A Non-Resonant Generalized Multi-Point Boundary Value Problem of Dirichlet-Neumann Type involving a p-Laplacian type operator

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Abstract

Let ϕ, θ be odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ satisfying $\phi(0) = \theta(0) = 0$, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying Carathéodory conditions and $e : [0,1] \to \mathbb{R}$ be a function in $L^1[0,1]$. Let $\xi_i, \tau_j \in (0,1)$, a_i, $b_j \in \mathbb{R}$, $i = 1, 2, \cdots, m-2$, $j = 1, 2, \cdots, n-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1$ be given. We study the problem of existence of solutions for the generalized multi-point boundary value problem

$$(\phi(x'))' = f(t, x, x') + e,\ 0 < t < 1,$$
$$x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i),\ \theta(x'(1)) = \sum_{j=1}^{n-2} b_j \theta(x'(\tau_j)), \tag{1}$$

in the non-resonance case. We say that this problem is non-resonant if the associated problem:

$$(\phi(x'))' = 0,\ 0 < t < 1,$$
$$x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i),\ \theta(x'(1)) = \sum_{j=1}^{n-2} b_j \theta(x'(\tau_j)), \tag{2}$$

has the trivial solution as its only solution. This is the case if

$$(1 - \sum_{j=1}^{n-2} b_j)(1 - \sum_{i=1}^{m-2} a_i) \neq 0.$$

Our methods consist in using topological degree and some a priori estimates.

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1 Introduction

Let ϕ, θ be odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ satisfying $\phi(0) = \theta(0) = 0$, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying Carathéodory conditions and $e : [0,1] \to \mathbb{R}$ be a function in $L^1[0,1]$. Let $\xi_i, \tau_j \in (0,1)$, a_i, $b_j \in \mathbb{R}$, $i = 1, 2, \cdots, m-2$, $j = 1, 2, \cdots, n-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1$ be given. We study the problem of existence of solutions for the generalized multi-point boundary value problem

$$\begin{gathered}(\phi(x'))' = f(t,x,x') + e,\ 0 < t < 1,\\ x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i),\ \theta(x'(1)) = \sum_{j=1}^{n-2} b_j \theta(x'(\tau_j)),\end{gathered} \tag{3}$$

in the non-resonance case. We say that this problem is non-resonant if the associated problem:

$$\begin{gathered}(\phi(x'))' = 0,\ 0 < t < 1,\\ x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i),\ \theta(x'(1)) = \sum_{j=1}^{n-2} b_j \theta(x'(\tau_j)),\end{gathered} \tag{4}$$

has the trivial solution as its only solution. This is the case if

$$(1 - \sum_{j=1}^{n-2} b_j)(1 - \sum_{i=1}^{m-2} a_i) \neq 0.$$

This problem was studied by Gupta, Ntouyas, and Tsamatos in [20] and by the author in [16] when the homeomorphisms ϕ, θ from $\mathbb{R}$ onto $\mathbb{R}$ are the identity homeomorphisms, i.e for second order ordinary differential equations. The study of multi-point boundary value problems for second order ordinary differential equations was initiated by Il'in and Moiseev in [22], [23] motivated by the works of Bitsadze and Samarskĩi on nonlocal linear elliptic boundary value problems, (see [2], [3], [4]) and has been the subject of many papers, see for example, [5], [6], [11], [12], [13], [14], [15], [17], [18], [19], [21], [24], [29] and [30]. More recently multipoint boundary value problems involving a p-Laplacian type operator or the more general operator $-(\phi(x'))'$ has been studied in [1], [7], [8], [9], [10], [25] to mention a few.

We present in Section 2 some a priori estimates for functions $x(t)$ that satisfy the boundary conditions in (3). Our a priori estimates are sharper versions of the corresponding estimates in [16] and explicitly utilize the non-resonance condition for the boundary value problem (3). In section 3, we present an existence theorem for the boundary value problem (3) using degree theory.

2 A Priori Estimates

We shall assume throughout that ϕ, θ are odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ satisfying $\phi(0) = \theta(0) = 0$. We shall also assume that the homeomorphisms ϕ, θ satisfy the following conditions:

(a) For any constant $M > 0$,

$$\lim \sup_{z \longrightarrow \infty} \frac{\phi(Mz)}{\phi(z)} \equiv \alpha(M) < \infty. \tag{5}$$

(b) For any σ, $0 \leq \sigma < 1$,

$$\widetilde{\alpha}(\sigma) \equiv \lim \sup_{z \longrightarrow \infty} \frac{(\phi \circ \theta^{-1})(\sigma z)}{(\phi \circ \theta^{-1})(z)} < 1. \tag{6}$$

The boundary value problem (3) is a non-resonant problem if the boundary value problem (4) has only the trivial solution. This holds if and only if

$$(1 - \sum_{j=1}^{n-2} b_j)(1 - \sum_{i=1}^{m-2} a_i) \neq 0. \tag{7}$$

We shall assume in the following that $\xi_i,\tau_j \in (0,1)$, a_i, $b_j \in \mathbb{R}$, $i = 1, 2, \cdots, m-2$, $j = 1, 2, \cdots, n-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1$ satisfy the condition (7). We observe that when condition (7) holds then $1 - \sum_{i=1}^{m-2} a_i \neq 0$ and $1 - \sum_{j=1}^{n-2} b_j \neq 0$. Now, for $a \in \mathbb{R}$, we set $a^+ = \max(a, 0)$, $a^- = \max(-a, 0)$ so that $a = a^+ - a^-$ and $|a| = a^+ + a^-$. Accordingly, we notice that

$$\sigma_1 \equiv \min \left\{ \begin{array}{l} \frac{\sum_{i=1}^{m-2} a_i^+}{1+\sum_{i=1}^{m-2} a_i^-}, \frac{1+\sum_{i=1}^{m-2} a_i^-}{\sum_{i=1}^{m-2} a_i^+} \} \in [0,1), \text{ if } \sum_{i=1}^{m-2} a_i^+ \neq 0 \\ 0, \text{ if } \sum_{i=1}^{m-2} a_i^+ = 0. \end{array} \right. \tag{8}$$

$$\sigma_2 \equiv \min \left\{ \begin{array}{l} \frac{\sum_{j=1}^{n-2} b_j^+}{1+\sum_{j=1}^{n-2} b_j^-}, \frac{1+\sum_{j=1}^{n-2} b_j^-}{\sum_{j=1}^{n-2} b_j^+} \} \in [0,1), \text{ if } \sum_{j=1}^{n-2} b_j^+ \neq 0 \\ 0, \text{ if } \sum_{j=1}^{n-2} b_j^+ = 0. \end{array} \right. \tag{9}$$

are well-defined. The a priori estimate obtained in the following proposition is similar to the a priori estimate of Lemma 4 of [16]. We repeat the details given in Lemma 4 of [16] for the sake of completeness.

Proposition 1. *Let $\xi_i \in (0,1)$, $a_i \in \mathbb{R}$, $i = 1, 2, \cdots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, with $(1 - \sum_{i=1}^{m-2} a_i) \neq 0$, be given. Also let the function $x(t)$ be such that $x(t)$, $x'(t)$ be absolutely continuous on $[0,1]$ and $x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i)$. Then*

$$\| x \|_\infty \leq M \| x' \|_\infty , \tag{10}$$

where

$$\begin{aligned} M &= \min \{ \frac{1}{| \sum_{i=1}^{m-2} a_i |} (\sum_{i=1}^{m-2} | a_i | \lambda_i + \frac{\sum_{i=1}^{m-2} | a_i \xi_i |}{| 1 - \sum_{i=1}^{m-2} a_i |}), \\ & \quad 1 + \frac{\sum_{i=1}^{m-2} | a_i \xi_i |}{| 1 - \sum_{i=1}^{m-2} a_i |}, \frac{1}{1 - \sigma_1} \} \end{aligned}$$

with $\lambda_i = \max(\xi_i, 1 - \xi_i)$ for $i = 1, 2, \cdots, m-2$, and σ_1 as defined in (8).

Proof. Since $(1-\sum_{i=1}^{m-2} a_i)$ is non-zero we see that $M<\infty$. Next, we see from $x(\xi_i)-x(0)=\int_0^{\xi_i} x'(s)ds$ for $i=1,\,2,\,\cdots,\,m-2$ and the assumption that $x(0)=\sum_{i=1}^{m-2} a_i x(\xi_i)$ that $(1-\sum_{i=1}^{m-2} a_i)x(0)=\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} x'(s)ds$. It then follows that

$$|\, x(0)\,| \le \frac{\sum_{i=1}^{m-2} |\, a_i\xi_i\,|}{|\, 1-\sum_{i=1}^{m-2} a_i\,|} \parallel x' \parallel_\infty . \tag{11}$$

Also, since $x(t)=x(\xi_i)+\int_{\xi_i}^t x'(s)ds$, we see that

$$(\sum_{i=1}^{m-2} a_i)x(t)=\sum_{i=1}^{m-2} a_i x(\xi_i)+\sum_{i=1}^{m-2} a_i \int_{\xi_i}^t x'(s)ds = x(0)+\sum_{i=1}^{m-2} a_i \int_{\xi_i}^t x'(s)ds.$$

Accordingly,

$$|\sum_{i=1}^{m-2} a_i\,||\, x(t)\,| \le |\, x(0)\,| + \sum_{i=1}^{m-2} |\, a_i\,||\int_{\xi_i}^t x'(s)ds\ \,|\,,$$

$$\le (\frac{\sum_{i=1}^{m-2} |\, a_i\xi_i\,|}{|\, 1-\sum_{i=1}^{m-2} a_i\,|}+\sum_{i=1}^{m-2}\lambda_i \,|\, a_i\,|) \parallel x' \parallel_\infty ,$$

in view of (11). It is now immediate that

$$\parallel x \parallel_\infty \le \frac{1}{|\sum_{i=1}^{m-2} a_i\,|}(\frac{\sum_{i=1}^{m-2} |\, a_i\xi_i\,|}{|\, 1-\sum_{i=1}^{m-2} a_i\,|}+\sum_{i=1}^{m-2}\lambda_i \,|\, a_i\,|) \parallel x' \parallel_\infty . \tag{12}$$

If we next use the equation $x(t)=x(0)+\int_0^t x'(s)ds$ and the estimate (11) we obtain

$$\parallel x \parallel_\infty \le (\frac{\sum_{i=1}^{m-2} |\, a_i\xi_i\,|}{|\, 1-\sum_{i=1}^{m-2} a_i\,|}+1) \parallel x' \parallel_\infty . \tag{13}$$

Next, since $x(0)=\sum_{i=1}^{m-2} a_i x(\xi_i)$ we see that

$$x(0)+\sum_{i=1}^{m-2} a_i^- x(\xi_i)=\sum_{i=1}^{m-2} a_i^+ x(\xi_i).$$

It follows that there must exist χ_1, χ_2 in $[0,1]$ such that

$$(1+\sum_{i=1}^{m-2} a_i^-)x(\chi_1)=(\sum_{i=1}^{m-2} a_i^+)x(\chi_2). \tag{14}$$

If, now, one of $x(\chi_1)$, $x(\chi_2)$ is zero or $\sum_{i=1}^{m-2} a_i^+=0$ (which would imply $x(\chi_1)=0$, in view of the assumption $0\neq 1-\sum_{i=1}^{m-2} a_i = 1-\sum_{i=1}^{m-2} a_i^+ + \sum_{i=1}^{m-2} a_i^- = 1-\sum_{i=1}^{m-2} a_i^-$) we see using one of the two equations

$$x(t)=x(\chi_k)+\int_{\tau_k}^t x'(s)ds,\ k=1,2;\ t\in[0,1] \tag{15}$$

that

$$\parallel x \parallel_\infty \le \parallel x' \parallel_\infty . \tag{16}$$

If both $x(\chi_1)$, $x(\chi_2)$ are non-zero we see that $x(\chi_1) \neq x(\chi_2)$ since $1 - \sum_{i=1}^{m-2} a_i \neq 0$, or equivalently $1 + \sum_{i=1}^{m-2} a_i^- \neq \sum_{i=1}^{m-2} a_i^+$. It then follows easily from (14) and (15) that

$$\| x \|_\infty \leq \frac{1}{1-\sigma_1} \| x' \|_\infty \, , \tag{17}$$

where σ_1 is as defined in (8).

The proposition is now immediate from (12), (13), (16), (17) and the definitions of σ_1 as given in (8). ∎

With σ_2 as given in (9), we see that

$$\tilde{\alpha}(\sigma_2) = \limsup_{z \to \infty} \frac{(\phi \circ \theta^{-1})(\sigma_2 z)}{(\phi \circ \theta^{-1})(z)} < 1 \tag{18}$$

in view of our assumption (6). Let $\varepsilon > 0$ be such that $\tilde{\alpha}(\sigma_2)+\varepsilon < 1$ and the constant C_ε be such that

$$(\phi \circ \theta^{-1})(\sigma_2 z) \leq (\tilde{\alpha}(\sigma_2) + \varepsilon)(\phi \circ \theta^{-1})(z) + C_\varepsilon, \text{ for every } z \in \mathbb{R}. \tag{19}$$

Proposition 2. *Let $\tau_j \in (0,1)$, $b_j \in \mathbb{R}$, $j = 1, 2, \cdots, n-2$, $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1$, with $1 - \sum_{j=1}^{n-2} b_j \neq 0$ be given. Also let the function $x(t)$ be such that $x(t)$, $x'(t)$ be absolutely continuous on $[0,1]$ with $(\phi(x'))' \in L^1(0,1)$ and $\theta(x'(1)) = \sum_{j=1}^{n-2} b_j \theta(x'(\tau_j))$. Then*

$$\| \phi(x') \|_\infty \leq \frac{1}{1-\tilde{\alpha}(\sigma_2)-\varepsilon} \| (\phi(x'))' \|_{L^1(0,1)} + \frac{C_\varepsilon}{1-\tilde{\alpha}(\sigma_2)-\varepsilon}, \tag{20}$$

where ε and C_ε are as in (19). Moreover, if $\sum_{j=1}^{n-2} b_j^+ = 0$, then

$$\| \phi(x') \|_\infty \leq \| (\phi(x'))' \|_{L^1(0,1)} \, . \tag{21}$$

Proof. If $\sum_{j=1}^{n-2} b_j^+ = 0$, then $b_j \leq 0$ for every $j = 1, 2, \cdots, n-2$. It then follows easily for our assumption $\theta(x'(1)) = \sum_{j=1}^{n-2} b_j \theta(x'(\tau_j))$ that there exists an $\eta_0 \in [0,1]$ such that $\theta(x'(\eta_0)) = 0$ which implies $x'(\eta_0) = 0$, $\phi(x'(\eta_0)) = 0$. Estimate (21) is now immediate from

$$\phi(x'(t)) = \int_{\eta_0}^t (\phi(x'(s)))' ds.$$

Next, suppose that $x'(t) = c$, for all $t \in [0,1]$, where c is a constant. We then see from our assumptions $1 - \sum_{j=1}^{n-2} b_j \neq 0$, $\theta(x'(1)) = \sum_{j=1}^{n-2} b_j \theta(x'(\tau_j))$ that $x'(t) = 0$ for all $t \in [0,1]$ and accordingly both the estimates (20), (21) are satisfied.

Suppose next that $\sum_{j=1}^{n-2} b_j^+ \neq 0$ which implies $\sigma_2 \neq 0$. Then from $\theta(x'(1)) = \sum_{j=1}^{n-2} b_j \theta(x'(\tau_j))$ we see that

$$\theta(x'(1)) + \sum_{j=1}^{n-2} b_j^- \theta(x'(\tau_j)) = \sum_{j=1}^{n-2} b_j^+ \theta(x'(\tau_j)),$$

and thus from the definition of σ_2 and the intermediate value property for continuous functions we find that there exist η_1, η_2 in $[0,1]$ such that

$$\theta(x'(\eta_1)) = \sigma_2 \theta(x'(\eta_2))$$

so that
$$x'(\eta_1) = \theta^{-1}(\sigma_2\theta(x'(\eta_2)))$$
and
$$\phi(x'(\eta_1)) = (\phi\circ\theta^{-1})(\sigma_2\theta(x'(\eta_2)))$$
We, next, use the equation
$$\begin{aligned}\phi(x'(t)) &= \phi(x'(\eta_1)) + \int_{\eta_1}^{t}(\phi(x'))'(s)ds\\ &= (\phi\circ\theta^{-1})(\sigma_2\theta(x'(\eta_2))) + \int_{\eta_1}^{t}(\phi(x'))'(s)ds.\end{aligned}$$
to get
$$\phi(\| x' \|_\infty) \leq (\phi\circ\theta^{-1})(\sigma_2\theta(\| x' \|_\infty)) + \| (\phi(x'))' \|_{L^1(0,1)} . \tag{22}$$
Now, for σ_2 as given in (9), let $\varepsilon > 0$ be such that $\tilde{\alpha}(\sigma_2)+\varepsilon < 1$. It follows from the definition of $\tilde{\alpha}(\sigma_2)$ that there exists a constant $\tilde{C}_\varepsilon$ such that for $z \in \mathbf{R}$ we have
$$(\phi\circ\theta^{-1})(\sigma_2|z|) \leq (\tilde{\alpha}(\sigma_2)+\varepsilon)(\phi\circ\theta^{-1})(|z|) + \tilde{C}_\varepsilon,$$
(see (19)). We thus get from (22) that
$$\phi(\| x' \|_\infty) \leq (\tilde{\alpha}(\sigma_2)+\varepsilon)(\phi\circ\theta^{-1})(\theta(\| x' \|_\infty)) + \| (\phi(x'))' \|_{L^1(0,1)} + \tilde{C}_\varepsilon.$$
Hence, we obtain the estimate
$$\phi(\| x' \|_\infty) \leq \frac{1}{(1-(\tilde{\alpha}(\sigma_2)+\varepsilon))} \| (\phi(x'))' \|_{L^1(0,1)} + C_\varepsilon,$$
where we have set $\frac{\tilde{C}_\varepsilon}{(1-(\tilde{\alpha}(\sigma_2)+\varepsilon))} = C_\varepsilon$.

This completes the proof of the proposition. ■

3 Existence Theorem

Let ϕ, θ be odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ satisfying $\phi(0) = \theta(0) = 0$, $f : [0,1]\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying Carathéodory conditions and $e : [0,1] \to \mathbb{R}$ be a function in $L^1[0,1]$. Let $\xi_i,\tau_j \in (0,1)$, a_i, $b_j \in \mathbb{R}$, $i = 1, 2, \cdots, m-2$, $j = 1, 2, \cdots, n-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1$ with $(1-\sum_{j=1}^{n-2} b_j)(1-\sum_{i=1}^{m-2} a_i) \neq 0$ be given.

Theorem 3. *Let $f : [0,1]\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying Carathéodory's conditions such that there exist non-negative functions $d_1(t)$, $d_2(t)$, and $r(t)$ in $L^1(0,1)$ such that*
$$|f(t,u,v)| \leq d_1(t)\phi(|u|) + d_2(t)\phi(|v|) + r(t),$$
for a. e. $t \in [0,1]$ and all u, $v \in \mathbb{R}$. Suppose, further,
$$\alpha(M)||d_1||_{L^1(0,1)} + ||d_2||_{L^1(0,1)} < 1 - \widetilde{\alpha}(\sigma_2) \tag{23}$$
where M is as defined in Proposition 1, $\alpha(M)$ is as defined in (5), σ_2 and $\widetilde{\alpha}(\sigma_2)$ are as defined in (9), (18). Then, for every given function $e(t) \in L^1[0,1]$, the boundary value problem (3) has at least one solution $x(t) \in C^1[0,1]$.

Proof. We consider the family of boundary value problems

$$\begin{gathered}(\phi(x'))' = \lambda f(t,x,x') + \lambda e,\ 0 < t < 1,\ \lambda \in [0,1]\\ x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i),\ \theta(x'(1)) = \sum_{j=1}^{n-2} b_j \theta(x'(\tau_j)).\end{gathered} \tag{24}$$

Also, we define an operator $\Psi : C^1[0,1] \times [0,1] \longrightarrow C^1[0,1]$ by setting for $(x,\lambda) \in C^1[0,1] \times [0,1]$

$$\begin{aligned}\Psi(x,\lambda)(t) &= x(0) + \int_0^t \phi^{-1}(\phi(x'(0) + \lambda \int_0^s (f(\tau, x(\tau), x'(\tau)) + e(\tau))d\tau)ds\\ &\quad + (x(0) - \sum_{i=1}^{m-2} a_i x(\xi_i)) + t(\theta(x'(1)) - \sum_{j=1}^{n-2} b_j \theta(x'(\tau_j)))\end{aligned} \tag{25}$$

Let us, suppose that $x(t) \in C^1[0,1]$ is a solution to the operator equation, for some $\lambda \in [0,1]$,

$$\begin{aligned}x &= \Psi(x,\lambda)\\ &= x(0) + \int_0^t \phi^{-1}(\phi(x'(0) + \lambda \int_0^s (f(\tau, x(\tau), x'(\tau)) + e(\tau))d\tau)ds\\ &\quad + (x(0) - \sum_{i=1}^{m-2} a_i x(\xi_i)) + t(\theta(x'(1)) - \sum_{j=1}^{n-2} b_j \theta(x'(\tau_j)))\end{aligned} \tag{26}$$

Evaluating the equation (26) at $t = 0$ we see that $x(t)$ satisfies the boundary condition

$$x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i).$$

Next, we differentiate the equation (26) with respect to t to get

$$\begin{aligned}x'(t) &= \phi^{-1}(\phi(x'(0) + \lambda \int_0^t (f(\tau, x(\tau), x'(\tau)) + e(\tau))d\tau\\ &\quad + \theta(x'(1)) - \sum_{j=1}^{n-2} b_j \theta(x'(\tau_j)).\end{aligned} \tag{27}$$

Evaluating, now, the equation (27) at $t = 0$ we see that $x(t)$ satisfies the boundary condition

$$\theta(x'(1)) = \sum_{j=1}^{n-2} b_j \theta(x'(\tau_j)),$$

and on differentiating the equation (27) with respect to t we get

$$(\phi(x'))' = \lambda f(t,x,x') + \lambda e,\ 0 < t < 1,\ \lambda \in [0,1].$$

Thus we see that if $x(t) \in C^1[0,1]$ is a solution to the operator equation $x = \Psi(x,\lambda)$ for some $\lambda \in [0,1]$ then $x(t)$ is a solution to the boundary value problems (24) for the corresponding $\lambda \in [0,1]$. Conversely, it is easy to see that if $x(t) \in C^1[0,1]$ is a solution to the boundary value problems (24) for some $\lambda \in [0,1]$ then $x(t) \in C^1[0,1]$ is a solution to the operator equation $x = \Psi(x,\lambda)$ for the corresponding $\lambda \in [0,1]$.

Next, it is easy to show, following standard arguments, that $\Psi : C^1[0,1] \times [0,1] \longrightarrow C^1[0,1]$ is a completely continuous operator.

We shall next show that there is a constant $R > 0$, independent of $\lambda \in [0,1]$, such that if $x(t) \in C^1[0,1]$ is a solution to (26), equivalently to the boundary value problems (24), for some $\lambda \in [0,1]$ then $||x||_{C^1[0,1]} < R$.

We note first that if $x(t) \in C^1[0,1]$ satisfies

$$x = \Psi(x, 0), \tag{28}$$

then $x(t) = 0$ for all $t \in [0,1]$. Indeed, from the definition of Ψ or from the boundary value problem (24), it follows that $x(t) = x(0) + x'(0)t$. It then follows from the two boundary conditions in (24) and the non-resonance assumption (7) that $x(0) = x'(0) = 0$, implying that $x(t) = 0$ for all $t \in [0,1]$.

We shall assume, in the following, that $\lambda \in (0,1]$. We shall also assume that σ_2, as defined in (9) is positive, since the proof for the case $\sigma_2 = 0$ is simpler. Let us choose $\varepsilon > 0$ such that $\widetilde{\alpha}(\sigma_2) + \varepsilon < 1$ and

$$(\alpha(M) + \varepsilon)||d_1||_{L^1(0,1)} + ||d_2||_{L^1(0,1)} < 1 - \widetilde{\alpha}(\sigma_2) - \varepsilon, \tag{29}$$

which is possible to do, in view of our assumption (23). Here M is as defined in Proposition 1 and $\alpha(M)$ is as defined in (5) so that for the $\varepsilon > 0$, chosen above, there exists a constant $C_\varepsilon^1 > 0$ such that

$$\phi(Mz) \leq (\alpha(M) + \varepsilon)\phi(z) + C_\varepsilon^1, \text{ for every } z \in \mathbb{R}. \tag{30}$$

Also, from Proposition 2 we see that there is a constant $C_\varepsilon^2 > 0$, for the chosen $\varepsilon > 0$, such that

$$\phi(||x'||_\infty) \leq \frac{1}{1 - \widetilde{\alpha}(\sigma_2) - \varepsilon}||(\phi(x'))'||_{L^1(0,1)} + C_\varepsilon^2. \tag{31}$$

We, now, see from the equation in (24), using our assumptions on the function f, Proposition 1, and estimates (30), (31) that

$$\begin{aligned}
||(\phi(x'))'||_{L^1(0,1)} &\leq \phi(||x||_\infty)||d_1||_{L^1(0,1)} + \phi(||x'||_\infty)||d_2||_{L^1(0,1)} \\
&\quad + ||r||_{L^1(0,1)} + ||e||_{L^1(0,1)} \\
&\leq \phi(M||x'||_\infty)||d_1||_{L^1(0,1)} + \phi(||x'||_\infty)||d_2||_{L^1(0,1)} \\
&\quad + ||r||_{L^1(0,1)} + ||e||_{L^1(0,1)} \\
&\leq ((\alpha(M) + \varepsilon)||d_1||_{L^1(0,1)} + ||d_2||_{L^1(0,1)})\phi(||x'||_\infty) \\
&\quad + ||r||_{L^1(0,1)} + ||e||_{L^1(0,1)} + C_\varepsilon^1||d_1||_{L^1(0,1)} \\
&\leq \frac{(\alpha(M) + \varepsilon)||d_1||_{L^1(0,1)} + ||d_2||_{L^1(0,1)}}{1 - \widetilde{\alpha}(\sigma_2) - \varepsilon}||(\phi(x'))'||_{L^1(0,1)} + C_\varepsilon,
\end{aligned}$$

where $C_\varepsilon = ||r||_{L^1(0,1)} + ||e||_{L^1(0,1)} + C_\varepsilon^1||d_1||_{L^1(0,1)} + C_\varepsilon^2[(\alpha(M) + \varepsilon)||d_1||_{L^1(0,1)} + ||d_2||_{L^1(0,1)}]$. It, now, follows from (29) that there exists a constant R_0, independent of $\lambda \in [0,1]$, such that if $x(t) \in C^1[0,1]$ is a solution to the boundary value problems (24) for some $\lambda \in [0,1]$ then

$$||(\phi(x'))'||_{L^1(0,1)} \leq R_0.$$

This combined with (31) and (10) give that there exists a constant $R > 0$ such that

$$||x||_{C^1[0,1]} < R.$$

This then implies that $\deg_{LS}(I - \Psi(\cdot, \lambda), B(0, R), 0)$ is well-defined for all $\lambda \in [0, 1]$, where $B(0, R)$ is the ball with center 0 and radius R in $C^1[0, 1]$.

Let, now, X denote the two-dimensional subspace of $C^1[0, 1]$ given by

$$X = \{A + Bt \mid \text{for } A,\ B \in \mathbb{R}\}. \tag{32}$$

Let us define the isomorphism $i : \mathbb{R}^2 \longrightarrow X$ by

$$i\begin{pmatrix} A \\ B \end{pmatrix} = i_{\begin{pmatrix} A \\ B \end{pmatrix}} \in X, \text{ for } \begin{pmatrix} A \\ B \end{pmatrix} \in \mathbb{R}^2, \tag{33}$$

where

$$i_{\begin{pmatrix} A \\ B \end{pmatrix}}(t) = A + Bt, \text{ for } t \in [0, 1]. \tag{34}$$

We note that for $v(t) = A + Bt \in X$ we have

$$(I - \Psi(\cdot, 0))(v) = -(1 - \sum_{i=1}^{m-2} a_i)A + (\sum_{i=1}^{m-2} a_i\xi_i)B - t(1 - \sum_{j=1}^{n-2} b_j)\theta(B), \tag{35}$$

Consider the following mappings from $\mathbb{R}^2$ onto $\mathbb{R}^2$:

$$F_1 : \begin{pmatrix} A \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} -(1 - \sum_{i=1}^{m-2} a_i) & \sum_{i=1}^{m-2} a_i\xi_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{36}$$

$$F_2 : \begin{pmatrix} A \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{37}$$

$$F_3 : \begin{pmatrix} A \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -(1 - \sum_{j=1}^{n-2} b_j) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \tag{38}$$

Now we see that

$$\begin{aligned}
&(F_3 \circ F_2 \circ F_1) \begin{pmatrix} A \\ B \end{pmatrix} \\
= &\begin{pmatrix} 1 & 0 \\ 0 & -(1 - \sum_{j=1}^{n-2} b_j) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix} \begin{pmatrix} -(1 - \sum_{i=1}^{m-2} a_i) & \sum_{i=1}^{m-2} a_i\xi_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \\
= &\begin{pmatrix} -(1 - \sum_{i=1}^{m-2} a_i)A + (\sum_{i=1}^{m-2} a_i\xi_i)B \\ -(1 - \sum_{j=1}^{n-2} b_j)\theta(B) \end{pmatrix}.
\end{aligned}$$

We thus see that

$$(I - \Psi(\cdot, 0)) \left(i_{\begin{pmatrix} A \\ B \end{pmatrix}} \right) = i_{(F_3 \circ F_2 \circ F_1)\begin{pmatrix} A \\ B \end{pmatrix}}$$

and it follows that

$$F_3 \circ F_2 \circ F_1 = i^{-1} \circ ((I - \Psi(\cdot, 0))|_X \circ i.$$

Now, we see from the homotopy invariance property of the Leray-Schauder degree that

$$\begin{aligned} \deg_{LS}(I - \Psi(\cdot, 1), B(0, R), 0) &= \deg_{LS}(I - \Psi(\cdot, 0), B(0, R), 0) \\ &= \deg_B(I - \Psi(\cdot, 0)|_X, X \cap B(0, R), 0) \\ &= \deg_B(F_3 \circ F_2 \circ F_1, \mathbb{B}(0, R), 0), \end{aligned}$$

where $\mathbb{B}(0, R)$ denotes the ball of radius R in $\mathbb{R}^2$ with center at the origin. Finally, we have, using standard results for Brouwer degree, (see [26], [27], [28]) that

$$\deg_B(F_3 \circ F_2 \circ F_1, \mathbb{B}(0, R), 0) \neq 0,$$

in view of the non-resonance assumption (7) i.e. $(1 - \sum_{i=1}^{m-2} a_i)(1 - \sum_{j=1}^{n-2} b_j) \neq 0$. Accordingly, we have $\deg_{LS}(I - \Psi(\cdot, 1), B(0, R), 0) \neq 0$ and there exists at least one $x(t) \in B(0, R) \subset C^1[0, 1]$ that satisfies

$$x = \Psi(x, 1),$$

or equivalently $x(t)$ is a solution to the boundary value (3). This completes the proof of the theorem. ∎

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