Examples of non-archimedean twisted nuclear Fréchet spaces

Wiesław Śliwa

Abstract

A Fréchet space is called *twisted* if it is not isomorphic to a countable product of Fréchet spaces with continuous norms. It is easy to show that no non-archimedean Fréchet space with a Schauder basis is twisted. We construct examples of non-archimedean twisted nuclear Fréchet spaces.

Introduction

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot| : \mathbb{K} \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [8], [10], [3] and [9].

Any finite-dimensional lcs of dimension n is isomorphic to the Banach space $\mathbb{K}^n$. Every infinite-dimensional Banach space E of countable type is isomorphic to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero with the sup-norm and any closed subspace of E is complemented ([9], Theorem 3.16). Nevertheless, the world of Fréchet spaces of countable type is very rich (see [11], [12], [13], [14]).

It is not hard to prove that any Fréchet space with a Schauder basis is isomorphic to a countable product of Fréchet spaces with continuous norms (see Proposition 1). In [11] we constructed many examples of Fréchet spaces of countable type without a Schauder basis but each of these spaces is a countable product of Fréchet spaces with continuous norms. It arises a natural question whether any Fréchet space of countable type is isomorphic to a countable product of Fréchet spaces with continuous norms. Recall that in [15] we have shown that any infinite-dimensional Fréchet space of countable type is homeomorphic to the Fréchet space $\mathbb{K}^{\mathbb{N}}$ with the product

2000 *Mathematics Subject Classification :* Primary 46S10, Secondary 46A04, 46A11.
Key words and phrases : Projective limits of Fréchet spaces, strong dual of Fréchet spaces.

topology ([15], Corollary 5). In this paper, developing some ideas of [7] and using the generalized Köthe spaces studied in [5], we shall prove that the answer to the above problem is negative, even for nuclear Fréchet spaces (Theorem 7 and Proposition 8).

Preliminaries

A *seminorm* on a linear space E is a function $p : E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x + y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\ker p = \{0\}$.

The set of all continuous seminorms on a lcs E is denoted by $\mathcal{P}(E)$. A family $\mathcal{B} \subset \mathcal{P}(E)$ is a *base* in $\mathcal{P}(E)$ if for every $p \in \mathcal{P}(E)$ there exists $q \in \mathcal{B}$ with $q \geq p$.

Any metrizable lcs E possesses a non-decreasing base (p_k) in $\mathcal{P}(E)$.

A *Fréchet space* is a metrizable complete lcs. Let (x_n) be a sequence in a Fréchet space E. The series $\sum_{n=1}^{\infty} x_n$ is convergent in E if and only if $\lim x_n = 0$.

A sequence (x_n) in a lcs E is a *basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$. If additionally the coefficient functionals $f_n : E \to \mathbb{K}$, $x \to \alpha_n$ $(n \in \mathbb{N})$ are continuous, then (x_n) is a *Schauder basis* in E.

Let E be a lcs. A sequence $(x_n) \subset E$ is *orthogonal with respect to* $\mathcal{B} \subset \mathcal{P}(E)$ if $p(\sum_{i=1}^{n} \alpha_i x_i) = \max_{1 \leq i \leq n} p(\alpha_i x_i)$ for all $p \in \mathcal{B}, n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$. Every Schauder basis in a Fréchet space F is orthogonal with respect to some (non-decreasing) base (p_k) in $\mathcal{P}(F)$ ([4], Proposition 1.7).

Put $B_{\mathbb{K}} = \{\alpha \in \mathbb{K} : |\alpha| \leq 1\}$. A subset B of a lcs E is *compactoid* if for each neighbourhood U of 0 in E there exists a finite subset $\{a_1, \ldots, a_n\}$ of E such that $B \subset U + \{\sum_{i=1}^{n} \alpha_i a_i : \alpha_1, \ldots, \alpha_n \in B_{\mathbb{K}}\}$

Let E and F be locally convex spaces. A linear map $T : E \to F$ is *compact* if there exists a neighbourhood U of 0 in E such that $T(U)$ is compactoid in F.

For any seminorm p on a lcs E the map $\overline{p} : E_p \to [0, \infty), x + \ker p \to p(x)$ is a norm on $E_p = (E/\ker p)$. Let $\varphi_p : E \to E_p, x \to x + \ker p$.

A lcs E is *nuclear* if for any $p \in \mathcal{P}(E)$ there exists $q \in \mathcal{P}(E)$ with $q \geq p$ such that the map $\varphi_{p,q} : (E_q, \overline{q}) \to (E_p, \overline{p})$, $x + \ker q \to x + \ker p$, is compact.

A lcs E is *of countable type* if for any $p \in \mathcal{P}(E)$ the normed space $(E_p, \overline{p})$ contains a linearly dense countable subset.

By a *Köthe matrix* we mean an infinite matrix $B = (b_{k,n})$ of positive real numbers such that $\forall k, n \in \mathbb{N} : b_{k,n} \leq b_{k+1,n}$. The *Köthe space* associated with the Köthe matrix B is the Fréchet space

$$K(B) = \{(\xi_n) \in \mathbb{K}^{\mathbb{N}} : b_{k,n}|\xi_n| \to_n 0 \text{ for any } k \in \mathbb{N}\}$$

with the base (p_k) of norms: $p_k((\xi_n)) = k \max_n b_{k,n}|\xi_n|, k \in \mathbb{N}$. The sequence (e_n) of coordinate vectors is a Schauder basis of $K(B)$ ([1], Proposition 2.2).

A Köthe matrix $B = (b_{k,n})$ is *nuclear* if $\forall k \in \mathbb{N}\, \exists m \in \mathbb{N} : (b_{k,n}/b_{m,n}) \to_n 0$. The Köthe space $K(B)$ is nuclear if and only if B is nuclear ([1], Proposition 3.5).

Let $a = (a_n)$ be a non-decreasing unbounded sequence of positive real numbers. Then the following Köthe spaces are nuclear:

(1) $A_1(a) = K(B)$ with $B = (b_{k,n}), b_{k,n} = (\frac{k}{k+1})^{a_n}$;
(2) $A_\infty(a) = K(B)$ with $B = (b_{k,n}), b_{k,n} = k^{a_n}$.

$A_1(a)$ and $A_\infty(a)$ are the *power series spaces* (*of finite* and *infinite type*, respectively)(see [1]).

Let E be a lcs. We write $\mathcal{B}(E)$ for the family of all bounded subset of E. The strong dual of E, that is the topological dual of E with the strong topology, will be denoted by E' (not by E'_b). If E is of countable type then E' separates points of E ([10], Theorem 4.4). The *polar* of $A \subset E$ is $A^\circ = \{f \in E' : |f(x)| \leq 1 \,\text{if}\, x \in A\}$. If M is a subspace of E then $M^\circ = \{f \in E' : f(x) = 0 \text{ for } x \in M\}$.

Let E and F be locally convex spaces. The space of all linear continuous maps from E to F is denoted by $L(E, F)$. An operator $T \in L(E, F)$ is an *isomorphism* if T is injective, surjective and the inverse map T^{-1} is continuous. E is *isomorphic* to F if there exists an isomorphism $T : E \to F$. If $T \in L(E, F)$ and $T(E) = F$ then the dual map $T' : F' \to E'$ is continuous, injective and $T'(F') = (\ker T)^\circ$. If $T \in L(E, F)$ then $(T(A))^\circ = (T')^{-1}(A^\circ)$ for $A \subset E$.

For fundamentals of projective and inductive limits of locally convex spaces we refer to [3].

Results

First we shall prove that no Fréchet space with a Schauder basis is twisted.

Proposition 1. *Let X be a Fréchet space with a Schauder basis (x_n). Then X is isomorphic to a countable product of Fréchet spaces with Schauder bases and continuous norms. In particular X is not twisted.*

Proof. The basis (x_n) is orthogonal with respect to some non-decreasing base (p_k) in $\mathcal{P}(X)$. Put $p_0(x) = 0$ for all $x \in X$ and $M_k = \{n \in \mathbb{N} : 0 = p_{k-1}(x_n) < p_k(x_n)\}$ for $k \in \mathbb{N}$. Denote by X_k the closed linear span of $\{x_n : n \in M_k\}$ for $k \in \mathbb{N}$ (if $M_k = \emptyset$, then $X_k = \{0\}$). Clearly, $(x_n)_{n \in M_k}$ is a Schauder basis in X_k and $p_k|X_k$ is a continuous norm on X_k for $k \in \mathbb{N}$. We shall prove that X is isomorphic to the product space $\prod_{k=1}^\infty X_k$. Let (f_n) be the sequence of coefficient functionals associated with the basis (x_n). For any $k \in \mathbb{N}$ the map

$$P_k : X \to X_k, P_k(x) = \sum_{n \in M_k} f_n(x) x_n$$

is a linear continuous projection from X onto X_k. For every $(x_k) \in \prod_{k=1}^\infty X_k$ the series $\sum_{k=1}^\infty x_k$ is convergent in X, since $p_k(x_n) = 0$ for all $n, k \in \mathbb{N}$ with $n > k$. Thus the linear map $P : X \to \prod_{k=1}^\infty X_k, Px = (P_k x)$, is continuous, injective and surjective. By the open mapping theorem P is an isomorphism. □

In the proof our main result we will need the following two lemmas.

Lemma 2. *Let Z be a closed subspace of a Fréchet space X of countable type. If Z° is complemented in X', then Z is complemented in X.*

Proof. Let S be a complement of Z° in X' and let $V = \bigcap_{f \in S} \ker f$. Clearly $Z \cap V = \{0\}$. Let $(z_n) \subset Z, (v_n) \subset V, z_0 \in Z, v_0 \in V$ and $z_n + v_n \to z_0 + v_0$ in X. Let $g \in Z'$. By the Hahn-Banach property ([10], Theorem 4.2), there exist $f_1 \in Z^\circ$ and $f_2 \in S$ such that $g(x) = (f_1 + f_2)(x) = f_2(x)$ for all $x \in Z$. We have

$f_2(z_n+v_n) \to f_2(z_0+v_0)$, so $f_2(z_n) \to f_2(z_0)$. Thus $g(z_n) \to g(z_0)$ for every $g \in Z'$, so (z_n) converges weakly to z_0 in Z. By [10], Theorem 4.11, $z_n \to z_0$ in Z, so in X. Hence $v_n \to v_0$ in X. Thus the subspace $W \doteq Z+V$ of X is isomorphic to the Fréchet space $Z \times V$. It follows that W is closed in X, so W is weakly closed in X ([10], Corollary 4.9). On the other hand it is easy to check that W is weakly dense in X. Thus $W = X$, so V is a complement of Z in X. □

Lemma 3. *Let X be a nuclear Fréchet space with a continuous norm p. Let $A = \{x \in X : p(x) \le 1\}$. Then the set A° is bounded and linearly dense in X'.*

Proof. Let $B \in \mathcal{B}(X)$. Then there exists $\alpha \in \mathbb{K}$ such that $p(b) < |\alpha|$ for all $b \in B$. Let $f \in A^\circ$. For $b \in B$ we have $|f(\alpha^{-1}b)| \le 1$, so $f \in \alpha B^\circ$. Thus $A^\circ \subset \alpha B^\circ$. It follows that A° is bounded in X'. Let M be the closed linear span of A° in X'. Suppose, by contradiction, that $M \ne X'$. X is reflexive ([10], Theorem 10.3) and X' is of countable type ([10], Corollary 8.7), so there exists a non-zero element x_0 in X such that $f(x_0) = 0$ for all $f \in M$ ([10], Corollaries 4.8 and 4.9 and Proposition 3.4(iv)). Let $\gamma, \beta \in \mathbb{K}$ with $0 < |\gamma| < p(x_0)$ and $|\beta| > 1$. Then there exists $g \in X'$ with $g(x_0) = \gamma$ such that $|g(x)| \le |\beta| p(x)$ for all $x \in X$ ([10], Theorem 4.2). Thus $g \in \beta A^\circ \subset M$, so $g(x_0) = 0$; a contradiction. □

We will also need the following proposition on projective limits of Fréchet spaces.

Proposition 4. *Let (E_n) be a sequence of nuclear Fréchet spaces and let π_n be a continuous linear map from E_{n+1} onto E_n with $\ker \pi_n \ne \{0\}$ for $n \in \mathbb{N}$. Then the strong dual of a projective limit of the projective system $\langle (E_n), (\pi_n) \rangle$ is isomorphic to an inductive limit of the inductive system $\langle (E_n'), (\pi_n') \rangle$.*

Proof. The closed subspace $G \doteq \{(x_n) \in \prod_{n=1}^\infty E_n : \pi_n(x_{n+1}) = x_n \text{ for } n \in \mathbb{N}\}$ of the product space $E \doteq \prod_{n=1}^\infty E_n$, with the coordinate maps $\varphi_n : G \to E_n, n \in \mathbb{N}$, is a projective limit of the projective system $\langle (E_n), (\pi_n) \rangle$ ([3], 1.3.2). Clearly, G is a nuclear Fréchet space and $\varphi_n(G) = E_n$ for $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ the adjoint operator $\pi_n' : E_n' \to E_{n+1}'$ is continuous and injective, and $\pi_n'(E_n')$ is a proper closed subspace of E_{n+1}'. Thus $\langle (E_n'), (\pi_n') \rangle$ is an inductive system ([3], Definition 1.1.1). Let F be the locally convex direct sum $\bigoplus_{n=1}^\infty E_n'$ of (E_n'). By [6], Proposition 2.8, F is isomorphic to E' and the linear map

$$J : F \to E', (J((x_n')))((x_n)) = \sum_{n=1}^\infty x_n'(x_n)$$

is an isomorphism. Clearly, $\hat{G} \doteq J^{-1}(G^\circ)$ is a closed subspace of F. Denote by H the linear subspace of F generated by the following subset of F

$$\{(z_1', -\pi_1'(z_1'), 0, \ldots) : z_1' \in E_1'\} \cup \{(0, z_2', -\pi_2'(z_2'), 0, \ldots) : z_2' \in E_2'\} \cup \ldots.$$

We shall prove that $\hat{G} = H$. Put $\pi_{n+1,n} = \pi_n$ and $\pi_{m,n} = \pi_n \circ \ldots \circ \pi_{m-1}$ for all $m, n \in \mathbb{N}$ with $m > n+1$. Let $x' = (x_n') \in \hat{G}$. For some $m \in \mathbb{N}$ we have $x_n' = 0$ for all $n > m$. Let $x = (x_n) \in G$. Then $0 = (Jx')(x) = \sum_{n=1}^m x_n'(x_n) = \sum_{n=1}^m x_n'(\pi_{m+1,n}(x_{m+1})) = (\sum_{n=1}^m \pi_{m+1,n}' x_n')(x_{m+1})$. Since $\varphi_{m+1}(G) = E_{m+1}$, we get $\sum_{n=1}^m \pi_{m+1,n}' x_n' = 0$. Hence $x' = (x_1', \ldots, x_m', -\sum_{n=1}^m \pi_{m+1,n}' x_n', 0, \ldots) =$

$(z'_1, -\pi'_1(z'_1), 0, \ldots) + (0, z'_2, -\pi'_2(z'_2), 0, \ldots) + \ldots + (0, \ldots, 0, z'_m, -\pi'_m(z'_m), 0, \ldots)$ where $z'_1 = x'_1, z'_2 = x'_2 + \pi'_1 x'_1, \ldots, z'_m = x'_m + \sum_{n=1}^{m-1} \pi'_{m,n} x'_n$. Thus $x' \in H$; so $\hat{G} \subset H$. Let $m \in \mathbb{N}, x'_m \in E'_m$ and $x' = (0, \ldots, 0, x'_m, -\pi'_m x'_m, 0, \ldots)$. For any $x = (x_n) \in G$ we have $(Jx')(x) = x'_m(x_m) - (\pi'_m x'_m)(x_{m+1}) = x'_m(x_m - \pi_m x_{m+1}) = 0$. Hence $x' \in \hat{G}$; so $H \subset \hat{G}$. Thus $\hat{G} = H$; in particular H is a closed subspace of F.

Let $n \in \mathbb{N}$. Let $J_n : E'_n \to F, x'_n \to (x'_k)$, where $x'_k = x'_n$ for $k = n$ and $x'_k = 0$ for $k \neq n$. Denote by T the quotient map from F onto F/H. It is easy to see that the linear map

$$\alpha_n : E'_n \to F/H, \alpha_n(x) = T(J_n(x))$$

is continuous and injective, and $\alpha_{n+1} \circ \pi'_n = \alpha_n$ for all $n \in \mathbb{N}$. Hence $\alpha_n(E'_n)$ is a proper subspace of $\alpha_{n+1}(E'_{n+1})$ for any $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} \alpha_n(E'_n) = \bigcup_{n=1}^{\infty}(\alpha_1(E'_1) + \ldots + \alpha_n(E'_n)) = T(\bigcup_{n=1}^{\infty}(J_1(E'_1) + \ldots + J_n(E'_n))) = T(\bigoplus_{k=1}^{\infty} E'_k) = F/H$.

By [3], 1.1.2, the quotient space F/H with the maps $\alpha_n : E'_n \to F/H, n \in \mathbb{N}$, is an inductive limit of the inductive system $\langle (E'_n), (\pi'_n) \rangle$; by [3], 1.1.4, F/H is the inductive limit of the inductive sequence $(\alpha_n(E'_n))$, where $\alpha_n(E'_n)$ inherits the topology of E'_n through α_n for any $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. By [2], Proposition 2.5, we get $\pi_n(\mathcal{B}(E_{n+1})) = \mathcal{B}(E_n)$ since any bounded subset of E_n is compactoid. For $B \in \mathcal{B}(E_{n+1})$ we get $\alpha_{n+1}(B^\circ) \cap \alpha_n(E'_n) = \alpha_{n+1}(B^\circ \cap \pi'_n(E'_n)) = \alpha_{n+1}(\pi'_n((\pi'_n)^{-1}(B^\circ)) = \alpha_n((\pi'_n)^{-1}(B^\circ)) = \alpha_n((\pi_n(B))^\circ)$. It follows that the topology of $\alpha_n(E'_n)$ agrees with the topology induced from $\alpha_{n+1}(E'_{n+1})$ onto $\alpha_n(E'_n)$. Thus the inductive sequence $(\alpha_n(E'_n))$ is strict. Moreover $\alpha_n(E'_n)$ is closed in $\alpha_{n+1}(E'_{n+1})$, since $\pi'_n(E'_n)$ is closed in E'_{n+1} $(n \in \mathbb{N})$.

Using [6], Theorem 5.12, we infer that the space G' is isomorphic to the quotient $F/\hat{G}$ of F and the linear map $\Psi : F/\hat{G} \to G', (\Psi(Ty))(x) = (Jy)(x)$ is an isomorphism ($T : F \to F/\hat{G}$ is the quotient map; $y \in F, x \in G$). Thus the strong dual of the projective limit G of the projective system $\langle (E_n), (\pi_n) \rangle$ is isomorphic to the inductive limit F/H of the inductive system $\langle (E'_n), (\pi'_n) \rangle$. □

By the proof of the last proposition we get the following

Corollary 5. (a) *$\alpha_n(E'_n)$ is a proper closed subspace of $\alpha_{n+1}(E'_{n+1}), n \in \mathbb{N}$;*
(b) *F/H is the inductive limit of the strict inductive sequence $(\alpha_n(E'_n))$;*
(c) *F/H is isomorphic to the strong dual of G.*

Moreover we have the following

Remark 6. *Let (Z_n) be a strict inductive sequence with the inductive limit Z. Assume that Z_n is closed in Z_{n+1} for any $n \in \mathbb{N}$. If a subspace Z_0 of Z contains a bounded subset B which is linearly dense in Z_0, then $Z_0 \subset Z_m$ for some $m \in \mathbb{N}$.*

Proof. By [3], Theorem 1.4.13, the inductive sequence (Z_n) is regular and Z_n is closed in Z for any $n \in \mathbb{N}$. Thus $B \subset Z_m$ for some $m \in \mathbb{N}$; hence $Z_0 \subset Z_m$. □

Let X and Y be nuclear Fréchet spaces with continuous norms such that there exists a continuous map Q from X onto Y which kernel $\ker Q$ is not complemented in X. For example, we can take $X = A_\infty(a)$ and $Y = A_1(b)$ provided $a = (a_n)$ and $b = (b_n)$ are increasing unbounded sequences of positive numbers with $\lim_n(a_n/b_n) = 0$ and $\sup_n(a_{2n}/a_n) < \infty$ (see [16], Proposition 21 and [1], Proposition 4.3; the assumption that $\mathbb{K}$ is spherically complete can be omitted in [1], Proposition 4.3). Let (r_k) and (p_k) be non-decreasing bases of continuous norms on X and Y, respectively. Let $B = (b_{k,n})$ be a nuclear Köthe matrix. The space $K(B,Y) = \{(y_n) \subset Y : b_{k,n}p_k(y_n) \to_n 0$ for every $k \in \mathbb{N}\}$with the base of norms $q_k((y_n)) = \max_n b_{k,n}p_k(y_n), k \in \mathbb{N}$, is a nuclear Fréchet space ([5], Lemma 2.2 and Theorem 7.1; $K(B,Y) = \Lambda_0(\mathbb{N}, P, Y)$ for $P = \{b_k = (b_{k,n})_{n\in\mathbb{N}} : k \in \mathbb{N}\}$). The strong dual of $K(B,Y)$ is nuclear ([10], Corollary 11.5). Put $p_A(y') = \sup_{y\in A} |y'(y)|$ for $A \in \mathcal{B}(Y)$ and $y' \in Y'$. Consider the linear space $S(B,Y') = \{(y'_n) \subset Y' : |\xi_n|p_A(y'_n) \to_n 0$ for all $\xi = (\xi_n) \in K(B), A \in \mathcal{B}(Y)\}$ with the locally convex topology generated by all seminorms of the form $r_{\xi,A}((y'_n)) = \max_n |\xi_n|p_A(y'_n)$, where $\xi = (\xi_n) \in K(B), A \in \mathcal{B}(Y)$. For any $\phi = (y'_n) \in S(B,Y')$ the linear functional $f_\phi : K(B,Y) \to \mathbb{K}, f_\phi((y_n)) = \sum_{n=1}^{\infty} y'_n(y_n)$ is continuous and the linear map $\Phi : S(B,Y') \to (K(B,Y))', \phi \to f_\phi$ is an isomorphism; thus $S(B,Y')$ is isomorphic to the strong dual space of $K(B,Y)$ ([5], Proposition 4.4 and Corollary 7.2; $S(B,Y') = \Lambda(\mathbb{N}, |\Lambda|, Y') = \Lambda_0(\mathbb{N}, |\Lambda_0|, Y'))$. Let $n \in \mathbb{N}$. The space $E_n = X^n \times K(B,Y)$ is a nuclear Fréchet space. The norms $q_{k,n} : E_n \to [0,\infty), q_{k,n}((x_1,\ldots,x_n),(y_1,y_2,\ldots)) = \max\{r_k(x_1),\ldots,r_k(x_n),q_k((y_n))\}, k \in \mathbb{N}$, form a base in $\mathcal{P}(E_n)$. Clearly, E'_n is isomorphic to $(X')^n \times S(B,Y')$; we will identified these spaces.

From now we will assume that the nuclear Köthe matrix $B = (b_{k,n})$ is *stable* that is

$$\forall k \in \mathbb{N}\ \exists l \in \mathbb{N} : \sup_n \left(\frac{b_{k,n+1}}{b_{l,n}} + \frac{b_{k,n}}{b_{l,n+1}}\right) < \infty.$$

For example, for an increasing unbounded sequence $(a_n) \subset (0,+\infty)$ the nuclear Köthe matrixes $A = (a_{k,n}) = (k^{a_n})$ and $C = (c_{k,n}) = ((\frac{k}{k+1})^{a_n})$ are stable, provided $\sup_n(a_{n+1}/a_n) < \infty$.

It is not hard to check that for any $(y_n) \subset Y$ we have $(y_1,y_2,\ldots) \in K(B,Y)$ if and only if $(y_2,y_3,\ldots) \in K(B,Y)$; if $k < l$ and $c_{k,l} \doteq \sup_n(b_{k,n+1}/b_{l,n}) < \infty$, and $p_k \circ Q \le r_l$ then $q_k(Qx,y_1,y_2,\ldots) \le (c_{k,l} + b_{k,1})\max\{r_l(x), q_l(y_1,y_2,\ldots)\}$ for all $x \in X, (y_n) \in K(B,Y)$. It follows that for any $n \in \mathbb{N}$ the map

$$\pi_n : E_{n+1} \to E_n, \pi_n((x_1,\ldots,x_{n+1}),(y_1,y_2,\ldots)) = ((x_1,\ldots,x_n),(Qx_{n+1},y_1,y_2,\ldots))$$

is well defined, linear, surjective and continuous.

Moreover, for any $(\xi_n) \subset \mathbb{K}$ we get $(\xi_1,\xi_2,\ldots) \in K(B)$ if and only if $(\xi_2,\xi_3,\ldots) \in K(B)$. Hence for every $(y'_n) \subset Y'$ we have $(y'_1,y'_2,\ldots) \in S(B,Y')$ if and only if $(y'_2,y'_3,\ldots) \in S(B,Y')$. Put $\pi_{k+1,k} = \pi_k$ and $\pi_{j,k} = \pi_k \circ \ldots \circ \pi_{j-1}$ for all $j,k \in \mathbb{N}$ with $j > k+1$. Let $j,k \in \mathbb{N}$ with $j > k$. Clearly, we have $\pi_{j,k} : E_j \to E_k, \pi_{j,k}((x_1,\ldots,x_j),(y_1,y_2,\ldots)) = ((x_1,\ldots,x_k),(Qx_{k+1},\ldots,Qx_j,y_1,y_2,\ldots))$ and $\pi'_{j,k} : E'_k \to E'_j$. For $(x'_1,\ldots,x'_k) \in (X')^k, (y'_n) \in S(B,Y'), (x_1,\ldots,x_j) \in X^j$ and $(y_n) \in K(B,Y)$ we have

$$(\pi'_{j,k}((x'_1,\ldots,x'_k),(y'_1,y'_2,\ldots)))((x_1,\ldots,x_j),(y_1,y_2,\ldots))$$

$$= ((x'_1,\ldots,x'_k),(y'_1,y'_2,\ldots))((x_1,\ldots,x_k),(Qx_{k+1},\ldots,Qx_j,y_1,y_2,\ldots))$$

$$= \sum_{i=1}^{k} x'_i(x_i) + \sum_{i=1}^{j-k} y'_i(Qx_{k+i}) + \sum_{i=1}^{\infty} y'_{j-k+i}(y_i)$$

$$= ((x'_1,\ldots,x'_k,Q'y'_1,\ldots,Q'y'_{j-k}),(y'_{j-k+1},y'_{j-k+2},\ldots))((x_1,\ldots,x_j),(y_1,y_2,\ldots)).$$

Thus

$$\pi'_{j,k} : E'_k \to E'_j, \pi'_{j,k}((x'_1, \ldots, x'_k), (y'_1, y'_2, \ldots))$$
$$= ((x'_1, \ldots, x'_k, Q'y'_1, \ldots, Q'y'_{j-k}), (y'_{j-k+1}, y'_{j-k+2}, \ldots)).$$

Hence we get $\pi'_{j,k}(E'_k) = (X')^k \times (Q'Y')^{j-k} \times S(B, Y')$.

Spaces $G, \hat{G}, F, H$ and maps $\Psi, \alpha_1, \alpha_2, \ldots$ are defined as in the proof of Proposition 4. By Lemma 3, Corollary 5 and Remark 6, no bounded subset of F/H is linearly dense in F/H and G has no continuous norm; nevertheless $\alpha_n(E'_n)$ contains a bounded and linearly dense subset for any $n \in \mathbb{N}$.

Now we can prove our main result.

Theorem 7. *The nuclear Fréchet space G is twisted.*

Proof. Suppose, by contradiction, that G is isomorphic to the product $\prod_{n=1}^{\infty} G_n$ of a sequence (G_n) of non-zero Fréchet spaces with continuous norms. Then G' is isomorphic to the direct sum $W \doteq \bigoplus_{n=1}^{\infty} G'_n$ ([6], Proposition 2.8).

Let $\Gamma : W \to G'$ be an isomorphism. For any $n \in \mathbb{N}$ the closed subspace $W_n \doteq \{(x_k) \in W : x_k = 0 \text{ for all } k > n\}$ of W is complemented; (W_n) is a strict inductive sequence and W is the inductive limit of (W_n)([3], Proposition 1.4.4). For any $n \in \mathbb{N}$, $L_n \doteq \Psi^{-1}\Gamma(W_n)$ is a closed and complemented subspace of $F/\hat{G} = \Psi^{-1}\Gamma(W)$, and, by Lemma 3, L_n contains a bounded and linearly dense subset. Moreover, (L_n) is a strict inductive sequence and $F/\hat{G}$ is the inductive limit of (L_n). By Corollary 5 and Remark 6, there exist $k, j \in \mathbb{N}$ with $1 < k < j$ such that

$$\alpha_1(E'_1) \subset L_k \subset \alpha_j(E'_j) \subset \alpha_{j+1}(E'_{j+1}) \subset F/\hat{G}.$$

Clearly, L_k is closed and complemented in $\alpha_{j+1}(E'_{j+1})$. Hence

$$\alpha_{j+1}^{-1}(\alpha_1(E'_1)) \subset \alpha_{j+1}^{-1}(L_k) \subset \alpha_{j+1}^{-1}(\alpha_j(E'_j)) \subset E'_{j+1}$$

and $M \doteq \alpha_{j+1}^{-1}(L_k)$ is a closed and complemented subspace of E'_{j+1}.
Since $\alpha_{j+1}\pi'_{j+1,1} = \alpha_1$ and $\alpha_{j+1}\pi'_{j+1,j} = \alpha_j$ we have

$$\pi'_{j+1,1}(E'_1) \subset M \subset \pi'_{j+1,j}(E'_j) \subset E'_{j+1},$$

so

$$X' \times (Q'Y')^j \times S(B, Y') \subset M \subset (X')^j \times Q'Y' \times S(B, Y') \subset (X')^{j+1} \times S(B, Y').$$

Let $C = \{(x'_2, \ldots, x'_j) \in (X')^{j-1} : ((0, x'_2, \ldots, x'_j), 0, (0, 0, \ldots)) \in M\}$. Then $M = X' \times C \times Q'Y' \times S(B, Y')$ and $(Q'Y')^{j-1} \subset C \subset (X')^{j-1}$.

Put $S : X' \to E'_{j+1} = (X')^{j+1} \times S(B, Y'), Sx = ((0, \ldots, 0, x), (0, 0, \ldots))$ and $R : X' \times C \times Q'Y' \times S(B, Y') \to Q'Y', R(x, c, y, s) = y$. Let P be a continuous linear projection from E'_{j+1} onto M. Then $R \circ P \circ S$ is a continuous linear projection from X' onto $Q'Y'$. Clearly $Q'Y' = (\ker Q)^\circ$. By Lemma 2 we infer that $\ker Q$ is complemented in X; a contradiction. □

The seminorms $\tilde{r_k} : X^{\mathbb{N}} \to [0, \infty), \tilde{r_k}((x_n)) = \max_{1 \le n \le k} r_k(x_n), k \in \mathbb{N}$, form a base of continuous seminorms on the product space $X^{\mathbb{N}}$. Consider the linear subspace $Z \doteq \{(x_n) \in X^{\mathbb{N}} : (Qx_n) \in K(B, Y)\}$ of $X^{\mathbb{N}}$ with the locally convex topology generated by the seminorms $s_k : Z \to [0, \infty), s_k((x_n)) = \max\{\tilde{r_k}((x_n)), q_k((Qx_n))\}$ $(k \in \mathbb{N})$. We shall prove the following

Proposition 8. *The locally convex space Z is isomorphic to G. Thus Z is a nuclear twisted Fréchet space.*

Proof. Put $\Phi_k : Z \to E_k, \Phi_k((x_n)) = ((x_1, \ldots, x_k), (Qx_{k+1}, Qx_{k+2}, \ldots))$ for $k \in \mathbb{N}$. The linear map $\Phi : Z \to \Pi_{k=1}^{\infty} E_k, \Phi(x) = (\Phi_k x)$ is injective. We shall prove that $\Phi(Z) = G$. Clearly $\Phi(Z) \subset G$. Let $(t_k) \in G$. Then $t_k = ((x_1^k, \ldots, x_k^k), (y_1^k, y_2^k, \ldots))$ for some $(x_1^k, \ldots, x_k^k) \in X^k, (y_1^k, y_2^k, \ldots) \in K(B, Y), k \in \mathbb{N}$, and $((x_1^{k+1}, \ldots, x_k^{k+1}), (Qx_{k+1}^{k+1}, y_1^{k+1}, y_2^{k+1}, \ldots)) = ((x_1^k, \ldots, x_k^k), (y_1^k, y_2^k, \ldots)), k \in \mathbb{N}$. Hence $x_n^{k+1} = x_n^k$ for $n, k \in \mathbb{N}$ with $n \leq k$, and $Qx_{k+1}^{k+1} = y_1^k, y_n^{k+1} = y_{n+1}^k$ for $n, k \in \mathbb{N}$. Thus $x_n^k = x_n^n$ for $n, k \in \mathbb{N}$ with $n \leq k$, and $y_n^k = y_1^{k+n-1} = Qx_{k+n}^{k+n}$ for $n, k \in \mathbb{N}$. It follows that $t_k = ((x_1^1, \ldots, x_k^k), (Qx_{k+1}^{k+1}, Qx_{k+2}^{k+2}, \ldots)), k \in \mathbb{N}$; so $(t_k) = \Phi((x_n^n)) \in \Phi(Z)$.

The seminorms $\tilde{q}_k : \Pi_{n=1}^{\infty} E_n \to [0, \infty), \tilde{q}_k((t_n)) = \max_{1 \leq n \leq k} q_{k,n}(t_n), k \in \mathbb{N}$, form a base of continuous seminorms on $\Pi_{n=1}^{\infty} E_n$. Let $k \in \mathbb{N}$. By the stability of B and the continuity of Q there exist $c_k > 1$ and $t \in \mathbb{N}$ with $t \geq k$ such that $b_{k,j} \leq c_k b_{t,j-1}$ for $j \in \mathbb{N}$ with $j \geq 2$; $b_{k,j} \leq c_k b_{t,j+n}$ for $n, j \in \mathbb{N}$ with $1 \leq n \leq k$, and $b_{k,1} p_k(Qx) \leq c_k r_t(x)$ for $x \in X$.
For $x = (x_n) \in Z$ we have

$$\tilde{q}_k(\Phi x) = \max_{1 \leq n \leq k} q_{k,n}((x_1, \ldots, x_n), (Qx_{n+1}, Qx_{n+2}, \ldots))$$

$$= \max_{1 \leq n \leq k} \max\{r_k(x_1), \ldots, r_k(x_n), q_k((Qx_{n+1}, Qx_{n+2}, \ldots))\}$$

$$= \max\{\max_{1 \leq i \leq k} r_k(x_i), \max_{1 \leq n \leq k} \max_{i \in \mathbb{N}} b_{k,i} p_k(Qx_{n+i})\}$$

$$= \max\{\tilde{r_k}((x_n)), \max_{1 \leq n \leq k} \max_{m > n} b_{k,m-n} p_k(Qx_m)\}$$

$$\leq \max\{\tilde{r_k}((x_n)), c_k \max_m b_{t,m} p_t(Qx_m)\} \leq c_k s_t((x_n)),$$

$$s_k(x) = \max\{\tilde{r_k}(x), \max_{i \in \mathbb{N}} b_{k,i} p_k(Qx_i)\} \leq c_k \max\{\tilde{r_k}(x), r_t(x_1), \max_{i \geq 2} b_{t,i-1} p_k(Qx_i)\}$$

$$\leq c_k \max\{\tilde{r_t}(x), \max_{1 \leq n \leq k} \max_{i \in \mathbb{N}} b_{t,i} p_t(Qx_{i+n})\} = c_k \tilde{q}_t(\Phi x).$$

Thus the map $\Phi : Z \to G$ is an isomorphism. □

References

[1] De Grande-De Kimpe, N., – Non-archimedean Fréchet spaces generalizing spaces of analytic functions, Indag. Mathem., 44(1982), 423–439.

[2] De Grande-De Kimpe, N., – Projective locally $\mathbb{K}$-convex spaces, Indag. Mathem., 46(1984), 247–254.

[3] De Grande-De Kimpe, N., Kąkol, J., Perez-Garcia, C. and Schikhof, W.H., – p-adic locally convex inductive limits, p-adic Functional Analysis (Nijmegen, 1996), 159–222, Lecture Notes in Pure and Appl. Math., 192, Dekker, New York (1997).

[4] De Grande-De Kimpe, N., Kąkol, J., Perez-Garcia, C. and Schikhof, W.H., – Orthogonal sequences in non-archimedean locally convex spaces, Indag. Mathem., (N.S.), 11(2000), 187–195.

[5] Katsaras, A.K., Non-archimedean Köthe spaces, Questiones Math., 19(1996), 483–503.

[6] Katsaras, A.K. and Benekas, V., – p-adic (dF)-spaces, p-adic Functional Analysis (Poznań, 1998), 127–147, Lecture Notes in Pure and Appl. Math. 207, Dekker, New York (1999).

[7] Moscatelli, V.B., – Fréchet spaces without continuous norms and without bases, Bull. London Math. Soc., 12(1980), 63–66.

[8] Prolla, J.B., – Topics in functional analysis over valued division rings, North-Holland Math. Studies 77, North-Holland Publ. Co., Amsterdam (1982).

[9] Rooij, A.C.M. van – Non-archimedean functional analysis, Marcel Dekker, New York (1978).

[10] Schikhof, W.H., – Locally convex spaces over non-spherically complete valued fields I-II, Bull. Soc. Math. Belgique, 38 (1986), 187–224.

[11] Śliwa, W., – Examples of non-archimedean nuclear Fréchet spaces without a Schauder basis, Indag. Mathem. (N.S.), 11(2000), 607–616.

[12] Śliwa, W., – Closed subspaces without Schauder bases in non-archimedean Fréchet spaces, Indag. Mathem., (N.S.), 12(2001), 261–271.

[13] Śliwa, W., – On closed subspaces with Schauder bases in non-archimedean Fréchet spaces, Indag. Mathem., (N.S.), 12(2001), 519–531.

[14] Śliwa, W., – On block basic sequences in non-archimedean Fréchet spaces, Contemporary Math., 319(2003), 389–404.

[15] Śliwa, W., – On topological classification of non-archimedean Fréchet spaces, Czechoslovak Math. J., 54(2004), 457–463.

[16] Śliwa, W., – On relations between non-archimedean power series spaces, Indag. Mathem., (N.S.), 17(2006), 627–639.

Faculty of Mathematics and Computer Science
Adam Mickiewicz University
ul. Umultowska 87, 61-614 Poznań, POLAND
e-mail: sliwa@amu.edu.pl