Spectra of quasisimilar operators

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Abstract

We show that whenever densely similar operators on a Banach spaces, their approximate point spectra must have non-empty intersection. Also, we introduce the class $\mathcal{A}$ that consists of those operators for which the Goldberg spectrum coincides with the right essential spectrum. We study spectral properties of quasisimilar operators satisfying Bishop's property (β) in the class $\mathcal{A}$. Finally, as an application to the class $\mathcal{N}$ that consists of those operators T whose range $R(T)$ is contained in the linear span of finite number of orbits of T, we show that any two quasisimilar operators such that are in $\mathcal{N}$ and satisfying property (β) must have the same approximate point spectrum.

1 Introduction

The notion of quasisimilarity plays an important role in the theory of operators on Complex Banach spaces. This equivalence relation between operators is, however, too weak a notion to preserve the distinguished parts of the spectrum, unless we add extra assumptions on the operators. For instance, it is well known that quasisimilar operators have the same spectrum if in addition they are both normal (in fact, in this case they will be unitarily equivalent [7]), or hyponormal [4], or decomposable [5]. Additional references include [1],[3],[12] and [15].

Of special interest are the spectral consequences of the weaker notion of dense similarity. For example, in 1992, Takahashi [16] showed that the essential spectrum of an operator T satisfying Bishop's property (β) is contained in the essential spectrum of any operator densely similar to T, and M. Putinar [14] proved that two

Received by the editors June 2006.

Communicated by F. Bastin.

2000 *Mathematics Subject Classification* : 47A, 47B.

Key words and phrases : multicyclic operators, spectra, quasisimilar operators, Bishop's property (β).

densely similar tuples of operators having Bishop's property (β) have equal essential spectra. More recently, Mbekhta and Zerouali showed that whenever densely similar operators S and T on a Hilbert space are cyclic, their approximate point spectra must have non-empty intersection [12, Theorem 3.5]. Our aim in this paper is, first, among others results, we show in Theorem 3.1 that the above mentioned result by Mbekhta and Zerouali is true even if T and S are not cyclic. Second, we improve the results of [12, Theorem 5.5] and [12, Theorem 5.6] for the smaller subclass of cyclic operators to the class $\mathcal{A}$ that consists of those operators for which the Goldberg spectrum coincides with the right essential spectrum. Finally, for X be a Banach space, we introduce the class $\mathcal{N}(X)$ that consists of those operators T whose range $R(T)$ is contained in the linear span of a finite number of orbits of T, an important consequence is that any two quasisimilar operators on the Banach spaces X and Y such that $S \in \mathcal{N}(Y)$ and $T \in \mathcal{N}(X)$ satisfying Bishop's property (β) must have the same approximate point spectrum.

2 . Preliminaries

Throughout This paper, X and Y are Banach spaces and $\mathcal{L}(X,Y)$ denotes the space of all bounded linear operators from X to Y. For a bounded linear operator $T \in \mathcal{L}(X) := \mathcal{L}(X,X)$, let as usual $\rho(T)$, $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_p(T)$ and T^* denote the resolvent set, the spectrum, the approximate point spectrum, the point spectrum and the adjoint operator of T, respectively. Next, we will write $N(T)$ and $R(T)$ for the null space and range of T, respectively. Also, let $\alpha(T) := \dim N(T)$ and $\beta(T) := \dim N(T^*)$.

The next sets of upper and lower semi-Fredholm operators are well-known as

$$\Phi_+(X) = \{T \in \mathcal{L}(X) : R(T) \text{ is closed and } \alpha(T) < \infty\}$$

and

$$\Phi_-(X) = \{T \in \mathcal{L}(X) : R(T) \text{ is closed and } \beta(T) < \infty\}.$$

The set of Fredholm operators is defined as $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$. For a semi-Fredholm operator $T \in \Phi_+(X) \cup \Phi_-(X)$, the index is given by :

$$\text{ind}(T) := \alpha(T) - \beta(T).$$

The left and right essential spectrum of T, respectively, are defined as :

$$\sigma_{le}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_+(X)\} \text{ and } \sigma_{re}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_-(X)\}.$$

The essential spectrum of T is defined as

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi(X)\} = \sigma_{le}(T) \cup \sigma_{re}(T).$$

Also, let $\rho_{ap}(T)$, $\rho_e(T)$, $\rho_{le}(T)$, and $\rho_{re}(T)$ denote the complement of $\sigma_{ap}(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$ and $\sigma_{re}(T)$, respectively. Finally, let $\rho_{SF}(T) = \rho_{le}(T) \cup \rho_{re}(T)$ and $\sigma_{SF}(T) = \mathbb{C} \setminus \rho_{SF}(T)$.

For an operator $T \in \mathcal{L}(X)$ the Goldberg spectrum is defined as $\sigma_f(T) = \{\lambda \in \mathbb{C} : R(T-\lambda) \text{ is not closed } \}$ (cf.[9]). Note that this spectrum is not necessarily closed or open subset of $\mathbb{C}$. The Kato resolvent set is defined as

$$\rho_K(T) = \{\lambda \in \mathbb{C} : R(T-\lambda I) \text{ is closed and } N(T-\lambda I) \subset \bigcap_{n\geq 0}(T-\lambda I)^n X\}.$$

As usual, let $\rho_f(T)$ and $\sigma_K(T)$ denote the complement of $\sigma_f(T)$ and $\rho_K(T)$, respectively.

Definition 2.1 ([11]). *An operator $T \in \mathcal{L}(X)$ is said to have the single-valued extension property, abbreviated SVEP, if for every non-empty open set $U \subseteq \mathbb{C}$, the equation $(T-\lambda I)f(\lambda) = 0$ admits the zero function $f \equiv 0$ as a unique analytic solution on U.*

Definition 2.2 ([11]). *An operator $T \in \mathcal{L}(X)$ is said to have Bishop's property (β), if, for each open subset U of $\mathbb{C}$ and every sequence of analytic functions $f_n : U \longrightarrow X$ for which $(T-\lambda I)f_n(\lambda) \to 0$ as $n \to \infty$, locally uniformly on U, it follows that $f_n(\lambda) \to 0$ as $n \to \infty$, again locally uniformly on U.*

Next, for a subset E of X, we set cl span$\{E\}$ the closure of the linear space generated by E and $\overline{E}$ for the closure of E.

Let T be a bounded linear operator on a Banach space X. We say for an integer n that T is a multicyclic operator of order n (n-multicyclic, for short) if there exist n vectors $x_1, \dots, x_n$ in X such that $X = \text{cl span}\{T^k x_i : 1 \leq i \leq n\ ,\ k \geq 0\}$ and for every $n-1$ vectors $z_1, \dots, z_{n-1}$ in X, the subspace cl span$\{T^k z_i : 1 \leq i \leq n-1\ ,\ k \geq 0\}$ is proper. The n-tuple $(x_1, \dots, x_n)$ is then called a cyclic n-tuple for T (see for example [8]).

Next, we denote by $\mathcal{M}(X)$ the set of all multicyclic operators on the Banach space X.

3 Densely similar operators

In this section, by an idea of the techniques of K. M. Laursen in [10], we establish Theorem 5.3 of [12] in the general case.

Recall that a bounded linear operator A is a quasi-affinity if A is one-to-one and $R(A)$ is dense.

Definition 3.1. *The operators $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ are densely similar (respectively, quasisimilar) if there exist $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$ for which*

$$SA = AT \qquad \text{and} \qquad TB = BS\ ,$$

and are with dense ranges (respectively, quasi-affinities).

Theorem 3.2. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be densely similar operators, Then*

$$\sigma_{ap}(T) \cap \sigma_{ap}(S) \neq \emptyset.$$

Moreover, there is $\lambda \in \sigma_{ap}(T) \cap \sigma_{ap}(S)$ which is a boundary point of either $\sigma(T)$ or of $\sigma(S)$.

To prove Theorem 3.2, we start with the next lemma and proposition and recall that the next lemma is the lemma 5.4 of [12] for Hilbert case; the proof is the same in Banach case. Recall that, for a subset $F \subseteq \mathbb{C}$ we denote ∂F for the boundary of F.

Lemma 3.3. *Let $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$, $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$ be operators such that A and B are with dense ranges. Then :*

1) $AT = SA \quad \Longrightarrow \sigma(S) \setminus \sigma_{ap}(S) \subseteq \sigma(T)$;

2) $TB = BS \quad \Longrightarrow \sigma(T) \setminus \sigma_{ap}(T) \subseteq \sigma(S)$.

Proposition 3.4. *Let T, S, A and B be operators as in lemma 3.3. If $L \subseteq \sigma(T)$ and $K \subseteq \sigma(S)$ are intersecting components of the respective spectra. Then, there exists*

$$\lambda \in \sigma_{ap}(S) \cap \sigma_{ap}(T) \cap K \cap L \quad \textit{with} \quad \lambda \in \partial\sigma(S) \textit{ or } \partial\sigma(T).$$

Proof. In the case when $L \not\subseteq K$, we have $L \cap K \neq \emptyset$ and $L \cap K^c \neq \emptyset$, where K^c is the complement of K. By connectedness we have

$$L \cap \partial K \neq \emptyset.$$

Now, let $\lambda \in L \cap \partial K$, hence $\lambda \in \sigma(T) \cap \partial\sigma(S)$ and so $\lambda \in \sigma_{ap}(S)$. Suppose that $\lambda \notin \sigma_{ap}(T)$, then λ is an interior point of $\sigma(T)$, and since $\lambda \in \partial\sigma(S)$ there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\sigma(T) \setminus \sigma(S)$ which converge to λ. On the other hand, by the preceding lemma, we have $\sigma(T) \setminus \sigma_{ap}(T) \subseteq \sigma(S)$ and so $\sigma(T) \setminus \sigma(S) \subseteq \sigma_{ap}(T)$. We deduce that

$$(\lambda_n)_{n\in\mathbb{N}} \subseteq \sigma_{ap}(T) \qquad \textit{and} \quad \lambda_n \longrightarrow \lambda \quad \text{as } n \longrightarrow +\infty$$

Since the approximate point spectrum is closed, then $\lambda \in \sigma_{ap}(T)$, a contradiction.
On the other case, when $L \subseteq K$, we have $\partial L \subseteq K$. Let $\lambda \in \partial L$ be given, consequently $\lambda \in \partial\sigma(T) \cap \sigma(S)$ and as in the preceding case, we show that $\lambda \in \sigma_{ap}(T) \cap \sigma_{ap}(S)$, as desired. ∎

Proof of Theorem 3.2. Since $SA = AT$ and A is non-trivial, $\sigma_{su}(S) \cap \sigma_{ap}(T) \neq \emptyset$ (cf. [11, Theorem 3.5.1(b), and Proposition 3.5.3]), where $\sigma_{su}(S)$ is the surjective spectrum of S. Consequently, Theorem 3.2 follows from Proposition 3.4. ∎

Remark. Generally we do not have the conclusion of the preceding theorem if we suppose only that $AT = SA$. see example 4 of [12].

Clearly, the preceding results for the approximate point spectrum (Lemma 3.3, Proposition 3.4 and Theorem 3.2) hold for the Kato spectrum if T and S are quasisimilar. We refine this result as follows

Lemma 3.5. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be operators such that A have dense range and B is one to one, and for which $AT = SA$ and $TB = BS$. Then $\sigma(S) \setminus \sigma_K(S) \subseteq \sigma(T)$ and vice versa.*

Proof. First, from the fact that $\sigma(T) = \sigma(T^*)$, and A^* and B are one to one, we obtain obviously that $\sigma(S) \setminus \sigma_K(S) \subseteq \sigma(T)$. Similarly, we deduce $\sigma(T^*) \setminus \sigma_K(T^*) \subseteq \sigma(S^*)$. And so, $\sigma(T) \setminus \sigma_K(T) \subseteq \sigma(S)$, since $\sigma_K(T^*) = \sigma_K(T)$. ∎

Theorem 3.6. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be operators such that A have dense range and B is one to one, and for which $AT = SA$ and $TB = BS$. Then there is $\lambda \in \sigma_K(T) \cap \sigma_K(S)$ which is a boundary point of either $\sigma(T)$ or of $\sigma(S)$.*

Proof. Obtained by using lemma 3.5 and an argument similar to the proof of Proposition 3.4 and Theorem 3.2. ∎

Corollary 3.7. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be operators such that A have dense range and B is one to one, and for which $AT = SA$ and $TB = BS$. Then there is $\lambda \in \sigma_{ap}(S) \cap \sigma_{ap}(T) \cap \sigma_{su}(T) \cap \sigma_{su}(S)$ which is a boundary point of either $\sigma(T)$ or of $\sigma(S)$.*

4 . Spectra and quasisimilarity

It is well-known that quasisimilar Banach space operators can have different parts of their distinguished part of the spectrum. (see for example [1]). And so, it seems interesting to consider the connection between various distinguished parts of the spectrum for quasisimilar operators.

In the cyclic case [12], and the multicyclic case [8], the authors established that if T is a cyclic or multicyclic operator and has SVEP, then there are certain parts of it is spectrum that are equal. Also, established in [12] that if T and S are quasisimilar operators with Bishop's property (β), then certain distinguished parts of their spectrum coincide. In this section, we show the same result for a class larger than multicyclic operators.

Next, for every $x \in X$, we set $X_x = \text{cl span}\{T^k x : k \geq 0\}$ and $T_x := T_{|X_x}$ be the restriction of T on X_x. It is not hard to see that for every n-tuple $(x_1, \ldots, x_n) \in X^n$, we have

$$\text{cl span}\{T^k x_i : 1 \leq i \leq n\ ,\ k \geq 0\} = \overline{X_{x_1} + \ldots + X_{x_n}}. \tag{4.1}$$

The following lemma will be useful in the sequel.

Lemma 4.1 ([8]). *Let $T \in \mathcal{L}(X)$ be an m-multicyclic operator, then*

$$codim\overline{R(T - \lambda I)} \leq m \qquad for\ all\ \lambda \in \mathbb{C}.$$

It is known that the quasisimilarity conserves the concept of multicyclic operators. For completeness, we include it's proof.

Proposition 4.2. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be densely similar operators. Then for all $n \geq 1$, T is n-multicyclic operator if and only if S is n-multicyclic operator. In particular, T is multicyclic operator if and only if S is multicyclic operator.*

Proof. It suffices to prove one implication. Let A and B be operators defined as above, and suppose that T is n-multicyclic. By $(4, 1)$, there exists $(x_1, \ldots, x_n) \in X^n$ such that

$$X = \overline{X_{x_1} + \ldots + X_{x_n}}.$$

It is clear that

$$AX_x \subseteq Y_{Ax} \qquad \text{for all } x \in X \tag{4.2}$$

Since A is continuous, then

$$AX = A\overline{X_{x_1} + \ldots + X_{x_n}} \subseteq \overline{AX_{x_1} + \ldots + AX_{x_n}}. \tag{4.3}$$

As $\overline{AX} = Y$, from (4.2) and (4.3) we obtain

$$\overline{Y_1 + \ldots + Y_n} = \overline{AX_{x_1} + \ldots + AX_{x_n}} = Y.$$

where $Y_i = Y_{Ax_i}$, for $i \in \{1, \ldots, n\}$. Now, suppose that there exists $(y_1, \ldots, y_{n-1}) \in X^{n-1}$ such that

$$Y = \overline{Y_1 + \ldots + Y_{n-1}};$$

hence, as above, we show that $X = \overline{X_{By_1} + \ldots + X_{By_{n-1}}}$. This contradicts the fact that T is n-multicyclic, which complete the argument. ■

In the following, we define the class $\mathcal{A}(X)$ as the set of all operators $T \in \mathcal{L}(X)$ satisfying the next relation

$$\rho_f(T) \subseteq \rho_{re}(T),$$

and note that for $T \in \mathcal{A}(X)$, it is clear that $\rho_f(T) = \rho_{re}(T)$.

Clearly, the class $\mathcal{A}(X)$ contain $\mathcal{M}(X)$ (Lemma 4.1), and more examples when X be an infinite dimensional are given in the following

i) If T is Riesz or in particular compact operator, then $T \in \mathcal{A}(X)$ if and only if $R(T)$ is not closed.

ii) If T is Rationally cyclic then $T \in \mathcal{A}(X)$([13, Proposition 1].

iii) If T is normal operator such that every isolated point of $\sigma(T)$ is an eigenvalue of finite multiplicity then $T \in \mathcal{A}(X)$([6, Proposition 4.5 and 4.6]).

The following result refines what has been proved in Theorem 5.5 of [12] for cyclic case on Hilbert spaces. For completeness, we include it's proof.

Theorem 4.3. *Let $T \in \mathcal{L}(X)$ and suppose that $T \in \mathcal{A}(X)$. Then the following statements hold.*

(1) $\rho_{SF}(T) = \rho_f(T) = \rho_{re}(T)$;

(2) $\rho_e(T) = \rho_{le}(T)$;

(3) *Moreover, if T has the SVEP, then*

$$\sigma_e(T) = \sigma_{SF}(T) = \sigma_f(T) = \sigma_{le}(T) = \sigma_{re}(T).$$

Proof. (1) It is obviously that $\rho_{re}(T) \subseteq \rho_{SF}(T) \subseteq \rho_f(T)$, hence (1) is an immediate consequence of the fact that $\rho_{re}(T) = \rho_f(T)$ for $T \in \mathcal{A}(X)$.

(2) From (1) we obtain $\rho_{le}(T) \subseteq \rho_{re}(T)$, and consequently

$$\rho_e(T) = \rho_{le}(T) \cap \rho_{re}(T) = \rho_{le}(T).$$

(3) Let $\lambda_0 \in \rho_{SF}(T)$, i.e $(T - \lambda_0 I)$ is semi-Fredholm operator, from the Kato decomposition, there exists $\eta > 0$ such that

$$D(\lambda_0, \eta) \setminus \{\lambda_0\} = \{\lambda \in \mathbb{C} \ : \ 0 <\mid \lambda - \lambda_0 \mid< \eta\} \subseteq \rho_K(T) \ ;$$

Since T has the SVEP, it follows from [11, corollary 3.1.7] that $\rho_K(T) = \rho_{ap}(T)$. Thus, for all $\lambda \in D(\lambda_0, \eta) \setminus \{\lambda_0\}$ we get $R(T - \lambda I)$ is closed and $N(T - \lambda I) = \{0\}$. This implies that $\text{ind}(T - \lambda I) \leq 0$ for all $\lambda \in D(\lambda_0, \eta) \setminus \{\lambda_0\}$. On the one hand, by the continuity of the index we obtain $\text{ind}(T - \lambda_0 I) \leq 0$. On the other hand, it follows from $T \in \mathcal{A}(X)$ that $\beta(T - \lambda_0 I) < \infty$. Consequently,

$$-\beta(T - \lambda_0 I) \leq \text{ind}(T - \lambda_0 I) \leq 0.$$

Thus, $\lambda_0 \in \rho_e(T)$ and so $\sigma_e(T) = \sigma_{SF}(T)$. Therefore, from (1) and (2) the proof is complete. ■

From the preceding lemma, we see that $\mathcal{A}(X)$ contains the class of multicyclic operators on X. As a consequence of this and Theorem 4.3, we obtain the following result

Corollary 4.4. *Let $T \in \mathcal{L}(X)$ be a multicyclic operator. Then the following statement hold.*

(1) $\rho_{SF}(T) = \rho_f(T) = \rho_{re}(T)$;

(2) $\rho_e(T) = \rho_{le}(T)$.

(3) *Moreover, if T have the SVEP, then*

$$\sigma_e(T) = \sigma_{SF}(T) = \sigma_f(T) = \sigma_{le}(T) = \sigma_{re}(T).$$

The following proposition will be useful in what follows.

Proposition 4.5 ([11], Theorem 3.7.15)**.** *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be densely similar operators with property (β). Then $\sigma(T) = \sigma(S)$, $\sigma_e(T) = \sigma_e(S)$ and ind$(T - \lambda I) =$ ind$(S - \lambda I)$ for all $\lambda \in \rho_e(T)$.*

Theorem 4.6. *If $T \in \mathcal{A}(X)$ and $S \in \mathcal{A}(Y)$ have property (β) and are densely similar, then $\sigma(T) = \sigma(S)$ and $\sigma_{re}(T) = \sigma_{le}(T) = \sigma_f(T) = \sigma_{SF}(T) = \sigma_e(T) = \sigma_e(S) = \sigma_{SF}(S) = \sigma_f(S) = \sigma_{le}(S) = \sigma_{re}(S)$.*
If, in addition, S and T are quasisimilar, then they also satisfy that $\sigma_p(T) = \sigma_p(S)$, and $\sigma_{ap}(T) = \sigma_{ap}(S)$.

Proof. Since S and T have property (β) and are densely similar, $\sigma_e(T) = \sigma_e(S)$ and $\sigma(T) = \sigma(S)$ [11, Theorem 3.7.15]. Also, S and T satisfy the SVEP, since they satisfy property (β) [11, Proposition 1.2.19]. Hence the first statement follows from Theorem 4.3. For the second statement, $\sigma_p(T) = \sigma_p(S)$ follows from S and T being quasisimilar, and hence by the first statement $\sigma_{ap}(T) = \sigma_p(T) \cup \sigma_f(T) = \sigma_p(S) \cup \sigma_f(S) = \sigma_{ap}(S)$. ■

Again, as a consequence of the fact that $\mathcal{A}(X)$ contains multicyclic operators on X and Theorem 4.6, we obtain the following result

Corollary 4.7. *If $T \in \mathcal{M}(X)$, $S \in \mathcal{M}(Y)$ have property (β) and are densely similar, then $\sigma(T) = \sigma(S)$ and $\sigma_{re}(T) = \sigma_{le}(T) = \sigma_f(T) = \sigma_{SF}(T) = \sigma_e(T) = \sigma_e(S) = \sigma_{SF}(S) = \sigma_f(S) = \sigma_{le}(S) = \sigma_{re}(S)$.*
If, in addition, S and T are quasisimilar, then they also satisfy that $\sigma_p(T) = \sigma_p(S)$, and $\sigma_{ap}(T) = \sigma_{ap}(S)$.

5 . Applications

In what follows, for a Banach space X an operator $T \in \mathcal{L}(X)$ is in $\mathcal{N}(X)$ if there exist an integer $n > 0$ and $(x_1, \dots, x_n) \in X^n$ for which

$$R(T) \subseteq \text{cl span}\{T^k x_i : 1 \leq i \leq n \ , \ k \geq 0\}.$$

It is clear that $\mathcal{M}(X) \subseteq \mathcal{N}(X)$. The next example shows that, the class $\mathcal{N}(X)$ may be larger than $\mathcal{M}(X)$.

Example 1. *Let H be a separable Hilbert space, with an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Let $N \in \mathcal{L}(H)$ be defined as follows :*

$$Ne_n = \begin{cases} e_1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

N is nilpotent and clearly that $N \in \mathcal{N}(H)$ but $N \notin \mathcal{M}(H)$.

Remark. If $T \in \mathcal{N}(X)$ such that codim$\overline{R(T)}$ is finite, then T is a multicyclic operator. In fact, let $n \geq 1$ and $(x_1, \dots, x_n) \in X^n$ for which $R(T) \subseteq \text{cl span}\{T^k x_i : 1 \leq i \leq n \ , \ k \geq 0\}$. Set E the subspace such that $X = \overline{R(T)} \oplus E$ and let $(x_{n+1}, \dots, x_{n+p})$ be a basis of E. And so, it is not hard to see that

$$\text{cl span}\{T^k x_i : 1 \leq i \leq n+p \ , \ k \geq 0\} = X.$$

It is known that if T is multicyclic and S densely similar with S, then so is S. We extend this result as follows.

Proposition 5.1. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be densely similar operators. Then $T \in \mathcal{N}(X)$ if and only if $S \in \mathcal{N}(Y)$.*

It is a simple consequence of proposition 4.2 and we omit the proof.

Next, for $T \in \mathcal{L}(X)$, we set $X_1 := \overline{R(T)}$, $T_1 := T_{|X_1}$ and $I_1 := I_{|X_1}$ viewed as operators in $\mathcal{L}(X_1)$. If $T \in \mathcal{N}(X)$, it is not hard to see that T_1 is a multicyclic operator on X_1. In fact, let $(x_1, \dots, x_n) \in X^n$ for which

$$X_1 \subseteq \text{cl span}\{T^k x_i : 1 \leq i \leq n \ , \ k \geq 0\},$$

and let I to be the subset of $\{1 \dots n\}$ for which $x_i \notin X_1$. Now, for each $i \in \{1 \dots n\}$, we set

$$y_i := \begin{cases} Tx_i & \text{if } i \in I \\ x_i & \text{if } i \notin I. \end{cases}$$

Thus, $(y_1, \dots, y_n) \in X_1^n$ and $X_1 = \text{cl span}\{T_1^k y_i : 1 \leq i \leq n \ , \ k \geq 0\}$.

Lemma 5.2 ([2])**.** *Let X be a Banach space and $T \in \mathcal{L}(X)$, let $\lambda \neq 0$ be a complex number. If $T \in \mathcal{N}(X)$ Then the following assertions hold.*

(*a*) *$R(T - \lambda I)$ is closed if and only if $R(T_1 - \lambda I_1)$ is closed.*

(*b*) *$R(T - \lambda I)$ has finite codimension if and only if $R(T_1 - \lambda I_1)$ has finite codimension.*

It suffices to apply in [2], Theorem 5 and Theorem 6 to Example 11.

Theorem 5.3. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Suppose that $T \in \mathcal{N}(X) \setminus \mathcal{M}(X)$. Then the following assertions hold.*

(1) *If $R(T)$ is not closed, then the conclusion of Theorem* 4.3 *holds.*

(2) *If $R(T)$ is closed, then*

a) $\rho_{SF}(T) \setminus \{0\} = \rho_f(T) \setminus \{0\} = \rho_{re}(T)$,

b) $\rho_e(T) = \rho_{le}(T) \setminus \{0\}$,

c) Moreover, if T has The SVEP, then

$$\sigma_e(T) = \sigma_{SF}(T) \cup \{0\} = \sigma_f(T) \cup \{0\} = \sigma_{le}(T) \cup \{0\} = \sigma_{re}(T).$$

Proof. (2) *a*) As $T \in \mathcal{N}(X) \setminus \mathcal{M}(X)$, we obtain through the remark preceding Proposition 5.1, that $\overline{R(T)}$ have infinite codimension and so $0 \notin \rho_{re}(T)$. Now, let $\lambda \neq 0$ in $\mathbb{C}$ be given. From the fact that $N(T - \lambda I) = N(T_1 - \lambda I_1)$ and by lemma 5.2, we obtain

$$\sigma_*(T) \setminus \{0\} = \sigma_*(T_1) \setminus \{0\} \qquad \text{for} \quad \sigma_* \in \{\sigma_e, \sigma_{le}, \sigma_{re}, \sigma_f, \sigma_{SF}\}.$$

Since T_1 is a multicyclic operator, it follows from Theorem 4.3 (1) and Lemma 5.2 that

$$\sigma_f(T) \setminus \{0\} = \sigma_{re}(T) \setminus \{0\} = \sigma_{SF}(T) \setminus \{0\}.$$

Consequently, (2)*a*) follows from the fact that $0 \notin \rho_{re}(T)$.

(2) *b*) Again, from Theorem 4.3(2) and Lemma 5.2, we obtain $\sigma_e(T) \setminus \{0\} = \sigma_{le}(T) \setminus \{0\}$, and since $0 \notin \rho_e(T)$, we deduce that $\rho_e(T) = \rho_{le}(T) \setminus \{0\}$.

(2) *c*) We know that T_1 inherits SVEP from T. And so, we obtain through Theorem 4.3(3), Lemma 5.2 and $0 \in \sigma_{re}(T)$, that

$$\sigma_e(T) = \sigma_{SF}(T) \cup \{0\} = \sigma_f(T) \cup \{0\} = \sigma_{le}(T) \cup \{0\} = \sigma_{re}(T).$$

(1) Assume that $R(T)$ is not closed, then

$$0 \in \sigma_*(T) \qquad \text{for} \quad \sigma_* \in \{\sigma_e, \sigma_{le}, \sigma_{re}, \sigma_f, \sigma_{SF}\}.$$

And so, from Lemma 5.2, we have

$$\sigma_e(T) = \sigma_{SF}(T) = \sigma_f(T) = \sigma_{le}(T) = \sigma_{re}(T).$$

This complete the argument. ∎

The next lemma shows that, the restriction to the closure of the range conserve the densely similarity and the quasisimilarity. Also, it will be useful in the proof of the last theorem.

Lemma 5.4. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be densely similar(respectively, quasisimilar). Then, $T_1 \in \mathcal{L}(X_1)$ and $S_1 \in \mathcal{L}(Y_1)$ are densely similar(respectively, quasisimilar).*

Again, I would omit its proof since this lemma is simple to verify.

Theorem 5.5. *Let $T \in \mathcal{N}(X)$ and $S \in \mathcal{N}(Y)$ have property (β).*

(a) *If T and S are densely similar, then $\sigma_{re}(T) = \sigma_{re}(S)$.*

(b) *If T and S are quasisimilar, then $\sigma_{ap}(T) = \sigma_{ap}(S)$.*

Proof. (a) From the fact that T_1 and S_1 are densely similar operators, and by the techniques that used in the proof of the preceding theorem, we show that $\sigma_{re}(T) \setminus \{0\} = \sigma_{re}(S) \setminus \{0\}$. Thus (a) follows, from Proposition 4.2 and Corollary 4.4 if $T \in \mathcal{M}(X)$, or from the fact that $0 \in \sigma_{re}(T) \cap \sigma_{re}(S)$ otherwise.

(b) Similarly, we obtain $\sigma_{ap}(T) \setminus \{0\} = \sigma_{ap}(S) \setminus \{0\}$, since T_1 and S_1 are quasisimilar operators. To prove the assertion (b), it suffices to show one implication. Now, Assume that $0 \notin \sigma_{ap}(T)$, and so $0 \notin \sigma_p(T)$, this implies that $0 \notin \sigma_p(S)$. It follows from the inclusion $\sigma_{ap}(T_1) \subseteq \sigma_{ap}(T)$, that $0 \notin \sigma_{ap}(T_1)$. On the one hand, T_1 and S_1 are quasisimilar multicyclic operators and satisfying property (β), hence Corollary 4.7 yields that $\sigma_{ap}(T_1) = \sigma_{ap}(S_1)$. Consequently $0 \notin \sigma_{ap}(S_1)$ and therefore $R(S_1)$ is closed because $0 \notin \sigma_p(S)$. And since S_1 is a multicyclic operator on $\overline{R(S)}$, by lemma 4.1 there exists a finite dimensional subspace F of $\overline{R(S)}$ such that

$$\overline{R(S)} = R(S_1) \oplus F.$$

Finally, since $R(S_1) \subseteq R(S)$, it is easy to see that

$$R(S) = R(S_1) \oplus F \cap R(S).$$

Thus $R(S)$ is closed ([9, Theorem 1.4.12]) and so $0 \notin \sigma_{ap}(S)$, as claimed. ■

Acknowledgments

The authors would like to thank the referee for his suggestions.

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