorname{span} P(m,n) \geq 3.$$

Proof. For the dimension we have $D = 2n + m \equiv 3 \pmod 4$. Since $m+n$ is even, $P(m,n)$ is nonorientable and $D = 2n + m \geq 2 \cdot 3 + 1 = 7$. By Theorem 2.14, we see that $w_{D-3}(P(m,n)) = 0$, hence necessarily $\beta^* w_{D-3}(P(m,n)) = 0$. Applying 2.16 we obtain $\operatorname{span} P(m,n) \geq 3$. ∎

Our results obtained above are summarized in the table; for typographical reasons, we abbreviate $\operatorname{span} P(m,n) = s$.

m mod 16, n mod 4	$n \equiv 0$	$n \equiv 1$	$n \equiv 2$	$n \equiv 3$
$m \equiv 1$	$s = 1$	$s = 2$	$s = 1$	$s \geq 3$
$m \equiv 3$	$s = 3$	$s \in \{3,4\}$	$s = 3$	$s \geq 3$
$m \equiv 5$	$s = 1$	$s = 2$	$s = 1$	$s \geq 3$
$m \equiv 7$	$s = 7$	$s \in \{7,8\}$	$s = 7$	$s \geq 7$
$m \equiv 9$	$s = 1$	$s = 2$	$s = 1$	$s \geq 3$
$m \equiv 11$	$s = 3$	$s \in \{3,4\}$	$s = 3$	$s \geq 3$
$m \equiv 13$	$s = 1$	$s = 2$	$s = 1$	$s \geq 3$
$m \equiv 15$	$s \geq 8$	$s \geq 8$	$s \geq 8$	$s \geq 8$
$m \equiv 2t$ (all t)	$s = 0$	$s = 0$	$s = 0$	$s = 0$

3 Bounds for the stable span of $P(m,n)$ and further results on $\operatorname{span} P(m,n)$

By [6, 2.2(a)], $\operatorname{stabspan} M > 0$ if and only if $\chi(M)$ is even. Combining this with (2.4), if m is even, then we have that

$$\operatorname{stabspan} P(m,n) = \operatorname{span} P(m,n) = 0 \qquad \text{for } n \text{ even, and}$$
$$\operatorname{stabspan} P(m,n) > \operatorname{span} P(m,n) = 0 \qquad \text{for } n \text{ odd.}$$

Thus the stable span may or may not be equal to the span.

By [17], $\operatorname{stabspan} \mathbb{C}P^n = 2\nu(n+1)$. By [6, 3.1.6(2)], we have $\operatorname{stabspan} E \leq \operatorname{stabspan} F + \dim B$ whenever $F \to E \to B$ is a smooth fibre bundle with F connected. Applying this to the fibre bundle (2.2), we obtain

$$\operatorname{stabspan} P(m,n) \leq \operatorname{stabspan} \mathbb{C}P^n + \dim \mathbb{R}P^m = 2\nu(n+1) + m. \tag{3.1}$$

Of course, the number $2\nu(n+1)+m$ is also an upper bound for $\operatorname{span} P(m,n)$, and this coincides with the bound given in [4] (where it was derived using the KU-theory). With the notation from Theorem 2.9, we have

$$\begin{aligned} 2\nu(n+1) + m &= 2\nu(n+1) + 2^{\nu(n+1)} \cdot k + l = \\ &= 2\nu(n+1) + 2^{\nu(n+1)}(r \cdot 2^{\nu(k+1)} - 1) + l, \end{aligned}$$

with odd r, which is for $r \geq 3$ greater than $2^{\nu(n+1)}(2^{\nu(k+1)}+1)-2$. Hence the bound from Theorem 2.9 is better in this case. Only for $r = 1$, the bound (3.1) may be better. This happens (with $r = 1$) whenever

$$(2^{\nu(n+1)}(2^{\nu(k+1)} + 1) - 2) - (2\nu(n+1) + m) = 2^{\nu(n+1)+1} - 2 - 2\nu(n+1) - l > 0.$$

This condition is fulfilled, for example, for $\nu(n+1) \geq 3$ with any l, or for $\nu(n+1) = 2$ with $l < 2$.

We now compare the bound from Theorem 2.9 with the bound obtained from [16, 3.2], which says that

$$(\text{stab})\operatorname{span} P(m,n) \leq m + 2n - \delta^*(m,n). \tag{3.2}$$

Here, $\delta^*(m,n)$ is defined by

$$\delta^*(m,n) = \begin{cases} \max\{s, 2\left\lfloor \frac{n}{2} \right\rfloor\} & \text{if } m \neq 0, \\ 2\left\lfloor \frac{n}{2} \right\rfloor & \text{if } m = 0, \end{cases}$$

where s is the largest integer for which $2^{s-1}\binom{m+n+1}{s}$ is not divisible by $2^{\phi(m)}$ and $\phi(m)$ is the number of integers t with $0 < t \leq m$ and $t \equiv 0, 1, 2,$ or $4 \pmod 8$. Clearly $s \leq \phi(m) \leq \frac{m}{2} + 2$, so

$$\delta^*(m,n) \leq \max\{\tfrac{m}{2} + 2, 2\left\lfloor \tfrac{n}{2} \right\rfloor\} \leq \max\{\tfrac{m}{2} + 2, n\}.$$

For the bound in (3.2), we then have

$$m + 2n - \delta^*(m,n) \geq m + 2n - \max\{\tfrac{m}{2} + 2, n\} = \min\{\tfrac{m}{2} + 2n - 2, m + n\}.$$

With the notation from Theorem 2.9, we have $n = a \cdot 2^{\nu(n+1)} - 1$ for some odd a, $m = 2^{\nu(n+1)}k + l$, and $k = r \cdot 2^{\nu(k+1)} - 1$ for some odd r. Substituting into the latter expression, one can readily verify that the bound (3.2) may be better than our bound from Theorem 2.9 only for $r = 1$ (and also in that case, there are additional restrictions on $\nu(k+1)$, $\nu(n+1)$, a and l). Hence, in most cases, the bound from Theorem 2.9 is better.

For the tangent bundle τ of $P(m,n)$, one has ([22]) the formula $\tau \oplus \xi \oplus \varepsilon^2 = (m+1)\xi \oplus (n+1)\eta$, where ξ is the only nontrivial line bundle and η is a certain 2-plane bundle over $P(m,n)$. Obstruction theory (see [3, 8.1.5]) implies that we can cancel one ξ from both sides. In this way we obtain a full description of the stable tangent bundle,

$$\tau \oplus \varepsilon^2 = m\xi \oplus (n+1)\eta.$$

Let us consider the case $m = 1$. The above formula gives then $\tau \oplus \varepsilon^2 = \xi \oplus (n+1)\eta$. Since $\dim(n+1)\eta = 2n+2 > 2n+1 = \dim P(1,n)$, necessarily $\operatorname{span}((n+1)\eta) \geq 1$. By [8, 1.1], we have, for any vector bundle β over any paracompact space, that $\operatorname{span}(k\beta) \geq \rho(k)$ whenever $\operatorname{span}(k\beta) \geq 1$. So we obtain $\operatorname{span}((n+1)\eta) \geq \rho(n+1)$. Hence

$$\operatorname{span}(\tau \oplus \varepsilon^2) = \operatorname{span}(\xi \oplus (n+1)\eta) \geq \operatorname{span}((n+1)\eta) \geq \rho(n+1),$$

and consequently $\operatorname{stabspan} P(1,n) \geq \rho(n+1) - 2$. By [10, 20.4], for any closed connected manifold M of odd dimension D we have

$$\operatorname{span} M \geq \min\{(D-1)/2, s(M), \operatorname{stabspan} M\},$$

and by [10, 20.6], $s(M) \geq \operatorname{span} S^D$; see [10] for the definition of $s(M)$. Applying this to $P(1,n)$, we have

$$\begin{aligned} \operatorname{span} P(1,n) &\geq \min\left\{n, \operatorname{span} S^{2n+1}, \operatorname{stabspan} P(1,n)\right\} \geq \\ &\geq \min\{n, \rho(2n+2)-1, \rho(n+1)-2\} = \rho(n+1) - 2. \end{aligned}$$

For $\nu(n+1) = c + 4d$ ($0 \leq c \leq 3$, $d \geq 0$), by combining the preceding inequality and the bound (3.1), we obtain

$$2^c + 8d - 2 \leq \operatorname{span} P(1,n) \leq 2c + 8d + 1.$$

In other words, there are only the following possibilities for $\operatorname{span} P(1,n)$:

c	possibilities for $\operatorname{span} P(1,n)$ with $\nu(n+1) = c + 4d$
0	$8d-1$, $8d$, $8d+1$
1	$8d$, $8d+1$, $8d+2$, $8d+3$
2	$8d+2$, $8d+3$, $8d+4$, $8d+5$
3	$8d+6$, $8d+7$

This improves on some of our results from Sec. 2. Indeed, for instance, for $n = 7$ we have now $\operatorname{span} P(1,7) \in \{6,7\}$, whilst by Sec. 2, we only had $\operatorname{span} P(1,7) \geq 3$, and by (2.3), $w(P(1,7)) = 1 + c$, hence using (2.8), we only had $\operatorname{span} P(1,7) \leq 14$.

The approach used above, based on [8, 1.1], can also be applied in more general situations, to improve on some of the results previously achieved. Indeed, let $d_1 = \gcd(m, n+1)$, $d_2 = \gcd(m-1, n+1)$. Then

$$\tau \oplus \varepsilon^2 = m\xi \oplus (n+1)\eta = \underbrace{d_1\left(\tfrac{m}{d_1}\xi \oplus \tfrac{n+1}{d_1}\eta\right)}_{\dim = 2n+m+2},$$

and by [8, 1.1] $\operatorname{stabspan} P(m,n) \geq \rho(d_1) - 2$. Analogously,

$$\tau \oplus \varepsilon^2 = \xi \oplus (m-1)\xi \oplus (n+1)\eta = \xi \oplus \underbrace{d_2\left(\tfrac{m-1}{d_2}\xi \oplus \tfrac{n+1}{d_2}\eta\right)}_{\dim = 2n+m+1},$$

so $\operatorname{stabspan} P(m,n) \geq \rho(d_2) - 2$. For odd $m \geq 3$, this is also true for $\operatorname{span} P(m,n)$, because by [10, 20.4, 20.6] we obtain

$$\begin{aligned} \operatorname{span} P(m,n) &\geq \min\left\{\tfrac{m+2n-1}{2}, \operatorname{span} S^{m+2n}, \operatorname{stabspan} P(m,n)\right\} \geq \\ &\geq \min\Big\{\underbrace{\tfrac{(m-1)+(n+1)+(n-1)}{2}}_{\geq d_2 \geq \rho(d_2)}, \underbrace{\rho(m+2n+1)-1}_{\geq \rho(d_2)-1,\ \text{since } d_2 \mid m+2n+1}, \rho(d_2)-2\Big\} = \rho(d_2)-2. \end{aligned}$$

To show a concrete example of an improvement achieved in this way, for $P(9,7)$ we now obtain $\operatorname{span} P(9,7) \geq \rho(8) - 2 = 6$, whilst by Proposition 2.18 we only had $\operatorname{span} P(9,7) \geq 3$.

4 Vector fields on $P(m,1)$

By Proposition 2.17, if $m \equiv 1 \pmod 4$, then $\operatorname{span} P(m,1) = 2 = \operatorname{span} S^m + 1$. Now we show that $\operatorname{span} P(m,1) \geq \operatorname{span} S^m + 1$ also for other odd values of m. Hence, excluding the case $m \equiv 15 \pmod{16}$, by Corollary 2.13, we obtain $\operatorname{span} P(m,1) = \operatorname{span} S^m + 1$ for odd m.

Proposition 4.1. *There is a homeomorphism $g\colon \mathbb{C}P^1 \to S^2$ such that if $g(z_1,z_2) = (x_1,x_2,x_3)$, then $g(\bar{z}_1,\bar{z}_2) = (x_1,x_2,-x_3)$. (We take the sphere S^2 as the set $\{(x_1,x_2, x_3) \in \mathbb{R}^3;\ x_1^2+x_2^2+x_3^2 = 1\}$.)*

Proof. A suitable map is

$$g(z_1,z_2) = \left(\frac{2\operatorname{Re}(z_1\bar{z}_2)}{|z_1|^2+|z_2|^2}, \frac{|z_2|^2-|z_1|^2}{|z_1|^2+|z_2|^2}, \frac{2\operatorname{Im}(z_1\bar{z}_2)}{|z_1|^2+|z_2|^2}\right).$$

One readily verifies that $g(z_1,z_2) \in S^2$ and the given condition is fulfilled. Thus it suffices to show that g is a homeomorphism. Clearly g is continuous, hence we are done if we find any continuous map $f\colon S^2 \to \mathbb{C}P^1$ such that $f \circ g$ and $g \circ f$ are the identity maps. A suitable f is the map defined by

$$f(x_1,x_2,x_3) = \begin{cases} (x_1+ix_3, 1+x_2), & \text{for } (x_1,x_2,x_3) \neq (0,-1,0), \\ (1-x_2, x_1-ix_3), & \text{for } (x_1,x_2,x_3) \neq (0,1,0). \end{cases}$$

A short calculation shows that for $(x_1,x_2,x_3) \notin \{(0,-1,0),(0,1,0)\}$ both formulae agree (the values are in $\mathbb{C}P^1$), and f is well defined. It is continuous on the entire S^2 since it is continuous on the sets $S^2 \setminus (0,-1,0)$ and $S^2 \setminus (0,1,0)$ and agrees on their intersection. Finally, one verifies directly that $f \circ g$ and $g \circ f$ are identities. ■

Theorem 4.2. *Let $k \geq 1$ and $v_1,\ldots,v_k\colon S^m \to TS^m$ be vector fields linearly independent at every point of S^m and equivariant with respect to the relation $x \sim -x$. Then there are at least $k+1$ vector fields on $S^m \times S^2$ linearly independent at every point and equivariant with respect to the relation $(x,(x_1,x_2,x_3)) \sim (-x,(x_1,x_2,-x_3))$.*

Proof. Take the vector fields $w_1, \dots, w_{k+1}\colon S^m \times S^2 \to T(S^m \times S^2)$ on $S^m \times S^2$ defined by

$$
\begin{aligned}
w_i(x,(x_1,x_2,x_3)) &= (v_i(x),(0,0,0)) \quad \text{for } i = 1,\dots,k-1,\\
w_k(x,(x_1,x_2,x_3)) &= (x_1 v_k(x),(x_1^2-1, x_1x_2, x_1x_3)),\\
w_{k+1}(x,(x_1,x_2,x_3)) &= (x_2 v_k(x),(x_1x_2, x_2^2-1, x_2x_3)).
\end{aligned}
$$

As can be readily verified, these maps actually are smooth vector fields on $S^m \times S^2$. They also are linearly independent at every point. Indeed, to prove this, if they are linearly dependent at a point $(x,(x_1,x_2,x_3))$, there are some coefficients $\alpha_1,\dots,\alpha_{k+1} \in \mathbb{R}$, not all zero, such that $\sum_{i=1}^{k+1} \alpha_i w_i = \vec{0}$. From this obviously

$$
\begin{aligned}
&\vec{0} = \sum_{i=1}^{k-1} \alpha_i v_i(x) + (\alpha_k x_1 + \alpha_{k+1}x_2) v_k(x),\\
&0 = \alpha_k(x_1^2-1) + \alpha_{k+1}x_1x_2 = x_1(\alpha_k x_1 + \alpha_{k+1}x_2) - \alpha_k,\\
&0 = \alpha_k x_1x_2 + \alpha_{k+1}(x_2^2-1) = x_2(\alpha_k x_1 + \alpha_{k+1}x_2) - \alpha_{k+1},\\
&0 = \alpha_k x_1x_3 + \alpha_{k+1}x_2x_3.
\end{aligned}
$$

Since $v_1,\dots,v_k$ are independent at every point, it necessarily follows from the first equality, that $\alpha_1 = \dots = \alpha_{k-1} = \alpha_k x_1 + \alpha_{k+1}x_2 = 0$. Substituting this into the second and third equality, we have $\alpha_k = \alpha_{k+1} = 0$. The contradiction proves our claim.

We are left with verifying the equivariance of our vector fields with respect to the relation $(x,z) \sim \Psi(x,z)$, where $z = (x_1,x_2,x_3)$, $\Psi(x,z) = (-x,\psi(z))$, $\psi(x_1,x_2,x_3) = (x_1,x_2,-x_3)$. By our assumption, the fields v_i are equivariant with respect to the relation $x \sim -x$. This means, that we can take a smooth curve $\gamma_x^i\colon \mathbb{R} \to S^m$ passing through $x = \gamma_x^i(0)$ with velocity $v_i(x)$, and the curve $\gamma_{-x}^i\colon \mathbb{R} \to S^m$, defined by $\gamma_{-x}^i(t) = -\gamma_x^i(t)$, then passes through $-x$ with velocity $v_i(-x)$.

Clearly, for $i = 1,2,\dots,k-1$, the vector $w_i(x,z)$ is the velocity (at $t=0$) of the curve $\Gamma_{(x,z)}^i\colon \mathbb{R} \to S^m \times S^2$, $\Gamma_{(x,z)}^i(t) = (\gamma_x^i(t), z)$. So we have

$$
\begin{aligned}
\Psi(\Gamma_{(x,z)}^i(t)) = \Psi(\gamma_x^i(t),z) = (-\gamma_x^i(t),\psi(z)) = (\gamma_{-x}^i(t),\psi(z)) =\\
= \Gamma_{(-x,\psi(z))}^i(t) = \Gamma_{\Psi(x,z)}^i(t).
\end{aligned}
$$

Hence, for $i = 1,2,\dots,k-1$, the vector fields w_i are equivariant.

The vector $w_k(x,z)$ is the velocity (at $t=0$) of the curve $\Gamma_{(x,z)}^k\colon \mathbb{R} \to S^m \times S^2$, $\Gamma_{(x,z)}^k(t) = (\gamma_x^k(x_1t), \beta_z(t))$, where $\beta_z\colon \mathbb{R} \to S^2 \subset \mathbb{R}^3$ is a curve passing through $(x_1,x_2,x_3) = z = \beta_z(0) \in S^2$ with velocity $(x_1^2-1, x_1x_2, x_1x_3)$. In other words, we have $\frac{d\beta_z}{dt}(0) = (x_1^2-1, x_1x_2, x_1x_3)$. For every point $(x_1,x_2,x_3) = z \in S^2$, the vector

$$
\begin{aligned}
\frac{d(\psi(\beta_z))}{dt}(0) = d\psi(\beta_z(0)) \cdot \frac{d\beta_z}{dt}(0) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1 \end{pmatrix} \cdot (x_1^2, x_1x_2, x_1x_3)^T =\\
= (x_1^2, x_1x_2, -x_1x_3)^T = \frac{d\beta_{\psi(z)}}{dt}(0)
\end{aligned}
$$

is the velocity, at $t = 0$, of the curve $\psi(\beta_z(t))$. Hence the flow on S^2 determined by the curves $\psi(\beta_z)$ generates the same vector field as the flow determined by the curves $\beta_{\psi(z)}$. Necessarily, $\psi(\beta_z(t)) = \beta_{\psi(z)}(t)$, whence

$$\begin{aligned}\Psi(\Gamma^k_{(x,z)}(t)) = \Psi(\gamma^k_x(x_1t), \beta_z(t)) = (-\gamma^k_x(x_1t), \psi(\beta_z(t))) = \\ = (\gamma^k_{-x}(x_1t), \beta_{\psi(z)}(t)) = \Gamma^k_{(-x,\psi(z))}(t) = \Gamma^k_{\Psi(x,z)}(t).\end{aligned}$$

Thus w_k is equivariant. The equivariance of the vector field w_{k+1} can be shown analogously. ■

Corollary 4.3. *If m is odd, then* $\operatorname{span} P(m,1) \geq \operatorname{span} S^m + 1$.

Proof. It is well known for odd m, that there are k everywhere linearly independent vector fields on S^m with $k = \operatorname{span} S^m \geq 1$. Moreover, these vector fields can be taken to be "linear", so we may assume they are equivariant with respect to $x \sim -x$. Combining 4.2 and 4.1, we produce $k+1$ everywhere independent vector fields on $P(m,1)$, hence we have $\operatorname{span} P(m,1) \geq k+1 = \operatorname{span} S^m + 1$. ■

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