Isometry Games in Banach Spaces

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Abstract

We study a property of extension of partial isometries in a Banach space. This property is formulated in game-theoretic language. It is weaker than transitivity, self-dual for reflexive spaces and it is related to a well-known open problem in functional analysis and to model-theoretic notions from mathematical logic.

1 Introduction

In this paper we attempt to relate and compare notions from functional analysis with notions from model theory in mathematical logic. The latter are in turn inspired by game theory. To be more precise, we formulate and consider a new property which is equivalent to *transitivity* in separable Banach spaces but is weaker in general. We call this property WS, for having a *Winning Strategy* in the game of extending partial isometries. More specifically, suppose from the previous moves of the game, we have two $n-$tuples $\bar{a}$ and $\bar{b}$ of vectors in a Banach space, so that between the subspaces they generate there is a linear isometric isomorphism T taking a_i to b_i for all $1 \leq i \leq n$. Now suppose there is given a new element a_{n+1}, we would like to respond by an element b_{n+1} so that T extends to a linear isometric isomorphism that takes a_{n+1} to b_{n+1}.

Property WS may provide more understanding into the following old problem posed by Banach and Mazur (see [22]): *Is every separable transitive Banach space necessarily a Hilbert space?* As far as we know, this problem remains unsolved.

The motivation behind the proposing of WS comes from similar considerations in model theory of first order logic, as mentioned in [17], and hence it is related to

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the *Banach space model theory* investigated by Ward Henson and Iovino ([13], [16]), as well as to Keisler's *BF* (*Back-and-Forth*) *property* ([19]) which we investigated in [4] and [5] in the context of Banach spaces. Actually we will compare WS and BF.

It is worthwhile to note that rather different notions of games in Banach spaces are considered in [15], [23] and [10].

Despite the motivation from logic, the current paper, with the exception of the minor result Theorem 9 (2), can be read without specific knowledge in logic. Although nonstandard hulls are mentioned, they can be replaced by Banach space ultraproducts given by [9], which are well-known among analysts. (However, for the interested reader, nonstandard analysis and nonstandard hull constructions can be found in [1] and [15]). The nonstandard hull construction is also briefly reviewed in § 5). We use [6], [20] and [18] as background in Banach space theory.

Here is an outline of the paper. In § 2 we introduce the definitions and notation (WS and others) and make some elementary observations. We also show that WS is weaker than transitivity. Then in § 3, we investigate whether there is any relationship between isometry games and convexity properties. In § 4, we prove our main result that WS is a self-dual property for reflexive spaces. Section 5 is devoted to the study of isometry games in classical Banach spaces. Eventually in § 6 we present some open problems.

2 Preliminaries

From now on, by *isometric isomorphism* we actually mean *linear isometric isomorphism.*

If X is a Banach space we denote by $\mathcal{S}_X$ its unit sphere, namely the set of unit vectors in X.

Recall that in a Banach space X a *smooth* point is a unit vector x normed by a unique functional of unit norm, i.e. there is a unique $f \in X'$ (the dual of X) such that $\|f\| = 1$ and $f(x) = 1$. And a space is smooth if every unit vector is smooth. Equivalently, every point is normed uniquely, i.e. for any nonzero vector x, there is a unique linear continuous functional f of norm 1 such that $\|x\| = f(x)$.

A Banach space X is *strictly convex* if every point on $\mathcal{S}_X$ is an extreme point, namely it cannot be written as midpoint of two distinct unit vectors. X is *uniformly convex* if for any $\epsilon > 0$ there is $\delta > 0$ such that, for all $x, y \in \mathcal{S}_X$, if $\|x - y\| \geq \epsilon$ then $\|\frac{x+y}{2}\| < 1 - \delta$.

We will deal with so called *Banach spaces structures* of the form $\mathcal{X} = (X, \bar{x})$, where $\bar{x}$ is a (possibly empty) sequence of elements from the Banach space X. Some of the results can be generalized for structures obtained from a Banach space together with additional constants, functions and relations. Given two Banach space structures $\mathcal{X} = (X, \bar{x})$ and $\mathcal{Y} = (Y, \bar{y})$, we write $\mathcal{X} \equiv_0 \mathcal{Y}$ if there is an isometric isomorphism between the subspaces spanned by $\{x_i\}$ and $\{y_i\}$, respectively, that maps x_i to y_i for all i. From a logical viewpoint (see [4]), this is the same as saying that $\bar{x}$ and $\bar{y}$ satisfy the same quantifier-free sentences of an appropriate first order language for Banach space structures in $\mathcal{X}$ and $\mathcal{Y}$ respectively, and hence we borrow this notation from logic.

We now give definitions of the main properties of our study.

- (Back-and-Forth) A Banach space structure $\mathcal{X} = (X, \bar{x})$ has BF_n if whenever $(\mathcal{X}, a_1, \ldots, a_n) \equiv_0 (\mathcal{X}, b_1, \ldots, b_n)$ and $a_{n+1} \in X$, there is $b_{n+1} \in X$ such that $(\mathcal{X}, a_1, \ldots, a_{n+1}) \equiv_0 (\mathcal{X}, b_1, \ldots, b_{n+1})$.

  $\mathcal{X}$ has BF if it has BF_n for all n.

- (Isometry Game) Given Banach space structures $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{X} \equiv_0 \mathcal{Y}$ we define a 2-player game $\Gamma = \Gamma[\mathcal{X}, \mathcal{Y}]$ by describing the $(n+1)$-th move in a play of the game.

  Suppose that, after n moves, $(\mathcal{X}, a_1, \ldots, a_n) \equiv_0 (\mathcal{Y}, b_1, \ldots, b_n)$ then player $\forall$ chooses some $a_{n+1} \in X$ and player $\exists$ responses by choosing some $b_{n+1} \in Y$ or $\forall$ chooses some $b_{n+1} \in Y$ and $\exists$ responses by choosing some $a_{n+1} \in X$.

  We say that player $\exists$ wins the play at move $n+1$ if

  $$(\mathcal{X}, a_1, \ldots, a_{n+1}) \equiv_0 (\mathcal{Y}, b_1, \ldots, b_{n+1}).$$

  We say that $\exists$ has a *winning strategy* if $\exists$ can win at every move n, no matter how $\forall$ plays.

- (Winning Strategy) The Banach space structure $\mathcal{X}$ has WS if for all (unit) vectors a, b satisfying $(\mathcal{X}, a) \equiv_0 (\mathcal{X}, b)$, player $\exists$ has a winning strategy in $\Gamma[(\mathcal{X}, a), (\mathcal{X}, b)]$.

- (Transitivity) (See [7]) A Banach space X is *transitive* if, for all $a, b \in \mathcal{S}_X$, there exists an isometric isomorphism of X taking a to b.

- (Almost transitivity) A Banach space X is *almost-transitive* if the orbits of the action of isometric isomorphisms of X are dense in $\mathcal{S}_X$. Otherwise said, X is almost-transitive if, for all $\epsilon > 0$ and all unit vectors a, b, there exists an isometric isomorphism T of X such that $\|T(a) - b\| \leq \epsilon$.

- For given rational $\lambda \geq 1$ we define $\bar{a} \sim_\lambda \bar{b}$ if

  $$\lambda^{-1}\|\sum \alpha_i a_i\| \leq \|\sum \alpha_i b_i\| \leq \lambda \|\sum \alpha_i a_i\| \quad \text{for all } \alpha \in \mathbf{Q}^n.$$

  So, for a Banach space X, $(X, \bar{a}) \equiv_0 (X, \bar{b})$ if and only if $\bar{a} \sim_1 \bar{b}$.

We now remark some elementary facts about the above properties.

Remark 1. *1. Note the main difference between BF and WS : in the latter, the partial isometry before the play of a game is not arbitrary; it must be constructed from previous moves.*

2. Let X be a Banach space with WS. Then, for all $a, b \in \mathcal{S}_X$, there exist a separable subspace $X_0 \subset X$ and an isometric isomorphism $T : X_0 \to X_0$ such that $T(a) = b$. For, from WS we have $(X, a_1, \ldots, a_n, \ldots) \equiv_0 (X, b_1, \ldots, b_n, \ldots)$, where $a_1 = a$, $b_1 = b$ and $\{a_i\}_{i \in \omega} = \{b_i\}_{i \in \omega}$. Then we let X_0 be the closed linear span of $\{a_i\}$, and $T : X_0 \to X_0$ given by $T(a_i) = b_i$.

3. *BF holds for a Banach space structure $\mathcal{X}$ if and only if, whenever $\bar{a}$ and $\bar{b}$ are tuples of the same length such that $(\mathcal{X}, \bar{a}) \equiv_0 (\mathcal{X}, \bar{b})$, player $\exists$ has a winning strategy in $\Gamma[(\mathcal{X}, \bar{a}), (\mathcal{X}, \bar{b})]$.*

4. *$BF \Rightarrow WS \Rightarrow BF_1$.*

5. *For a Banach space of finite dimension $n+1$, BF_n is equivalent to BF.*

6. *All Hilbert spaces satisfy BF.*

7. *Let X be a Banach space. It is not difficult to check that, for all unit vectors $a, b \in X$, the game $\Gamma[(\mathcal{X}, a), (\mathcal{X}, b)]$ is closed. Hence a celebrated theorem of game theory due to Gale and Stewart ensures the existence of a winning strategy for one of the two players. Consequently a (possibly of little use) criterion for WS is the non existence of a winning strategy for player $\forall$ in $\Gamma[(\mathcal{X}, a), (\mathcal{X}, b)]$ for all unit vectors a, b.*

8. *WS and transitivity are equivalent in separable Banach spaces.*

Recall that the only finite dimensional transitive spaces are Hilbert (see [2] or [7]). Hence a finite dimensional Banach space has WS if and only if it is Hilbert. Also, properties WS and BF are equivalent in finite dimensional spaces and they hold precisely in finite dimensional Hilbert spaces.

Recall the following characterization: a Banach space is smooth if and only if its norm is Gâteaux differentiable. We note the following:

Proposition 2. *Let X be a Banach space with norm θ. Suppose that θ satisfies the following assumption: for all $u, v \in X$ such that $\theta(u) = \theta(v)$ there exists a linear and continuous mapping $T : X \to X$ such that*

1. *$T(u) = v$;*

2. *$\theta T(x) = \theta(x)$ for all x.*

Let $L : X \to X$ be a linear homeomorphism and let $\eta = \theta \circ L$. Then η is a norm on X that satisfies the same assumption as θ. Moreover, whenever θ is smooth, so is η.

Proof. First note that Gâteaux differentiability of η follows from

$$\eta'_x(y) = \lim_{\epsilon \to 0} \frac{\eta(x + \epsilon y) - \eta(x)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\theta(L(x) + \epsilon L(y)) - \theta(L(x))}{\epsilon} = \theta'_{L(x)}(L(y)).$$

Let u and v be such that $\eta(u) = \eta(v)$ and let $T : X \to X$ be as in the assumptions such that $TL(u) = L(v)$. We let $S = L^{-1}TL$. Clearly, S is linear and continuous.

Moreover, $S(u) = L^{-1}TL(u) = L^{-1}L(v) = v$ and, for all x,

$$\eta(S(x)) = \theta L(S(x)) = \theta TL(x) = \theta L(x) = \eta(x). \qquad \blacksquare$$

Remark 3. *Condition 2 in the statement of the previous proposition implies that* $\theta'_x(y) = \theta'_{T(x)}(T(y))$ *for all* x *and* y.

Notice also that if θ *as above is Hilbertian, then so is* η.

We prove now that WS and BF, although not preserved by equivalent renorming, behave well with respect to certain renorming.

Corollary 4. *Let* $L : X \to X$ *be a linear homeomorphism of a Banach space* X *with norm* $\|\cdot\|$ *and let* $|\!|\!|\cdot|\!|\!|$ *be the norm defined by* $|\!|\!|x|\!|\!| = \|L(x)\|$. *Then*

1. $(X, \|\cdot\|)$ *has* WS *if and only if* $(X, |\!|\!|\cdot|\!|\!|)$ *has* WS.
2. $(X, \|\cdot\|)$ *has* BF *if and only if* $(X, |\!|\!|\cdot|\!|\!|)$ *has* BF.

Proof. Follows from Proposition 2. ∎

We remark that no finite dimensional vector space X can be equipped with a unique norm $\|\cdot\|$ such that $(X, \|\cdot\|)$ has BF : without loss of generality, let $X = \mathbb{R}^n$ and let $\|\cdot\|$ be the euclidean norm. Then $(\mathbb{R}^n, \|\cdot\|)$ has BF (being a Hilbert space). Let L be any linear isomorphism of $\mathbb{R}^n$ and let $|\!|\!|\cdot|\!|\!|$ be as in Corollary 4. Then $(\mathbb{R}^n, |\!|\!|\cdot|\!|\!|)$ has BF.

We now prove that WS is weaker than transitivity.

Proposition 5. *WS does not imply transitivity.*

Proof. In the sequel $[0,1]^I$ is always equipped with the product measure of the Lebesgue measure on $[0,1]$, for all sets I.

Let

$$X = L_p([0,1]^{\omega_1}) \qquad Y = \bigoplus_{\alpha<\omega_1} L_p([0,1]^{\omega_2}) \qquad Z = X \bigoplus Y,$$

where by $\bigoplus$ we denote p-sum and $1 \le p < \infty$, $p \neq 2$. It is proved in [12] that Z is not transitive. Nevertheless we claim that it has WS.

Note that each element $[\theta] \in L_p([0,1]^\kappa)$, as an equivalence class of measurable functions, is determined by a countable set of coordinates. More precisely, for $A \subset \kappa$, let π_A denote the projection of $[0,1]^\kappa$ onto $[0,1]^A$ via $(x_i)_{i<\kappa} \mapsto (x_i)_{i\in A}$. Then for all $[\theta] \in L_p([0,1]^\kappa)$, there is a countable $A \subseteq \kappa$ and $[\psi] \in L_p([0,1]^A)$ such that $[\theta] = [\psi \circ \pi_A]$.

Note also that $X \subset L_p([0,1]^{\omega_2})$ as a subspace via $[\theta] \mapsto [\theta \circ \pi_{\omega_1}]$. In particular, as a subspace, $Z \subset Y \simeq L_p([0,1]^{\omega_2}) \bigoplus Y$.

We denote by $G(Y)$ the group of isometric isomorphisms of Y onto itself. Again by [12], we have that $G(Y)$ acts transitively on the unit sphere of Y.

In order to prove that Z has WS let $a_1, b_1 \in Z$ (denoting both functions and equivalence classes) with $a_1 \sim_1 b_1$ and let $T_1 \in G(Y)$ be such that $T_1(a_1) = b_1$.

Assume now inductively that $(a_1, \ldots, a_n) \sim_1 (b_1, \ldots, b_n)$ and $T_1, \ldots, T_n$ in $G(Y)$ are constructed with $a_i, b_i \in Z$ and $T_j(a_i) = b_i$, $i \le j \le n$. Write $a_i = a_i^1 \oplus a_i^2$ and $b_i = b_i^1 \oplus b_i^2$, where $a_i^1, b_i^1 \in X \subset L_p([0,1]^{\omega_2})$ and $a_i^2, b_i^2 \in Y$. Now let $a_{n+1} \in Z \subset Y$ be given and let $b = T_n(a_{n+1})$. Regard $b \in L_p([0,1]^{\omega_2}) \bigoplus Y$, i.e. we decompose b as $c \oplus d$, where $c \in L_p([0,1]^{\omega_2})$ and $d \in Y$. Let $A \subset \omega_2$ be a countable set of coordinate indices on which c depends on.

Let $B \subset \omega_1$ be a countable set of coordinate indices on which $a_1^1, \ldots, a_n^1, b_1^1, \ldots, b_n^1$ depend on and let $\rho : \omega_2 \to \omega_2$ be a bijection that fixes B pointwise and $\rho(A) \subset \omega_1$. Let $\hat{\rho} : [0,1]^{\omega_2} \to [0,1]^{\omega_2}$ be the corresponding bijection.

Let $T_{n+1} \in G(Y)$ be defined as follows: for all $a \in Y$, regarding $T_n(a)$ as an element of $L_p([0,1]^{\omega_2}) \bigoplus Y$, with $T_n(a) = r \oplus s$, where $r \in L_p([0,1]^{\omega_2})$ and $s \in Y$, define $T_{n+1}(a) = r \circ \hat{\rho}^{-1} \oplus s$.

Define $b_{n+1} = c \circ \hat{\rho}^{-1} \oplus d$. Since $c \circ \hat{\rho}^{-1}$ depends on $\rho(A) \subset \omega_1$, we have: $b_{n+1} \in Z$, $T_{n+1}(a_i) = b_i$ for $i \leq n+1$, and therefore $(a_1, \ldots, a_{n+1}) \sim_1 (b_1, \ldots, b_{n+1})$. ■

3 Isometry games, convexity and smoothness

Hilbert spaces are uniformly convex and uniformly smooth. At the same time they have good isometry game properties, so it is natural to consider the connections between these properties.

The following shows that convexity does not necessarily imply good behavior in isometry games.

Remark 6. *(A uniformly convex but not BF_1 space.) Let $\mathbb{R}^2$ be normed so that it is strictly convex (equivalently uniformly convex) but not Hilbert. Such norm can be obtained by using a sufficiently rounded but not symmetric convex set as the unit disc. Then BF_1 cannot be satisfied in this space, for otherwise it satisfies BF and so it has to be Hilbert (see the remarks in the previous section).*

Notice that by rotating the above $2-$dimensional unit disc in $\mathbb{R}^n$ along an axis, we have example of an $n-$dimensional space which is uniformly convex but BF_1 fails.

Along the lines of the proof of [8, Theorem 2.1], we can prove the following

Theorem 7. *For a Banach space X with BF_1 the following are equivalent:*

1. *X has the Radon-Nikodým property;*
2. *X is reflexive;*
3. *X is superreflexive;*
4. *X is uniformly convex.*

Proof. By well-known results, every reflexive space has Radon-Nikodým property and every uniformly convex space is superreflexive. Therefore it remains to prove that 1. implies 4. For $0 \leq \epsilon \leq 1$ and $x \in X$ let

$$\Delta(x, \epsilon) = \inf\{1 - \lambda : \text{ there exists } y \in X \text{ such that } \|\lambda x \pm y\| \leq 1 \text{ and } \|y\| \geq \epsilon\}.$$

It is known that if $\inf\{\Delta(x, \epsilon) : \ x \in \mathcal{S}_X\} > 0$ for all $0 < \epsilon \leq 1$, then X is uniformly convex (see [8] and [11]).

By Radon-Nikodým property there exists $x_0 \in \mathcal{S}_X$ such that $\Delta(x_0, \epsilon) > 0$ for all $0 < \epsilon \leq 1$ (see [8]). An easy calculation shows that BF_1 implies $\Delta(x_0, \epsilon) = \Delta(x, \epsilon)$ for all $x \in \mathcal{S}_X$ and all $0 < \epsilon \leq 1$. Then X is uniformly convex. ■

Corollary 8. *Properties (1) to (4) in the statement of Theorem 7 are equivalent for Banach spaces with WS.*

Proof. WS implies BF_1. ∎

The following gives dichotomy results with respect to smoothness and convexity property.

Theorem 9. *Let X be a Banach space.*

1. *If X has WS then either all points on $\mathcal{S}_X$ are smooth or none is smooth;*

2. *If X has BF_1 then either all points on $\mathcal{S}_X$ are extreme or none is extreme.*

Proof.
1. For sake of contradiction, suppose u, v are unit vectors, such that u is smooth but v is not. Let w be a unit vector witnessing the nonsmoothness of v. (By this we mean that there exist two unit functionals ϕ and θ both norming v but $\phi(w) \neq \theta(w)$.) Using WS we can find a closed separable subspace X_0 of X including elements u, v, w, and an isometric isomorphism $T : X_0 \to X_0$ such that $T(u) = v$. (See Remark 1(2).)

   By Hahn-Banach, u is a smooth point in X_0 and, by including the witness w, we conclude that v is nonsmooth in X_0. We thus get a contradiction since there can be no isometric isomorphism mapping a smooth point to a nonsmooth point.

2. (This is the only part of the paper where we use a little bit of logic. See [4] for background.) Notice that BF_1 is equivalent to saying that whenever two vectors of X have the same qfpb-type (i.e. the same norm), they also satisfy the same existential pb formulas with one existential quantifier.

   Now we can prove the dichotomy by noticing that, for some $\epsilon > 0$, either all unit vectors satisfy the existential pb formula

   $$\exists y(\|x\| = \|y\| = \|2x - y\| = 1 \wedge \|x - y\| \geq \epsilon)$$

   or none does. ∎

4 Isometry games in dual spaces

The following shows that, for reflexive Banach spaces, WS is a self-dual property.

Theorem 10. *Let X be a reflexive Banach space. Then X has WS if and only if the dual space X' has WS.*

Proof. Suppose X has WS. Then, by Theorem 7, it is strictly convex and super-reflexive. It follows from the results in [11] (see Definition 7 and Proposition 9) that there are smooth points on the unit sphere of X (Fréchet differentiability is stronger than Gâteux differentiability). So, by Theorem 9, X is smooth.

Recall also that, for a reflexive space, strict convexity is equivalent to smoothness of its dual.

Let ϕ_1, θ_1 be unit functionals in X': we want to show that player $\exists$ has a winning strategy in the game $\Gamma[(X', \phi_1), (X', \theta_1)]$.

Let a_1 and b_1 be unit vectors in X such that $\|a_1\| = \|\phi_1\| = \phi_1(a_1) = 1$ and $\|b_1\| = \|\theta_1\| = \phi_1(b_1) = 1$.

The main idea of the proof is that, for each play in the game $\Gamma[(X', \phi_1), (X', \theta_1)]$, we keep track of a simultaneous play in the game $\Gamma[(X, a_1), (X, b_1)]$.

Suppose that, for some $\bar{\phi}, \bar{\theta} \in (X')^n$ and some $\bar{a}, \bar{b} \in X^n$, we already have the following from the first $n-1$ moves of a play in the game $\Gamma[(X', \phi_1), (X', \theta_1)]$:

- $(X', \phi_1, \ldots, \phi_n) \equiv_0 (X', \theta_1, \ldots, \theta_n)$;
- $(X, a_1, \ldots, a_n) \equiv_0 (X, b_1, \ldots, b_n)$;
- (norming) $\|a_i\| = \|\phi_i\| = \phi_i(a_i) = 1, \quad \|b_i\| = \|\theta_i\| = \theta_i(b_i) = 1, \quad i \le n.$

Let $\phi_{n+1} \in X'$ with $\|\phi_{n+1}\| = 1$. We are going to respond by producing $\theta_{n+1} \in X'$ such that the above properties extend to some $(n+1)$-tuples.

By smoothness and reflexivity of X', let $a_{n+1} \in X$ be the unique vector so that $\|a_{n+1}\| = \phi_{n+1}(a_{n+1}) = 1$. By WS in X, choose $b_{n+1} \in X$ so that

$$(X, a_1, \ldots, a_{n+1}) \equiv_0 (X, b_1, \ldots, b_{n+1}).$$

Now, by smoothness and reflexivity of X, there is $\theta_{n+1} \in X'$ so that

$$\|b_{n+1}\| = \|\theta_{n+1}\| = \theta_{n+1}(b_{n+1}) = 1.$$

Claim: $(X', \phi_1, \ldots, \phi_{n+1}) \equiv_0 (X', \theta_1, \ldots, \theta_{n+1})$.

First we construct a closed separable subspace $X_0 \subset X$ and an isometric isomorphism $T : X_0 \to X_0$ so that

(i) $T(a_i) = b_i, \quad i \le n+1$;

(ii) $\phi_i = \theta_i \circ T$ on $X_0, \quad i \le n+1$;

(iii) $\| \sum_{i=1}^{n+1} \alpha_i \phi_i \|_{X_0} = \| \sum_{i=1}^{n+1} \alpha_i \phi_i \|_X$ and $\| \sum_{i=1}^{n+1} \alpha_i \theta_i \|_{X_0} = \| \sum_{i=1}^{n+1} \alpha_i \theta_i \|_X$, for all $\alpha \in \mathbf{R}^{n+1}$ (i.e. the norms are the same whether one views the functionals as functionals on X or as functionals on the subspace X_0).

Define $\mathcal{A} = \{\alpha \in \mathbb{Q}^{n+1} : \sum_{i=1}^{n+1} \alpha_i = 1\}$. Enumerate $\mathcal{A}$ as $\alpha^{(m)}$, $m < \omega$. Using norming property on X', for each $\alpha = \alpha^{(m)} \in \mathcal{A}$, let $x_m, y_m \in X$ be unit vectors such that

$$\| \sum_{i=1}^{n+1} \alpha_i \phi_i \| = \sum_{i=1}^{n+1} \alpha_i \phi_i(x_m), \quad \| \sum_{i=1}^{n+1} \alpha_i \theta_i \| = \sum_{i=1}^{n+1} \alpha_i \theta_i(y_m).$$

Next we define a bijection T_0 whose domain is a countable subset of X as follows:
step 0: Define $k_0 = n+1, c_i = a_i, T_0(c_i) = b_i, i \le k_0$.
step $m+1$: Suppose we have $(X, c_1, \ldots, c_{k_m}) \equiv_0 (X, T_0(c_1), \ldots, T_0(c_{k_m}))$.

Let $c_{k_m+1}, \ldots, c_{k_{m+1}}$ be a list consisting of x_m, y_m and images from previous steps $T_0(c_i)$, $i \le k_m$, which are distinct from c_j, $j \le k_m$. By WS we can find vectors $T_0(c_i)$, $k_m < i \le k_{m+1}$ so that

$$(X, c_1, \ldots, c_{k_{m+1}}) \equiv_0 (X, T_0(c_1), \ldots, T_0(c_{k_{m+1}})).$$

Let the domain of T_0 be $\{c_i : i < \omega\}$. Since $(X, c_i)_{i<\omega} \equiv_0 (X, T_0(c_i))_{i<\omega}$, T_0 is a bijection. Now let $X_0 = \overline{\text{Span}}\{c_i : i < \omega\}$, the closed linear span. Extend T_0 to an isometric isomorphism $T : X_0 \to X_0$ by linearity and continuity.

Conditions (i) and (iii) follow from the construction.

For (ii), note that, for $i \leq n+1$, $\phi_i(a_i) = 1 = \theta_i(b_i)$ and $b_i = T(a_i)$, therefore $\phi_i(a_i) = \theta_i(T(a_i))$. So a_i is normed by both ϕ_i and $\theta_i \circ T$. Hence $\phi_i = \theta_i \circ T$ on X_0 by uniqueness. (X_0 is smooth and $\phi_i, \theta_i \circ T$ are unit vectors in X_0'.)

Now, to prove the claim, it suffices to let $\alpha = \alpha^{(m)} \in \mathcal{A}$ and show that

$$\| \sum_{i=1}^{n+1} \alpha_i \phi_i \| = \| \sum_{i=1}^{n+1} \alpha_i \theta_i \|.$$

But

$$\begin{aligned} \| \textstyle\sum_{i=1}^{n+1} \alpha_i \phi_i \| &= \textstyle\sup_{\|x\|=1,\, x \in X_0} \sum_{i=1}^{n+1} \alpha_i \phi_i(x) && \text{(by (iii))} \\ &= \textstyle\sup_{\|x\|=1,\, x \in X_0} \sum_{i=1}^{n+1} \alpha_i \theta_i \circ T(x) && \text{(by (ii))} \\ &= \textstyle\sup_{\|x\|=1,\, x \in X_0} \sum_{i=1}^{n+1} \alpha_i \theta_i(x) && \text{(by isometric isomorphism } T) \\ &= \| \textstyle\sum_{i=1}^{n+1} \alpha_i \theta_i \| && \text{(by (iii)).} \end{aligned}$$

The same arguments apply to prove that WS is inherited from X' to X. ■

The following is a consequence of the previous theorem and of the equivalence of WS and transitivity for separable spaces.

Corollary 11. *Let X be a reflexive, separable Banach space. Then X is transitive if and only if X' is transitive.*

Proof. We notice that reflexivity and separability of X imply separability of X'. (Indeed, it is known that a separable Banach space has separable dual if and only if the dual has Radon-Nikodým property, and, as already noticed, reflexivity implies the Radon-Nikodým property.) Therefore one can apply the previous theorem. ■

Remark 12. *The previous corollary is indeed strengthened in [8] where it is proved that transitivity is a self-dual property for reflexive spaces.*

5 Isometry games in some classical spaces

In this section we discuss whether some classical spaces have the property WS.

Let us consider first the L_p spaces, where $1 \leq p \leq \infty$.

For $1 \leq p < \infty$, $p \neq 2$, the only transitive infinite dimensional L_p spaces are of the form

$$\bigoplus_{\alpha<\lambda} L_p([0,1]^\kappa),$$

where $\bigoplus$ stands for p-sum, κ and λ are infinite cardinals and λ is uncountable (see [12]).

Recall that two elements of an L_p space are said to be *disjoint* if they have disjoint supports. We denote by $\text{supp}(f)$ the support of element f.

Remark 13. *For $p \neq 2, \infty$ a characterization of disjointness is the following (see [3], page 175): elements f, g of $L_p[0,1]$ or of $\ell_p(\mathbf{N})$ are disjoint if and only if*

$$\|\alpha f + \beta g\|^p = |\alpha|^p \|f\|^p + |\beta|^p \|f\|^p \quad \textit{for all } \alpha, \beta \in \mathbb{R}.$$

By using the previous characterization, we can give an easy proof of the following:

Proposition 14. *Let X be either $L_p[0,1]$ or $L_p(\mathbf{R})$. Then X does not have WS for $1 \leq p < \infty$, $p \neq 2$.*

Proof. In X consider unit vectors f_1, g_1, g_2, where f_1 is given by a constant function on $[0,1]$, g_1 a constant function on $[0, \frac{1}{2}]$, zero elsewhere, g_2 a constant function on $(\frac{1}{2}, 1]$, zero elsewhere. Then, by the previous remark, there is no f_2 in X such that $(f_1, f_2) \sim_1 (g_1, g_2)$. ■

Note that the Hilbert spaces $L_2[0,1]$ and $\ell_2(\mathbf{N})$ are transitive, hence they have WS.

Remark 15. *Similarly to Proposition 14, one can prove that the following do not have WS:*

1. *$\ell_p(X)$, where X is any set with at least two elements, and where $1 \leq p \leq \infty$ and $p \neq 2$.*

   For p in the finite range, $p \neq 2$, let $u \neq v$ be elements of X and let $a_1 = 1_{\{u\}} + 1_{\{v\}}$, $a_2 = 1_{\{u\}}$, $b_1 = \|a_1\| 1_{\{u\}}$. There is no $b_2 \in \ell_p(X)$ such that $(a_1, a_2) \sim_1 (b_1, b_2)$: from $\mathrm{supp}(a_1 - a_2) \cap \mathrm{supp}(a_2) = \emptyset$ we have that b_2 must satisfy $\mathrm{supp}(b_1 - b_2) \cap \mathrm{supp}(b_2) = \emptyset$. Then $\emptyset \neq \mathrm{supp}(b_2) \subset \mathrm{supp}(b_1)$, which is impossible.

   For $p = \infty$, let a be the constant function 1 and b, c unit vectors with disjoint support. For every unit vector d we have $\|\alpha b + \beta c\| = \max(|\alpha|, |\beta|)$ but $\|\alpha a + \beta d\| = \max(|\alpha + \beta|, |\alpha - \beta|)$. So $\|\alpha b + \beta c\| = \|\alpha a + \beta d\|$ fails for some $\alpha, \beta \in \mathbb{Q}$.

2. *The space $\mathcal{C}(X)$ of bounded real valued continuous functions on a compact Hausdorff topological space X which has at least two isolated points, equipped with supremum norm.*

Note that the above argument can be used to prove in $\ell_p(X)$ that, under the assumption $(a_1, \dots, a_n) \sim_1 (b_1, \dots, b_n)$, the boolean algebra generated by $\{\mathrm{supp}(a_i)\}_{i \leq n}$ is isomorphic to the boolean algebra generated by $\{\mathrm{supp}(b_i)\}_{i \leq n}$. Furthermore, if $\|a_i\| = \|b_i\|$, $i = 1, \dots, n$, then the converse implication also holds.

Now we briefly recall the construction of *nonstandard hull* of a Banach space due to Luxemburg [21]. (See [1] for a more updated reference. One can also replace nonstandard hulls with the Banach space ultraproducts - due to Krivine, Dacunha-Castelle and Stern - introduced in [9].) The nonstandard hull construction takes place in a *nonstandard universe* that is at least ω_1-saturated.

Once fixed a nonstandard universe, let X be an internal Banach space with norm $\| \ \|$ and let fin(X) be the $\mathbb{R}$-vector subspace of norm-finite elements of X. The nonstandard hull $\hat{X}$ of X is, as $\mathbf{R}$-vector space, the quotient of fin(X) by the $\mathbb{R}$-vector subspace of elements having infinitesimal norm. We let $^\circ : \mathrm{fin}(^*\mathbb{R}) \to \mathbb{R}$ denote the *standard part* map. Then norm of the equivalence class $[a]$ of $a \in \mathrm{fin}(X)$ is given by $^\circ\|a\|$. This turns $\hat{X}$ into a Banach space (completeness is ensured by saturation, even if we start from an internal normed space only).

If X is a Banach space, we write $\hat{X}$ for $\widehat{^*X}$, where $^*X$ is the nonstandard extension of X. As customary, we say that $\hat{X}$ is the nonstandard hull of the Banach space X.

There is a canonical isometric embedding of a Banach space X as closed subspace of its nonstandard hull $\hat{X}$. In general X is a proper subspace of $\hat{X}$ (for, unless X is finite-dimensional, $\hat{X}$ is nonseparable).

The following is indeed an easy remark:

Proposition 16. *The nonstandard hull $\hat{X}$ of an almost-transitive Banach space X is transitive.*

Proof. Let a, b be unit vectors in $^*X$ and let ϵ be a positive infinitesimal. By Transfer there exists *-linear isometric isomorphism $T : {}^*X \to {}^*X$ such that $\|T(a) - b\| \leq \epsilon$.

Let $^\circ T : \hat{X} \to \hat{X}$ be defined by $^\circ T([x]) = [T(x)]$. It is straightforward to see that $^\circ T$ is a well-defined linear isometric isomorphism such that $T([a]) = [b]$. ■

Corollary 17. *Let μ be a homogeneous measure and $1 \leq p < \infty$. Then $\widehat{L_p[\mu]}$ is transitive.*

Proof. $L_p[\mu]$ is almost-transitive (see [12]). ■

Note that Proposition 14 and Corollary 17 together with the isometric embeddability of a Banach space into its nonstandard hull imply that WS is not a *superproperty*. It also shows that WS is not preserved under subspace.

6 Open problems

The main open problem, of course, is the one posed by Banach and Mazur over 60 years ago, which we restated equivalently as

(1) Is every separable WS Banach space a Hilbert space?

Here are some weakenings of this problem:

(2) Is a (separable) WS Banach space reflexive?

(3) Is WS a self-dual property for (separable) Banach spaces? I.e. does the statement of Theorem 10 hold without the additional assumption of reflexivity?

(4) Is a (separable) transitive Banach space reflexive?

Finally, it would be of interest to clarify further the properties in this paper.

(5) Is WS the same as BF?

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