Continuous convexity and canonical partial metrics in normed spaces

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Abstract

The aim of this paper is to study the canonical partial metric associated to the norm of a normed space, whose related non-translation-invariant topology can be used to characterize the convexity properties of the original space. In order to do this, we define and characterize a new intermediate geometric property that we call continuous convexity, which appears in a natural way in the context of the canonical partial metric topology.

1 Introduction

The notion of partial metric was introduced by Matthews in [7] as a part of the study of programming language semantics. Later on, this concept was also used in other contexts, as in the complexity analysis of algorithms and programs (see [3, 5, 10, 11, 13, 14]). Actually, several structures that provide models for the complexity analysis can be identified with certain subsets of linear spaces with particular topologies. However, these examples do not fit with the usual scheme of topological linear spaces. For instance, the topologies defined in this context are not in general Hausdorff, or they are non-translation-invariant (see [1, 4, 8, 9]).

In this paper we show that partial metrics also provide an adequate framework to define new topologies on linear spaces that can be used to investigate geometrical properties of normed spaces. Our motivation is given by the fact that strict convexity

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and related geometric notions of the theory of Banach spaces can be characterized by comparing topologies on linear spaces which do not provide topological linear space structures. Some preliminary results in this direction can be found in [9]; in this paper we continue this research by introducing and characterizing the notion of continuous convexity. We conclude in this way our investigation on the geometrical applications of partial metrics defined on linear spaces.

We will present these ideas in three sections. Section 1 is devoted to recall several definitions and basic results on partial metrics. In Section 2 we define the canonical partial metric associated to a norm that induces the topology of the space and prove several fundamental results. Since some of these results also work for the case of translation invariant topologies defined by metrics, we also explain the suitable extension of these ideas when we replace norms by a more general class of functions φ associated to distances (Remark 2.4). Finally, in Section 3 we characterize the notion of continuous convexity in terms of the convexity properties of the norm using the canonical partial metric. Our aim is to show that it provides a natural way to define a (non-translation-invariant) topology on normed spaces that in fact contains all the information about the convexity properties of the norm. Actually, we show that convergence properties of sequences with respect to the canonical partial metric can be used to characterize strict and uniform convexity; the notion of continuous convexity is the main tool for our analysis.

In all the paper, R will denote the set of real numbers, R^+ the set of the non-negative real numbers, and N the set of natural numbers. A *quasi-metric* on a set X is a function $d : X \times X \to R^+$ such that for all $x, y, z \in X$:

(1) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$;

(2) $d(x, y) \leq d(x, z) + d(z, y)$.

If d is a quasi-metric on X, we say that (X, d) is a quasi-metric space. If $d(x, y) = d(y, x) = 0$ does not necessarily imply $x = y$ in the definition above, then d is said to be a *quasi-pseudo-metric*. In this case we say that (X, d) is a *quasi-pseudo-metric space*. A quasi-pseudo-metric d generates a topology $\mathcal{T}(d)$ on X which has as a base the family of open d-balls

$$B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}, \quad x \in X, \ \varepsilon > 0.$$

Note that if d is a quasi-metric, the function $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ is a metric on X. In this case, $\mathcal{T}(d)$ is clearly a T_0-topology on X.

Definition 1.1. *A partial metric on X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$,*

(1) $p(x, y) = p(x, x) = p(y, y) \Leftrightarrow x = y$,

(2) $p(x, x) \leq p(x, y)$,

(3) $p(x, y) = p(y, x)$, *and*

(4) $p(x, y) + p(z, z) \leq p(x, z) + p(z, y)$.

If the equalities of (1) hold for some x and y which are not necessarily equal, we say that p is a partial pseudo-metric. Each partial pseudo-metric defines a quasi-pseudo-metric by means of the formula $d(x, y) := p(x, y) - p(x, x)$, and the topology generated by p is the one given by d, i.e. by the base $\mathcal{B} = \{V_{\varepsilon,p}(x) : x \in X, \varepsilon > 0\}$, where

$$V_{\varepsilon,p}(x) = \{y \in X : p(x,y) < \varepsilon + p(x,x)\}$$

(see [2]). Clearly, if p is a partial metric then d is a quasi-metric. If p is a partial pseudo-metric on X we say that (X,p) is a partial pseudo-metric space (a partial metric space if p is a partial metric). If X is a linear space, we will also call (X,p) a partial pseudo-metric linear space. If x is an element of a linear space X, we will write $[0,x]$ for the set $\{\alpha x \mid 0 \leq \alpha \leq 1\}$.

Through the paper the set X will be a real linear space. We will use several definitions of the classical framework of the Functional Analysis. Our basic reference is [12]. For notions related to convexity in Banach spaces and lattices we refer to [6]. If φ is a norm on X, $\varepsilon > 0$ and $x \in X$, we denote by $B_{\varepsilon,\varphi}(x)$ the open ball $\{y \in X | \, \varphi(x-y) < \varepsilon\}$.

The following lemma, which is a direct consequence of Lemma 2.2 in [7], gives information about the basic sets $V_{\varepsilon,p}(x)$ that will be useful in this paper.

Lemma 1.2. *Let $x \in X$. Then $\mathcal{B} = \{V_{\varepsilon,p}(x) : \varepsilon > 0\}$ is a base of neighborhoods of x for the space (X,p).*

2 The canonical partial metric topology

In this section we define and characterize the canonical partial metric p_φ associated to a norm φ.

Lemma 2.1. *Every norm φ on a linear space X defines a partial metric p_φ by means of the formula*

$$p_\varphi(x,y) := \varphi(x-y) + \varphi(x) + \varphi(y), \qquad x,y \in X.$$

Moreover, p_φ satisfies

(1) $p_\varphi(x,0) = p_\varphi(-x,0)$ *for every $x \in X$ and*

(2) $p_\varphi(x,y) = 0$ *if and only if $x = y = 0$.*

Proof. Consider the function p_φ defined as above. The following calculations show that p_φ is a partial metric. For (1) in Definition 1.1, note that $p_\varphi(x,x) = p_\varphi(x,y) = p_\varphi(y,y)$, if and only if

$$2\varphi(x) = \varphi(x-y) + \varphi(x) + \varphi(y) = 2\varphi(y),$$

i.e. $\varphi(x-y)+\varphi(x)-\varphi(y) = 0$ and $\varphi(x-y)-\varphi(x)+\varphi(y) = 0$. This is also equivalent to $\varphi(x-y) = 0$. Since φ is a norm, it follows that $p_\varphi(x,x) = p_\varphi(x,y) = p_\varphi(y,y)$ if and only if $x = y$.

Let be $x,y \in X$. An easy calculation shows that $p_\varphi(x,x) \leq p_\varphi(x,y)$. It is also clear that $p_\varphi(x,y) = p_\varphi(y,x)$ for every $x,y \in X$, as a consequence of the equality $\varphi(z) = \varphi(-z)$ for every $z \in X$. To prove the last condition in Definition 1.1, consider $x,y,z \in X$. Then

$$p_\varphi(x,y) + p_\varphi(z,z) = \varphi(x-y) + \varphi(x) + \varphi(y) + 2\varphi(z)$$

$$\leq \varphi(x - z + z - y) + \varphi(x) + \varphi(y) + 2\varphi(z)$$
$$\leq \varphi(x - z) + \varphi(x) + \varphi(z) + \varphi(z - y) + \varphi(y) + \varphi(z) = p_\varphi(x, z) + p_\varphi(y, z).$$

So, p_φ defines a partial metric. The statements (1) and (2) follow as direct consequences of the properties of the norms. ∎

We will call the partial metric p_φ the *canonical partial metric associated to a norm* φ, and we will denote by τ_{p_φ} the topology generated by the quasi-metric associated to p_φ. An analogous definition has been made by Matthews in the last section of [7]. Some preliminary results regarding the canonical partial metric associated to a norm can be found in [9].

Remark 2.2. *The definition of the basic neighborhoods of* 0 *given by* p_φ *makes clear that the local structure at* 0 *is equivalent for* τ_{p_φ} *and for the norm topology* τ_φ. *It is also easy to see that this is not the case for elements* $x \in X$ *such that* $x \neq 0$.

The norm topology τ_φ is finer than τ_{p_φ}. To see this, consider $\varepsilon > 0$, $x \in X$ and the neighborhood $V_{\varepsilon,p_\varphi}(x) = \{y : \varphi(x - y) + \varphi(y) < \varphi(x) + \varepsilon\}$. Let us show that $B_{\frac{\varepsilon}{2},\varphi}(x) \subset V_{\varepsilon,p_\varphi}(x)$. Take $y \in B_{\frac{\varepsilon}{2},\varphi}(x)$. We can find $z \in B_{\frac{\varepsilon}{2},\varphi}(0)$ such that $y = x + z$. Then $\varphi(x - (x + z)) + \varphi(x + z) < \frac{\varepsilon}{2} + \varphi(x) + \frac{\varepsilon}{2}$, and we obtain the result.

Let us finish this section providing a representation theorem for topologies with a particular local structure by means of the canonical partial metric.

Theorem 2.3. *Let* X *be a linear space, let* τ *be a topology on* X *and let* φ *be a norm on* X. *The following statements are equivalent.*

(1) *For every* $x \in X$, *the family of subsets*

$$\cup_{\alpha+\beta=\varphi(x)+\varepsilon}(B_{\alpha,\varphi}(0) \cap B_{\beta,\varphi}(x)), \qquad \varepsilon > 0,\ \alpha \geq 0,\ \beta \geq 0,$$

defines a base of neighborhoods of x *for the topology* τ.

(2) $\tau = \tau_{p_\varphi}$.

Proof. Let us prove first (1) → (2). Consider the canonical partial metric p_φ. Take $x \in X$ and $\varepsilon > 0$. Since

$$V_{\varepsilon,p_\varphi}(x) = \{y \in X : p_\varphi(x, y) < \varepsilon + p_\varphi(x, x)\} = \{y \in X : \varphi(y - x) + \varphi(y) < \varphi(x) + \varepsilon\},$$

we have that for each element $y \in X$ such that there are $\alpha \geq 0$ and $\beta \geq 0$ satisfying $\varphi(y) < \alpha$, $\varphi(y - x) < \beta$, and $\alpha + \beta = \varphi(x) + \varepsilon$, we obtain that $y \in V_{\varepsilon,p_\varphi}(x)$. This proves $\cup_{\alpha+\beta=\varphi(x)+\varepsilon}(B_{\alpha,\varphi}(0) \cap B_{\beta,\varphi}(x)) \subset V_{\varepsilon,p_\varphi}(x)$. The reverse inclusion is given by the following argument. Consider an element $y \in V_{\varepsilon,p_\varphi}(x)$, i.e. $\varphi(x - y) + \varphi(y) < \varphi(x) + \varepsilon$. There are always $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha + \beta < \varphi(x) + \varepsilon$, $\varphi(y) < \alpha$ and $\varphi(x - y) < \beta$. It follows that $y \in B_{\alpha,\varphi}(0) \cap B_{\beta,\varphi}(x)$ and then

$$y \in \cup_{\alpha+\beta=\varphi(x)+\varepsilon}(B_{\alpha,\varphi}(0) \cap B_{\beta,\varphi}(x)), \qquad \text{for every} \quad \varepsilon > 0.$$

Thus, τ and τ_{p_φ} coincide.

For the converse, if $\tau = \tau_{p_\varphi}$, then for every $x \in X$ the sets $V_{\varepsilon,p_\varphi}(x)$ define a base of neighborhoods of x for the topology τ. A direct calculation using the arguments of the paragraph above gives that $V_{\varepsilon,p_\varphi}(x) = \cup_{\alpha+\beta=\varphi(x)+\varepsilon}(B_{\alpha,\varphi}(0) \cap B_{\beta,\varphi}(x))$ for every $\varepsilon > 0$. ∎

Remark 2.4. *The construction above can also be considered in the case that the topology on the linear space is given by a metric. If* $\varphi : X \to R^+$ *is just a real function that satisfies that for every* $x, y \in X$*,* $\varphi(x+y) \leq \varphi(x)+\varphi(y)$*,* $\varphi(x) = \varphi(-x)$*, and* $\varphi(x) = 0$ *if and only if* $x = 0$ *—i.e., if we do not require homogeneity to the function* φ*— the same definition of* p_φ *provides a topology on* X*. A natural way of defining such a function* φ *is by using a metric* $d : X \times X \to R^+$ *that satisfies* $d(x, 0) = d(-x, 0)$*,* $x \in X$*. But even if we define a translation invariant topology generated by such a function just considering the neighborhoods of* $x \in X$ *given by the sets* $B_{\varepsilon,\varphi}(x) = x + B_{\varepsilon,\varphi}(0)$ *—where* $B_{\varepsilon,\varphi}(0) := \{y \in X : \varphi(y) < \varepsilon\}$*— we do not obtain in general a topological linear space structure for* X*; the continuity of the linear operations may fail, as can be easily seen by considering the function* φ *given by the discrete metric on* 0*. Other example of a function* φ *that satisfies the requirements above and is not a norm is the following. Let* $(X, \|.\|)$ *be a normed space and consider the function given by* $\varphi(x) = \log(\|x\| + 1)$*,* $x \in X$*; the triangle inequality of* φ *is a consequence of*

$$\|x+y\|+1 \leq \|x\|+\|y\|+1 \leq \|x\|+\|y\|+1+\|x\|\|y\| = (\|x\|+1)(\|y\|+1), \qquad x, y \in X,$$

and the other requirements clearly hold. Other examples of functions φ *are given by topological linear spaces whose topology is described by a countable base of neighborhoods of* 0*; it can be used to define a balanced (countable) local base that provides such a function* φ *following a standard construction (see for instance the proof of Th.1.24 in [12]).*

3 Continuous convexity in normed spaces

In this section we develop a new geometrical concept that is related to the convexity properties of the norm. We will call it continuous convexity, since it involves convergence of sequences with respect to the canonical partial metric topology and the norm topology. We will also show the relations between this concept and the classical convexity properties: strict convexity and uniform convexity. If $(x_n)_n$ is a sequence in X, we will say that it *p_φ-converges to* $x \in X$ if $\lim_n p_\varphi(x, x_n) = p_\varphi(x, x)$. We also write in this case $x \in p_\varphi - \lim_n x_n$.

Definition 3.1. *Let* φ *be a norm on a linear space* X*. Let* $x \in X$*. We say that* φ *is continuously convex at* x *if every sequence that* p_φ*-converges to* x *satisfies that there is a subsequence that* φ*-converges to an element* $y \in [0, x]$*. We will say that* φ *is continuously convex if it is continuously convex at* x *for every* $x \in X$*. We will also say that the normed space* (X, φ) *is continuously convex.*

Simple examples of norms that satisfy this property are given by the natural norms of the finite dimensional spaces ℓ_p^n, $n \in N$, $1 < p < \infty$, i.e. the n-dimensional spaces R^n endowed with the usual p-norm. Since every p_φ-convergent subsequence is bounded, there is a subsequence that converges in the norm topology (recall that each bounded subset of a finite dimensional normed space is relatively compact). But the limit y of this subsequence satisfies $\varphi(x - y) + \varphi(y) = \varphi(x)$. Therefore, if the space is strictly convex —and this happens whenever $1 < p < \infty$, see Definition 3.7 below— we obtain that $y = \alpha x$ for some $0 \leq \alpha \leq 1$.

Proposition 3.2. *Let φ be a norm on X and $x \in X$. The following statements are equivalent.*

(1) *φ is continuously convex at x.*

(2) *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\varphi(x - y) + \varphi(y) < \varphi(x) + \delta$ implies*
$$\inf\{\varphi(\alpha x - y) : 0 \le \alpha \le 1\} < \varepsilon$$
for every $y \in X$.

Proof. Let us show first (2) $\to$ (1). Consider a sequence $(y_n)_n$ that p_φ-converges to $x \in X$, i.e. $x \in p_\varphi - \lim_n y_n$. Note that for each pair $x, y \in X$ the function $f_{x,y} : [0,1] \to R^+$ given by $f_{x,y}(\alpha) := \varphi(\alpha x - y)$ is continuous; then f_{x,y_n} is continuous for each $n \in N$; thus, there is a sequence $(\alpha_n)_n$ in $[0,1]$ such that
$$\varphi(\alpha_n x - y_n) = \inf\{\varphi(\alpha x - y_n) : 0 \le \alpha \le 1\}.$$
Note that there are also a subsequence $(\alpha_{n'})_{n'}$ of $(\alpha_n)_n$ and an $\alpha_0 \in [0,1]$ such that $\alpha_{n'} \to \alpha_0$, and so $\varphi(\alpha_0 x - \alpha_{n'} x) \to 0$.

Now choose an arbitrary $\varepsilon > 0$. Then for its associated $\delta \in]0, \varepsilon[$ of condition (2), there is $n'_0 \in N$ such that for $n' \ge n'_0$,
$$\varphi(\alpha_0 x - \alpha_{n'} x) = |\alpha_0 - \alpha_{n'}|\varphi(x) < \delta \quad \text{and} \quad \varphi(x - y_{n'}) + \varphi(y_{n'}) < \varphi(x) + \delta.$$
It follows from the assumption that $\varphi(\alpha_{n'} x - y_{n'}) < \varepsilon$ and hence
$$\varphi(\alpha_0 x - y_{n'}) \le \varphi((\alpha_0 - \alpha_{n'})x) + \varphi(\alpha_{n'} x - y_{n'}) < \delta + \varepsilon < 2\varepsilon,$$
whenever $n' \ge n'_0$. So that the subsequence $(y_{n'})_{n'}$ converges to $\alpha_0 x \in [0, x]$.

To prove that (1) $\to$ (2) we check that if the condition (2) does not hold, then neither does (1). Suppose that (2) does not hold. Then there is $\varepsilon_0 > 0$ such that for every $\delta > 0$ there is y_δ that satisfies that
$$\varphi(x - y_\delta) + \varphi(y_\delta) < \varphi(x) + \delta \quad \text{but} \quad \varphi(\alpha_\delta x - y_\delta) > \varepsilon_0,$$
where α_δ is the element of the interval $[0,1]$ for which $\inf\{\varphi(\alpha x - y_\delta) : 0 \le \alpha \le 1\}$ is attained. Note that this α_δ exists as a consequence of the fact that for every $x, y \in X$ the function $f_{x,y}$ defined at the beginning of the proof is continuous.

Take now $\delta = \frac{1}{n}$ for every $n \in N$. Then the sequence $(y_n)_n$ defined in this way p_φ-converges to x. But for every α_0 and $n \in N$,
$$\varphi(\alpha_0 x - y_n) \ge \inf\{\varphi(\alpha x - y_n) : 0 \le \alpha \le 1\} > \varepsilon_0.$$
Then there is no α_0 such that $\alpha_0 x = \lim_{n'} y_{n'}$ for any subsequence $(y_{n'})_{n'}$ of $(y_n)_n$. ■

In what follows we characterize continuous convexity in terms of the base of neighborhoods of the topology τ_{p_φ} given by the sets $V_{\varepsilon, p_\varphi}$. We need to introduce a new type of topology on linear spaces.

Definition 3.3. *Let τ be a topology on a linear space X. We say that τ is a radial topology if*

(1) 0 *has a countable local base* $\mathcal{V} = \{V_\eta(0) : \eta \in N\}$, *and*

(2) *for every* $x \in X$, *the family*

$$\cup_{0 \leq \alpha \leq 1}(\alpha x + V_\eta(0)), \qquad \eta \in N,$$

defines a local base of x.

Lemma 3.4. *Let φ be a norm on a linear space X. Then the class of families*

$$\mathcal{R}_x = \{R_\varepsilon(x) := \cup_{0 \leq \alpha \leq 1}(\alpha x + B_{\varepsilon,\varphi}(0)) \mid \varepsilon > 0\}$$

when considered as a base of neighborhoods for each $x \in X$ defines a radial topology.

Proof. It is enough to prove that for every $x \in X$ and $\varepsilon > 0$, every $y \in R_{\frac{\varepsilon}{2}}(x)$ satisfies that $R_{\frac{\varepsilon}{2}}(y) \subset R_\varepsilon(x)$. Take $y \in R_{\frac{\varepsilon}{2}}(x)$ and $z \in R_{\frac{\varepsilon}{2}}(y)$. Then there are $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, and $v, w \in X$ such that $y = \alpha x + v$ and $z = \beta y + w$, and $\varphi(v) < \frac{\varepsilon}{2}$, $\varphi(w) < \frac{\varepsilon}{2}$. Note that

$$z = \beta y + w = \beta(\alpha x + v) + w = \beta\alpha x + (\beta v + w).$$

Since $0 \leq \beta\alpha \leq 1$ and $\varphi(\beta v + w) \leq \varphi(\beta v) + \varphi(w) \leq \beta\varphi(v) + \varphi(w) < \varepsilon$, we have a representation of z as $\gamma x + r$, where $\gamma := \beta\alpha$ and $r := \beta v + w$. Since $\varphi(r) < \varepsilon$ we obtain the result. ■

Lemma 3.5. *Let φ be a norm on a linear space X. For every $x \in X$ and $\varepsilon > 0$,* $R_{\frac{\varepsilon}{2}}(x) \subset V_{\varepsilon,p_\varphi}(x)$.

We omit the straightforward proof. The following result gives a characterization of continuous convexity for a normed space as a geometric condition. This will be also useful to motivate the introduction of uniform convexity as an associated property.

Proposition 3.6. *Let (X, φ) be a normed space. The following statements are equivalent.*

(1) (X, φ) *is continuously convex.*

(2) *For every $x \in X$ and every $\varepsilon > 0$ there exists a $\delta(x) > 0$ such that*

$$\varphi(x - y) + \varphi(y) < \varphi(x) + \delta(x)$$

implies $\inf\{\varphi(\alpha x - y) : 0 \leq \alpha \leq 1\} < \varepsilon$ *for every* $y \in X$.

(3) *For every $z \in X$ and for every $\varepsilon > 0$ there is a $\delta(z) > 0$ such that for every $x, y \in X$ such that $x + y = z$, if $\varphi(x) + \varphi(y) < \varphi(z) + \delta(z)$ then*

$$\inf\{\varphi(\alpha x - (1 - \alpha)y) : 0 \leq \alpha \leq 1\} < \varepsilon.$$

(4) *The topology τ_{p_φ} is radial.*

Proof. The equivalence between (1) and (2) is given by applying Proposition 3.2 for every $x \in X$. Let us show now that (2) implies (3). Let $z \in X$ and $\varepsilon > 0$. Consider a decomposition $z = x + y$. If we apply (2) to the element z, we obtain that there exists a δ (depending on z and ε) such that for every $v \in X$ satisfying $\varphi(z - v) + \varphi(v) < \varphi(z) + \delta$, we have $\inf\{\varphi(\alpha z - v) : 0 \leq \alpha \leq 1\} < \varepsilon$. Now take $y = v = z - x$ and $\delta(z) = \delta$, and suppose that $\varphi(x) + \varphi(y) < \varphi(z) + \delta(z)$. Then $\varphi(z - v) + \varphi(v) < \varphi(z) + \delta$. Consequently, we obtain

$$\varepsilon > \inf\{\varphi(\alpha z - v) : 0 \leq \alpha \leq 1\} = \inf\{\varphi(\alpha(x + y) - y) : 0 \leq \alpha \leq 1\}$$
$$= \inf\{\varphi(\alpha x - (1 - \alpha)y) : 0 \leq \alpha \leq 1\}.$$

For the converse, let $x \in X$ and $\varepsilon > 0$. Take $z = x$ in (3). Then there is a $\delta(x) > 0$ such that if $y \in X$ satisfies that $\varphi(x - y) + \varphi(y) < \varphi(x) + \delta(x)$, then

$$\inf\{\varphi(\alpha(x - y) - (1 - \alpha)y) : 0 \leq \alpha \leq 1\} < \varepsilon.$$

But this inequality can be rewritten as $\inf\{\varphi(\alpha x - y) : 0 \leq \alpha \leq 1\} < \varepsilon$. This proves that (3) implies (2).

Let us prove now that (2) implies (4). Note that the sets $\{B_{\varepsilon,\varphi}(0) : \varepsilon > 0\}$ define a local base at 0 for the topology τ_{p_φ}, since $p_\varphi(x, x) = 2\varphi(x)$ for every $x \in X$. Suppose that φ satisfies the requirements of the statement. Let $x \in X$ and $\varepsilon > 0$. As a consequence of Lemma 3.5, we only need to prove that there is a $\delta > 0$ such that $V_{\delta,p_\varphi}(x) \subset R_\varepsilon(x)$. Since φ satisfies (2), we have that there is $\delta > 0$ such that, if y satisfies $\varphi(x - y) + \varphi(y) < \varphi(x) + \delta$, then there is $0 \leq \alpha \leq 1$ such that $\varphi(y - \alpha x) < \varepsilon$. Then y can be written as $y = \alpha x + z$, where $\varphi(z) < \varepsilon$. This proves the desired inclusion.

For the converse, suppose that the topology is radial and fix $x \in X$. Since the family of subsets $\{V_{\gamma,p_\varphi}(x) : \gamma > 0\}$ is a local base for τ_{p_φ} in x, for every $R_\varepsilon(x)$ there is a $\delta > 0$ such that $V_{\delta,p_\varphi}(x) \subset R_\varepsilon(x)$. Consider $y \in V_{\delta,p_\varphi}(x) \subset R_\varepsilon(x)$. Then $\varphi(x - y) + \varphi(y) < \delta + \varphi(x)$ implies that for an $\alpha_0 \in [0, 1]$, $y - \alpha_0 x \in B_{\varepsilon,\varphi}(0)$. Then

$$\inf\{\varphi(\alpha x - y) : 0 \leq \alpha \leq 1\} \leq \varphi(\alpha x_0 - y) < \varepsilon.$$

This finishes the proof. ∎

In the rest of the paper we analyze the relation between strict convexity, uniform convexity and continuous convexity. We will introduce a new sequential property for norms on linear spaces that is closely related to the definition of continuous convexity. Let us recall first several classical definitions and characterizations.

Definition 3.7. *We say that a norm φ is strictly convex if for every $x, y \in X$, if $\varphi(x) \neq 0$ and $\varphi(x + y) = \varphi(x) + \varphi(y)$, then $y \in \langle x\rangle$.*

It is well known that the definition above is equivalent to the following condition. We say that the norm φ —equivalently, the normed space (X, φ)— is strictly convex if for every pair of norm one elements $x, y \in X$, $\varphi\Big(\frac{x+y}{2}\Big) = 1$ implies $x = y$.

Proposition 3.8. *A norm φ is strictly convex if and only if for each $x \in X$,*

$$\bigcap_{\varepsilon>0} V_{\varepsilon,p_\varphi}(x) \subset \langle x\rangle.$$

Proof. Let us show first that the inclusion implies that φ is strictly convex. Let $x, y \in X$ such that $x \neq 0$ and $\varphi(x) + \varphi(y) = \varphi(x+y)$. A direct computation shows that

$$\bigcap_{\varepsilon>0} V_{\varepsilon,p_\varphi}(x+y) = \{z : \varphi(x+y-z) + \varphi(z) = \varphi(x+y)\}.$$

Therefore, $y \in \cap_{\varepsilon>0} V_{\varepsilon,p_\varphi}(x+y)$. Since the inclusion in the statement holds, we have that there is a real number λ such that $y = \lambda(x+y)$. Note that $\lambda \neq 1$ —in other case $x = 0$, which gives a contradiction with $\varphi(x) \neq 0$—. Thus, $(1-\lambda)y = \lambda x$, and then $y = \frac{\lambda}{1-\lambda}x$.

For the converse, consider an element $y \in \cap_{\varepsilon>0} V_{\varepsilon,p_\varphi}(x)$. If $x = y$, then obviously $y \in \langle x \rangle$. Then we can assume that $y \neq x$. As a consequence of the calculations above, we obtain $\varphi(x-y) + \varphi(y) = \varphi(x)$. If $x = 0$, this formula implies $y = 0$, and then the inclusion holds. Thus we can assume that $x \neq 0$. The strict convexity of φ gives that there is a real number λ such that $y = \lambda(x-y)$, and so $y = \frac{\lambda}{1+\lambda}x$ (note that $\lambda \neq -1$; in other case $x = 0$). Therefore $y \in \langle x \rangle$. This proves the proposition. ■

Proposition 3.9. *If the space (X, φ) is continuously convex, then it is strictly convex.*

Proof. Let $x \neq 0$ and consider an element $y \in \cap_{\varepsilon>0} V_{\varepsilon,p_\varphi}(x)$. Then the constant sequence (y) converges to x with respect to p_φ. Since φ is continuously convex, we obtain that there is an $\alpha \in [0,1]$ such that $\alpha x = y$. Thus, $\cap_{\varepsilon>0} V_{\varepsilon,p_\varphi}(x) \subset \langle x \rangle$, and (X, φ) is strictly convex as a consequence of Proposition 3.8. ■

A direct application of Proposition 3.9 provides more examples of spaces which are not continuously convex. For instance, the Banach spaces $(\ell_1, \|.\|_1)$ and $(c_0, \|.\|_\infty)$ do not have this property.

A straightforward argument shows that the strict convexity for a norm φ is equivalent to the following property: for every $x, y \in X$, if $x \neq 0$ and $\varphi(x-y) + \varphi(y) = \varphi(x)$, then $y \in \langle x \rangle$. Note that in this case, $\varphi(y) \leq \varphi(x)$. This and the proposition above motivate the following definition.

Definition 3.10. *We say that a norm satisfies the sequential convex-compact property —SCC property for short— if for every sequence $(x_n)_n$ that p_φ-converges to an element $x \in X$, there is a subsequence that φ-converges.*

Example 3.11. (i) *Every finite dimensional normed space satisfy the SCC property. This is a straightforward consequence of the fact that convergence with respect to the canonical partial metric implies boundedness of the sequence; if $(x_n)_n$ converges in this sense to an element $x \in X$, then in particular $\varphi(x-x_n)+\varphi(x_n) < \varphi(x)+1$ for every $n \geq n_0$ for a certain $n_0 \in N$. Therefore, $(x_n)_n$ is bounded. Since the closure of each bounded set in a finite dimensional normed space is compact, we obtain a convergent subsequence.*

(ii) *Every continuously convex norm satisfies also the SCC property, as a direct consequence of the definition of continuous convexity.*

However, there are also simple examples of spaces which do not satisfy this property.

Example 3.12. *Consider the Banach space* $(\ell_\infty, \|.\|_\infty)$*, the element* $x = (2, 0, 0, 0, ...)$ *and the sequence* $(y_i)_i$*,* $y_i = (1, 1, ..., 1, 0, 0, ...)$*, where the last* 1 *is situated in the i-th position,* $i \in N$*. It is easy to see that for every* $i \in N$*,* $\|y_i\|_\infty + \|x - y_i\|_\infty = 1 + 1 = \|x\|_\infty$*. This obviously means that* $(y_i)_i$ *converges to* x *with respect to the canonical partial metric* $p_{\|\cdot\|_\infty}$*. However, for every* $i \neq j$ *we have that* $\|y_i - y_j\|_\infty = 1$*, and therefore there is no norm convergent subsequence for the sequence* $(y_i)_i$*.*

Theorem 3.13. *Let* (X, φ) *be a normed space. Then the following statements are equivalent.*

(1) φ *is continuously convex.*

(2) φ *is strictly convex and satisfies the SCC property.*

Proof. Let us show first that (2) implies (1). Consider a sequence $(y_n)_n$ that p_φ-converges to an element $x \in X$. Since φ satisfies the SCC property, we have that there is a subsequence —which we still denote by $(y_n)_n$— that converges with respect to φ. Let us call y to the limit of this subsequence. Then, since for every δ there is an index n_0 such that for every $n \geq n_0$,

$$\varphi(x - y_n) + \varphi(y_n) < \varphi(x) + \frac{\delta}{3} \quad \text{and} \quad |\varphi(y) - \varphi(y_n)| \leq \varphi(y - y_n) < \frac{\delta}{3},$$

we have that

$$\varphi(x-y)+\varphi(y) \leq \varphi(x-y_n)+\varphi(y_n-y)+\varphi(y) < \varphi(x-y_n)+\frac{\delta}{3}+\frac{\delta}{3}+\varphi(y_n) < \varphi(x)+\delta.$$

These inequalities hold for every $\delta > 0$, and so we obtain that $\varphi(x-y)+\varphi(y) = \varphi(x)$. Therefore $y \in \cap_{\varepsilon>0} V_{\varepsilon,p_\varphi}(x)$. Since φ is strictly convex, we have that $y \in [0, x]$, and thus φ is continuously convex. The converse is a direct consequence of Proposition 3.9. ■

Example 3.14. *Let us show another example of the continuous convexity property by using Proposition 3.6 in order to motivate the study of the relation between this property and the uniform convexity of a norm. Consider a Hilbert space* H *with norm* $\|.\|_2$*. Then the following calculations show that* $\|.\|_2$ *defines a continuously convex norm.*

Let $x \in H$. Take a scalar $\delta > 0$ and suppose that a vector $y \in H$ satisfies $\|x - y\|_2 + \|y\|_2 < \|x\|_2 + \delta$. Consider the projection λx of y on the subspace $\langle x \rangle$, and let us define $z := y - \lambda x$. Note that z is orthogonal to x. We distinguish two cases.

(i) $0 \leq \lambda \leq 1$. Then we have that

$$\|x\|_2 + \delta \geq \sqrt{(1-\lambda)^2\|x\|_2^2 + \|z\|_2^2} + \sqrt{\lambda^2\|x\|_2^2 + \|z\|_2^2} \geq \sqrt{\|x\|_2^2 + 4\|z\|_2^2}.$$

Therefore $\|z\|_2 \leq \sqrt{\frac{\|x\|_2\delta}{2} + \frac{\delta^2}{4}}$.

Let $\varepsilon > 0$. Then, it is enough to take a $\delta > 0$ such that $\varepsilon > \sqrt{\frac{\|x\|_2\delta}{2} + \frac{\delta^2}{4}}$ to obtain that $\|y - \lambda x\|_2 < \varepsilon$, and this implies that

$$\inf\{\|\alpha x - y\|_2 \,|\, 0 \leq \alpha \leq 1\} < \varepsilon.$$

(ii) $\lambda > 1$ or $\lambda < 0$. In the first case let us consider z' such that $y = x + z'$. A direct calculation shows that $\|x\|_2 \leq \|x + z'\|_2$. Then

$$\|z'\|_2 + \|x\|_2 \leq \| - z'\|_2 + \|x + z'\|_2 < \|x\|_2 + \delta.$$

Thus, $\|y - x\|_2 = \|z'\|_2 < \delta$, which clearly implies that $\inf\{\|\alpha x - y\|_2 \,|\, 0 \leq \alpha \leq 1\} < \delta$. The same kind of argument shows that for the other case $\|y - 0\|_2 = \|y\|_2 < \delta$. Therefore, if we take $\delta > 0$ such that $\max\{\delta, \sqrt{\frac{\|x\|_2\delta}{2} + \frac{\delta^2}{4}}\} < \varepsilon$, we obtain the scalar δ for which the desired property is satisfied.

Note that the relation between the ε and the δ of the characterization of continuous convexity in Example 3.14 depends on the norm of the element $x \in H$. It can be easily seen that this would happen in any space having this property. Consequently, and in order to define the *uniform version* of this convexity property for normed spaces, we need to restrict the definition using a boundedness condition for the elements of the space. Taking into account this requirement, we can define the uniform continuous convexity property as follows. Actually, Theorem 3.17 states that this uniform version of the continuous convexity property gives in fact a new characterization of the uniform convexity.

Definition 3.15. *We say that a norm φ is uniformly continuously convex if for every $\varepsilon > 0$ there exists a $\delta > 0$ (depending only on ε) such that for every $z \in X$ that can be written as a sum $x + y = z$ of two norm one elements $x, y \in X$, $2 = \varphi(x) + \varphi(y) < \varphi(z) + \delta$ implies* $\inf\{\varphi(\alpha x + (1 - \alpha)(-y)) : 0 \leq \alpha \leq 1\} < \varepsilon$.

Definition 3.16. *Recall that a norm φ is uniformly convex if for every $\varepsilon > 0$ there is a $\delta > 0$ such that, if $x, y \in X$, $\varphi(x) = \varphi(y) = 1$ and $1 - \delta < \varphi(\frac{x+y}{2})$, then $\varphi(x - y) < \varepsilon$.*

Theorem 3.17. *Let (X, φ) be a normed space. The following statements are equivalent.*

(1) *φ is uniformly continuously convex.*

(2) *φ is uniformly convex.*

(3) *For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x, y \in X$ are norm one elements and $y \in V_{\delta, p_\varphi}(x + y)$, then $y \in V_{\varepsilon, p_\varphi}(x)$.*

Proof. Let us prove first that (1) $\to$ (2), i.e. if φ is uniformly continuously convex then φ is uniformly convex. Consider $\varepsilon' > 0$ and two elements $x, y \in X$ such that $\varphi(x) = \varphi(y) = 1$. Let us define $\varepsilon = \varepsilon'/4$. Then there exists a $\delta > 0$ such that $\varphi(\frac{x+y}{2}) > 1 - \delta$ implies that $\inf\{\varphi(\alpha x + (1 - \alpha)(-y)) : 0 \leq \alpha \leq 1\} < \varepsilon$. Hence, there is an $0 \leq \alpha \leq 1$ such that

$$\varepsilon > \varphi(\alpha x + (1 - \alpha)(-y)) \geq |\varphi(\alpha x) - \varphi((1 - \alpha)(-y))| = |\alpha\varphi(x) - (1 - \alpha)\varphi(y)| = |2\alpha - 1|.$$

Thus $|2\alpha - 1| < \varepsilon$. Let us define now τ such that $\alpha = \frac{1}{2} + \tau$. Then $|\tau| < \frac{1}{2}$, and thus the condition above is equivalent to the following one: $|2(\frac{1}{2} + \tau) - 1| = 2|\tau| < \varepsilon$. Therefore, if $\varphi(\frac{x+y}{2}) > 1 - \delta$ then $2|\tau| < \varepsilon$. Then the following holds

$$\varepsilon > \inf_{0 \leq \alpha \leq 1}\{\varphi(\alpha x + (1 - \alpha)(-y))\} = \inf_{|\tau| \leq \frac{1}{2}} \left\{\varphi\left(\left(\frac{1}{2} + \tau\right) x + \left(\frac{1}{2} - \tau\right)(-y)\right)\right\} =$$

$$= \inf_{|\tau|\leq\frac{1}{2}} \left\{\varphi\left(\frac{1}{2}x - \frac{1}{2}y + \tau(x+y)\right)\right\} = \inf_{|\tau|\leq\frac{1}{2}} \left\{\varphi\left(\frac{1}{2}(x-y) + \tau(x+y)\right)\right\}.$$

Fixing an adequate τ, we obtain

$$\varepsilon > \inf_{|\tau|\leq\frac{1}{2}} \left\{\varphi\left(\frac{1}{2}(x-y) + \tau(x+y)\right)\right\} \geq \left|\varphi\left(\frac{1}{2}(x-y)\right) - |\tau|\varphi(x+y)\right| \geq$$

$$\geq \varphi\left(\frac{1}{2}(x-y)\right) - |\tau|2 \geq \varphi\left(\frac{1}{2}(x-y)\right) - \varepsilon.$$

We conclude that $\varphi(x-y) < 4\varepsilon = \varepsilon'$, so φ is uniformly convex.

For the converse, suppose that φ is uniformly convex. Consider an $\varepsilon > 0$. Then there is a $\delta > 0$ such that, if $\varphi(u) = \varphi(v) = 1$ and $2 - 2\delta < \varphi(u+v)$, then $\varphi(u-v) < \varepsilon$. Take two norm one elements $x, y \in X$ such that

$$2 = \varphi(x) + \varphi(y) < \varphi(z) + \delta < \varphi(z) + 2\delta,$$

where $x+y = z$. The uniform convexity condition at the beginning of this paragraph gives that $\varphi(x-y) < \varepsilon$. Therefore

$$\inf\{\varphi(\alpha x - (1-\alpha)y) : 0 \leq \alpha \leq 1\} \leq \varphi(\frac{x-y}{2}) \leq \varphi(x-y) < \varepsilon,$$

where the first inequality is obtained by the evaluation of $\varphi(\alpha x - (1-\alpha)y)$ for $\alpha = 1/2$.

Let us prove now (2) $\rightarrow$ (3). First note that the uniform convexity is equivalent to the following property: given $\varepsilon > 0$, there is a $\delta > 0$ such that $\varphi(\frac{x+y}{2}) \geq 1 - \delta$ implies $\varphi(x-y) < \varepsilon$ for each couple of norm one elements $x, y \in X$. Suppose now that φ is uniformly convex and consider $\varepsilon > 0$. This ε gives a $\delta > 0$ satisfying the condition that appears in the definition of uniform convexity. Let $\delta' = 2\delta$ and $x, y \in X$ such that $\varphi(x) = \varphi(y) = 1$ and

$$\varphi(x) + \varphi(y) < \varphi(x+y) + \delta',$$

i.e. $y \in V_{\delta',p_\varphi}(x+y)$. Thus $1 - \delta < \frac{1}{2}\varphi(x+y) = \varphi(\frac{x+y}{2})$, and hence, $\varphi(x-y) < \varepsilon$ and then $y \in V_{\varepsilon,p_\varphi}(x)$. The same argument proves also the converse. ■

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