Numerical Solution of Korteweg-de Vries Equation by the Fourier Pseudospectral Method*

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Abstract

In this paper, we present a numerical solution of one-dimensional Korteweg-de Vries equation with variant boundary conditions by the Fourier pseudospectral method. Four test problem with known exact solutions were studied to demonstrate the accuracy of the present method. An artificial viscosity was proposed to improve the accuracy of the numerical scheme. The obtained results were compared with the exact solution of each problem and found to be in good agreement with each other.

1 Introduction

A famous equation which arises in the study of nonlinear dispersive waves is the Korteweg-de Vries (KdV) equation. KdV equation was derived in 1895 by Korteweg and de Vries to model water in shallow canal [1]. We considered the KdV equation which is a nonlinear partial differential equation of third order given by

$$\frac{\partial U}{\partial t}(x,t) + \varepsilon U(x,t)\frac{\partial U}{\partial t}(x,t) + \mu\frac{\partial^3 U}{\partial x^3}(x,t) = 0, \quad a \le x \le b, \quad t > 0, \tag{1.1}$$

where ε and μ are positive parameters and a, b are real constants.

The KdV equation is generic equation for the study of weakly nonlinear long waves. The KdV type of equation have been an important class of nonlinear evolution equations with numerous applications in physical sciences and engineering

*This work is supported by Chinese Academy of Sciences and TWAS-UNESCO.
Received by the editors August 2006.
Communicated by A. Bultheel.
2000 *Mathematics Subject Classification :* 76M22, 35Q35, 65M70.
Key words and phrases : Korteweg-de Vries equation, Fourier pseudospectral method, artificial viscosity.

field. For example, in plasma physics these equations give rise to the ion acoustic solitons [2]; in geophysical fluid dynamics, they describe a long wave in shallow seas and deep oceans [3, 5]. Their strong presence is exhibited in cluster physics, super deformed nuclei, fission, thin films, radar and rheology [7, 16].

Several authors mainly had paid attention to study solutions of nonlinear equation by using various methods, such as tanh method [8], the sine-cosine method [9] and the homogeneous balance method [17]. This equation is of interest of the numerical point of view, because in general, analytical solution is not available. For appropriate initial condition, Gardner et al. [13] have shown the existence and uniqueness of solutions of the KdV equation. For KdV equation, there has been several quite successful numerical methods such as finite element methods [14], finite difference methods [12] and spectral methods [15].

This paper presents a new numerical Fourier pseudospectral scheme to solve third order KdV equation with a set of initial and boundary conditions. An artificial viscosity was used to reduce the round-off error of pseudospectral method. The numerical results were compared with the exact solution of four model problem to demonstrate the accuracy of the method. The numerical scheme was also compared with earlier work and shown to be accurate and efficient.

This paper is organized as follows: In section 2, we proposed pseudospectral method. The numerical results are presented in section 3. Finally the conclusion is given in section 4.

2 The Fourier Pseudospectral Discretization

In this section, we will apply the Fourier pseudospectral method to (1.1). For simplicity, we will consider the spatial domain $[0, L]$. The Fourier pseudospectral method involve two basic steps. First, we construct the discrete representation of the solution through interpolate trigonometric polynomial of the solution at collocation points. Secondly, equation for the discrete values of the solution are obtained from the original equation. This second step involves finding an approximation for the differential operator in terms of the discrete values of the solution at collocation points. For details, see [4, 10, 21].

We approximate $u(x,t)$ by $u_N(x,t)$, which interpolate $u(x,t)$ at the following set of collocation points

$$x_j = \frac{L}{N} j, \quad j = 0, 1, \cdots, N-1,$$

where N is an even number.

The approximation $u_N(x,t)$ have the form

$$u_N(x,t) = \sum_{j=0}^{N-1} u_j g_j(x), \tag{2.1}$$

where $u_j = u(x_j, t)$, and $g_j(x_k) = \delta_j^k$. Therefore, we have $u_N(x_j, t) = u_j$, $j = 0, 1, \cdots, N-1$. In fact $g_j(x)$ can be given explicitly by

$$g_j(x) = \frac{1}{N} \sum_{\ell=-N/2}^{N/2} \frac{1}{c_\ell} e^{i\ell\mu(x-x_j)}, \tag{2.2}$$

where $c_\ell = 1(|\ell| \neq N/2), c_{-N/2} = c_{N/2} = 2, \mu = \dfrac{2\pi}{L}$. By direct computations, we can easily verify that $g_j(x_k) = \delta_j^k$. Substituting (2.2) into (2.1), we obtain

$$u_N(x,t) = \sum_{\ell=-N/2}^{N/2} \frac{1}{c_\ell} e^{ij\mu x} \frac{1}{N} \sum_{\ell=0}^{N-1} u_\ell e^{-i\ell\mu x_j}, \tag{2.3}$$

with the following definition

$$\widehat{u}_\ell = \frac{1}{Nc_\ell} \sum_{j=0}^{N-1} u_j e^{-i\ell\mu x_j}, \tag{2.4}$$

(2.3) becomes

$$u_N(x,t) = \sum_{\ell=-N/2}^{N/2} \widehat{u}_\ell e^{i\ell\mu x}. \tag{2.5}$$

Therefore

$$u_j = u_N(x_j,t) = \sum_{\ell=-N/2}^{N/2} \widehat{u}_\ell e^{i\ell\mu x_j}. \tag{2.6}$$

In order to obtain the equation for u_ℓ, we substitute (2.5) into (1.1) and requiring that (2.1) is satisfied exactly at collocation points, i.e.,

$$\frac{\partial}{\partial t} u_N(x_j,t) + \varepsilon u_N(x_j,t) \frac{\partial}{\partial x} u_N(x_j,t) + \mu \frac{\partial^3}{\partial x^3} u_N(x_j,t) = 0, \quad j = 0,1,\cdots,N-1. \tag{2.7}$$

In order to obtain the values for the kth derivatives $\dfrac{\partial^k}{\partial x^k} u_N(x_j,t)$ at the collocation points in terms of the value u_j. We can differentiate (2.1) and evaluating the resulting expression at the point x_j

$$\frac{\partial^k}{\partial x^k} u_N(x_j,t) = \sum_{n=0}^{N-1} u_n \frac{\partial^k}{\partial x^k} g_n(x_j,t) = (D_k \mathbf{u})_j, \quad j = 0,1,\cdots,N-1, \tag{2.8}$$

where (D_k) is an $N \times N$ matrix with elements $(D_k)_{j,n} = \dfrac{d^k}{dx^k} g_n(x_j)$. We call (D_k) the kth order spectral differentiation matrix and $\mathbf{u} = (u_0, u_1, \cdots, u_{N-1})^T$.

Remark 1

We can also evaluate the derivatives by using the FFT algorithm instead of spectral differential matrix in $O(N \log N)$ operation rather than $O(N^2)$ operations. However it is more convenient to investigate the Fourier pseudospectral discretizations of (1.1) by using the spectral differential matrices.

2.1 Artificial Viscosity

The drawback of pseudospectral method is aliasing error could cause nonlinear instability. In order to weaken the nonlinear instability in computation and improve the accuracy of numerical solutions, we use the artificial viscosity as defined in [6]. Let $\gamma \geq 1$ and R_γ be the artificial viscosity. It means that if

$$u_N(x,t) = \sum_{j=0}^{N-1} u_j g_j(x) \tag{2.9}$$

is the exact solution of the scheme. Then

$$R_\gamma u_N(x,t) = \sum_{j=0}^{N-1} \left(1 - \left(\left|\frac{j}{N-1}\right|^\gamma\right)\right) u_j g_j(x). \tag{2.10}$$

Remark 2

The term "artificial viscosity" R_γ refers to a "filtering" term or more precisely an "artificial damping" term, which improves the stability of the pseudospectral method, especially in case when the solution of the partial differential equations change rapidly. A suitable value of γ must be chosen to get good results. How to choose parameter γ suitably is relative to the smoothness of the exact solution. Generally speaking, if the exact solution change rapidly we should take small γ and conversely take large γ.

2.2 Treatment of Nonlinear Terms

Let

$$u_N(x) = \sum_{j=0}^{N-1} u_j g_j(x), \qquad v_N(x) = \sum_{k=0}^{N-1} v_k g_k(x),$$

The circle convolution is defined by

$$u_N * v_N = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} u_k v_{j-k} g_j(x).$$

We denote here by V_N the space of trigonometric polynomials of degree up to $N-1$:

$$V_N = \text{span}\left\{\frac{1}{\sqrt{L}}\exp(i2\pi jx/L) : j = 0, 1, \cdots, N-1\right\},$$

where $i = \sqrt{-1}$. In order to approximate the nonlinear term $u_N \dfrac{\partial u_N}{\partial x}$ of (2.7) reasonably [15], we define the operator $J : V_N \times V_N \longrightarrow V_N$ as

$$J(u_N, v_N) = \frac{1}{3} P_c\left(\frac{\partial u_N}{\partial x} * v_N\right) + \frac{1}{3}\frac{\partial}{\partial x}\left(P_c(u_N * v_N)\right), \tag{2.11}$$

where P_c is the interpolation operator. Using (2.10) we can rewrite the nonlinear terms as follows:

$$J(u_N, v_N) = R_\gamma\left\{\frac{1}{3} P_c\left(\frac{\partial u_N}{\partial x} * R_\gamma v_N\right) + \frac{1}{3}\frac{\partial}{\partial x}\left(P_c(u_N * R_\gamma v_N)\right)\right\}. \tag{2.12}$$

2.3 Solution of Korteweg-de Vries equation

In this section, we use the spectral differentiation matrix and artificial viscosity to solve the Korteweg-de Vries equation, we obtain from (2.7)

$$\frac{d}{dt}u_N(x_j,t)+\varepsilon J(u_N(x_j,t),u_N(x_j,t))+\mu(D_3u_N(x_j,t))=0,\quad j=0,1,\cdots,N-1, \tag{2.13}$$

where

$$\begin{aligned}&J(u_N(x_j,t),u_N(x_j,t))\\&=R_\gamma\left\{\frac{1}{3}P_c(D_1u_N(x_j,t)*R_\gamma u_N(x_j,t))+\frac{1}{3}D_1(P_c(u_N(x_j,t)*R_\gamma u_N(x_j,t))\right\}.\end{aligned}$$

The equation (2.13) is the Fourier pseudospectral discretization for the Korteweg-de Vries equation in the form (1.1). Now consider the time discretization of the Fourier pseudospectral discretization (2.13). Using the modified leapfrog scheme such that the linear part is treated implicitly and the nonlinear part explicitly, we get a fully discrete Fourier pseudospectral scheme for (1.1): find $u(t)\in V_N$ such that

$$\begin{cases}\dfrac{u_N(x_j,t+\tau)-u_N(x_j,t-\tau)}{2\tau}+\varepsilon J\left(u_N(x_j,t),u_N(x_j,t)\right)\\\qquad\qquad+\dfrac{\mu}{2}D_3(u_N(x_j,t+\tau)+u_N(x_j,t-\tau)=0,\\u_N(x_j,\tau)=P_N\left(U(x_j,0)+\tau\dfrac{\partial U}{\partial t}(x_j,0)\right),\\u_N(x_j,0)=P_NU(x_j,0),\end{cases} \tag{2.14}$$

where P_N is the orthogonal projection operator. The values of $u_N(x_j,0)$ and $u_N(x_j,\tau)$ are known, then we can evaluate the values of $u_N(x_j,2\tau)$ etc. Since the nonlinear term $u_N\dfrac{\partial u_N}{\partial x}$ is approximated explicitly by $J(u_N(x_j,t),u_N(x_j,t))$, the solution of the above scheme can be obtained by solving with an octa diagonal matrix at each time level $t_k,k=0,1,\cdots,N-1$. We refer to [19] for details.

Remark 3

When time-dependent partial differential equations are solved numerically by spectral methods, spectral differentiation is used in space, while finite differences are used in the time direction. In principle, we have to sacrifice spectral accuracy in time, but in practice a small time step with a formula of order two or higher often leaves the global accuracy quite satisfactory. Small time steps are much more affordable than small space steps, for they affect the computation time, but not the storage. The details are given in [10, 21].

3 Numerical Results

In this section, we present some numerical results of our scheme for the KdV equation. We used single solitary wave propagation and double solitary waves interaction to test the good accuracy of our method. All computations were carried out in FORTRAN 90 and all figures were obtained by using Matlab 6.5 on Compaq Branded system, Processor 3.2 GHz, Memory 1 GB, Hard Disk 80 GB. For describing the error, we defined maximum error and the discrete L^2-normed error given as follows:

$$\|E\|_\infty = \max_{0\le j\le N-1} |U(x_j,t) - u_N(x_j,t)|$$

and

$$\|E\|_2 = \sum_{j=0}^{N-1} (U(x_j,t) - u_N(x_j,t))^{1/2},$$

where $u_N(x_j,t)$ is the solution of numerical scheme (2.14) and $U(x_j,t)$ is the exact solution of (1.1).

Problem (a): For this problem we considered the KdV equation (1.1) with $\varepsilon = 1$ and $\mu = 4.84 \times 10^{-4}$. The homogenous boundary conditions are

$$U(0,t) = U(2,t) = 0, \qquad t > 0,$$

and initial condition is

$$U(x,0) = 3r\ \mathrm{sech}^2(Ax + D), \qquad 0 \le x \le 2.$$

The problem has an exact solution of the form [14]:

$$U(x,t) = 3r\ \mathrm{sech}^2(Ax - Bt + D), \qquad 0 \le x \le 2,$$

where $A = \frac{1}{2}\sqrt{\varepsilon c/\mu}$, $B = \varepsilon Ac$, $r = 0.3$ and $D = -6$. This problem represents a single soliton with amplitude $3r$ which moves to the right with constant speed εr. The calculation is carried out at $c = 0.5$, and $\tau = 0.001$. The approximate solution obtained for $N = 8$ and $N = 16$ was compared with the exact solution. As seen in Table 1 and Table 2, these solutions are in good agreement with each other. The effect of artificial viscosity on the accuracy of the scheme (2.14) is clear. The value of γ in the artificial viscosity must be chosen suitably small . If γ is too large the filtering technique will be weakened. If γ is too small, the approximation accuracy will be lowered. The suitable choice of γ is between 5 and 10. But the best choice of γ varies in different cases. In our case the best choice of γ is 5 (see Table 1 and Table 2). Numerical results show that numerical precision depends on the choice of parameter γ.

The difference between the exact and the numerical solution obtained, using the present method, EFD[12] and ANM[18] are shown in Figure 1 to observe the error distribution over the space domain $0 \le x \le 2$. The graphs of the analytical and the numerical solution of single soliton at $t = 0.01$ are given in Figure 2.

Table 1: Maximum and L^2- Errors of the problem (a) at N=8

	$\|E\|_\infty$			$\|E\|_2$		
t	$\gamma = 5$	$\gamma = 10$	$\gamma = \infty$	$\gamma = 5$	$\gamma = 10$	$\gamma = \infty$
1	3.268E-5	4.984E-4	4.778E-3	3.388E-6	1.420E-4	5.009E-3
2	6.526E-5	2.109E-4	9.708E-3	6.910E-6	5.385E-4	2.652E-3
3	2.078E-4	1.909E-3	1.511E-2	1.197E-5	1.279E-3	5.793E-2
4	1.484E-4	2.961E-3	2.861E-1	3.894E-5	2.886E-3	1.139E-1
5	2.225E-4	4.367E-3	3.751E+0	8.491E-5	6.223E-3	2.168E+0

Table 2: Maximum and L^2- Errors of the problem (a) at N=16

	$\|E\|_\infty$			$\|E\|_2$		
t	$\gamma = 5$	$\gamma = 10$	$\gamma = \infty$	$\gamma = 5$	$\gamma = 10$	$\gamma = \infty$
1	5.167E-5	1.478E-4	1.506E-3	4.780E-6	1.131E-5	1.831E-3
2	1.157E-5	3.012E-4	3.066E-2	9.774E-6	5.981E-5	6.770E-2
3	1.989E-4	5.033E-3	5.384E-2	1.543E-5	1.706E-4	2.211E-2
4	2.997E-4	7.827E-3	1.026E-1	2.205E-5	3.625E-3	6.136E-1
5	4.131E-4	1.190E-2	7.002E+0	2.966E-5	6.957E-3	5.420E+0

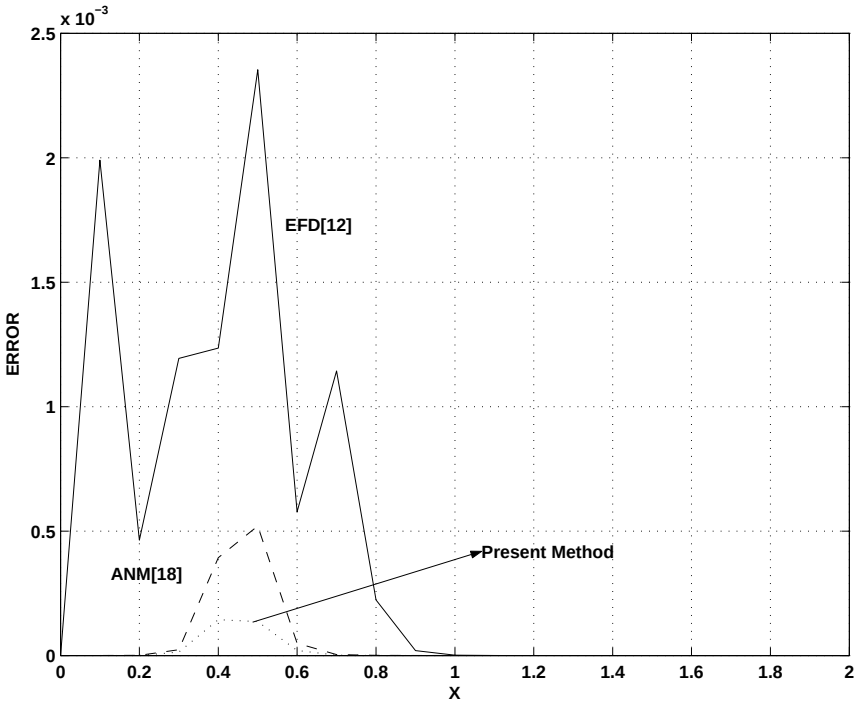

Figure 1: Graph of the errors=|Exact-Numerical| of problem (a) at t=0.005

Problem (b): For this problem we considered the KdV equation (1.1) with $\varepsilon = 6$ and $\mu = 1$ with the homogeneous boundary conditions

$$U(0,t) = U(4,t) = 0, \qquad t > 0,$$

and the initial condition obtained from the exact solution [14]

$$U(x,t) = 12\mu \frac{\partial^2}{\partial x^2}(\log f), \qquad 0 \le x \le 4,$$

where

$$f = 1 + \exp(\eta_1) + \exp(\eta_2) + \left(\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}\right)^2 \exp(\eta_1 + \eta_2),$$
$$\eta_i = \alpha_i x - \alpha_i^3 \mu t + b_i, i = 1, 2.$$
$$\alpha_1 = \sqrt{\frac{0.3}{\mu}}, \alpha_2 = \sqrt{\frac{0.1}{\mu}}, b_1 = -0.48\alpha_1 \quad \text{and} \quad b_2 = -1.07\alpha_2.$$

We studied the interaction of two solitary waves to the third order nonlinear KdV equation. The numerical solutions of KdV third order equation for this problem obtained by the present method have been compared with analytical solution in Table 3 and Table 4 for $N = 8$ and $N = 16$ at $\tau = 0.001$. It can be seen that numerical solutions are in good agreement with the analytical one. Table 3 and Table 4 show that accuracy of the numerical solutions improve rapidly as the value of γ in artificial viscosity is chosen suitably small.

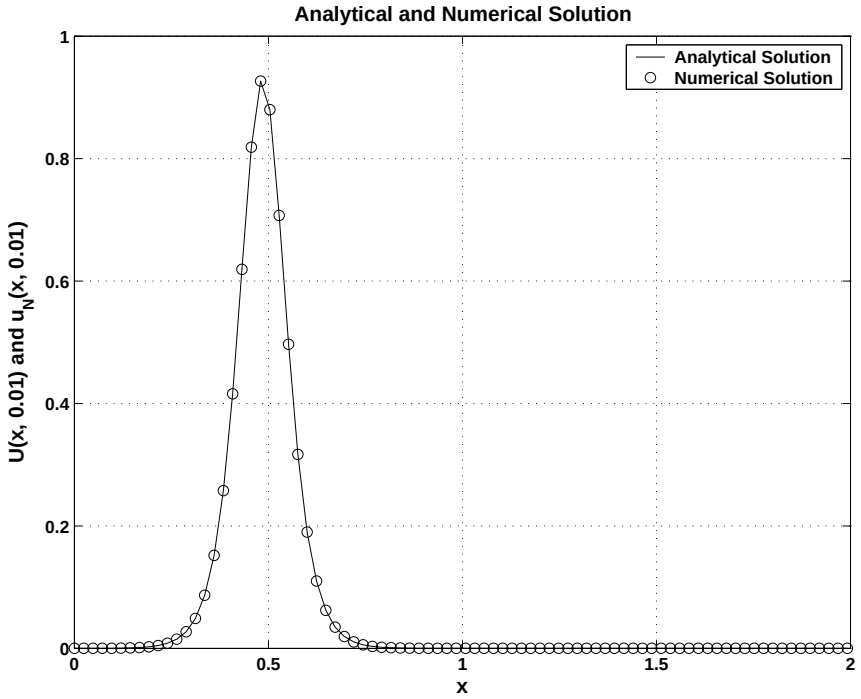

Figure 2: Plot of single soliton for problem (a) at t=0.01

In Figure 3 we compare the difference between the exact and the numerical solution obtained, from the Fourier pseudospectral method, EFD[12] and ANM[18] over the space domain $0 \leq x \leq 4$. The graphs of the analytical and the numerical solution of double soliton at $t = 0.01$ are given in Figure 4.

Table 3: Maximum and L^2- Errors of the problem (b) at N=8

	$\|E\|_\infty$			$\|E\|_2$		
t	$\gamma = 5$	$\gamma = 10$	$\gamma = \infty$	$\gamma = 5$	$\gamma = 10$	$\gamma = \infty$
0.1	2.055E-3	1.018E-3	4.810E-2	2.412E-3	1.390E-3	7.928E-2
0.2	4.482E-3	2.250E-2	1.075E-1	1.636E-3	8.713E-2	4.327E-1
0.3	1.040E-3	5.508E-2	2.677E+0	5.846E-3	3.110E-2	1.498E+0

Table 4: Maximum and L^2 Errors of the problem (b) at N=16

	$\|E\|_\infty$			$\|E\|_2$		
t	$\gamma = 5$	$\gamma = 10$	$\gamma = \infty$	$\gamma = 5$	$\gamma = 10$	$\gamma = \infty$
0.1	1.038E-3	1.301E-3	5.204E-2	2.604E-3	1.361E-3	6.515E-2
0.2	2.319E-3	2.961E-2	1.178E-1	5.949E-3	2.988E-2	1.522E-1
0.3	6.472E-3	8.778E-2	3.484E+0	7.186E-3	1.006E-1	5.908E+0

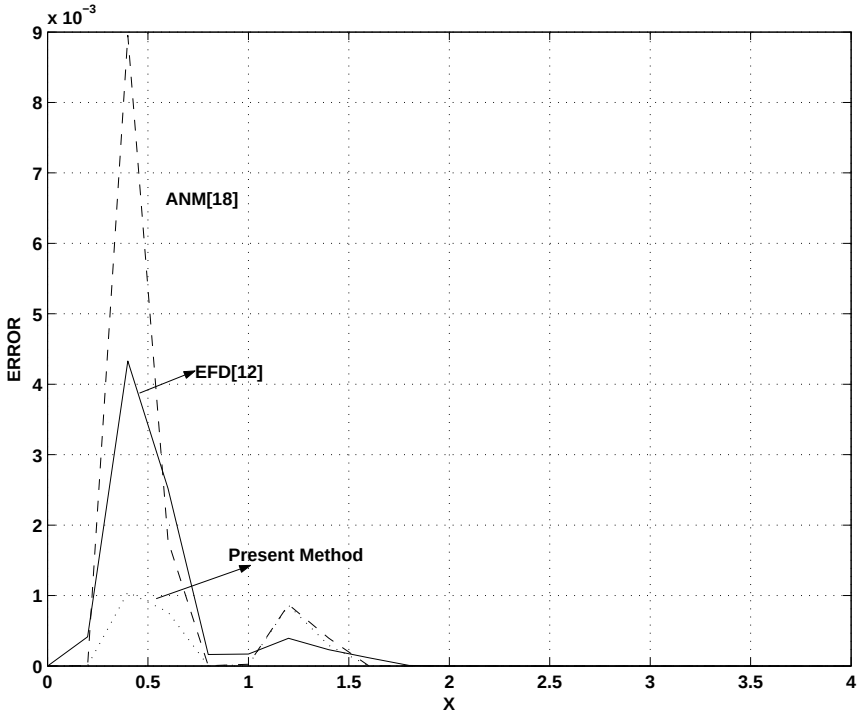

Figure 3: Graph of the errors=|Exact-Numerical| of problem (b) at t=0.01

Problem (c): In this problem, we considered the KdV equation (1.1) with $\varepsilon = 6$, $\mu = 1$ with single solitary wave solution. The homogenous boundary conditions are

$$U(0,t) = U(40,t) = 0, \qquad t > 0,$$

and initial condition is given by

$$U(x,0) = \frac{r}{2}\text{sech}^2\left(\frac{\sqrt{r}}{2}x - 7\right), \qquad r = 0.5.$$

The exact solution [20] is

$$U(x,t) = \frac{r}{2}\text{sech}^2\left(\frac{\sqrt{r}}{2}(x - rt) - 7\right), \qquad r = 0.5.$$

The calculation is carried out with parameter $\tau = 0.001$, $\gamma = 5$ and $N = 8$. To show the efficiency of the present method with the problem (c) in comparison with exact solution, we report maximum error and the discrete L^2-normed errors in Table 5 for $t = 1, 2, 3, 4$ and 5. According to the results presented in Table 5, we can say the present scheme provides better results than the results obtained from [20]. Hence our scheme (2.14) is efficient and reliable. .

Problem (d): In this problem, we studied the interaction of two solitary waves to the third order KdV equation (1.1) with $\varepsilon = 6$, $\mu = 1$. The homogeneous boundary conditions are

$$U(0,t) = U(15,t) = 0, \qquad t > 0,$$

and initial condition is

$$U(x,0) = 12\left\{\frac{3 + 4\cosh(2x) + \cosh(4x)}{(3\cosh(x) + \cosh(3x))^2}\right\}.$$

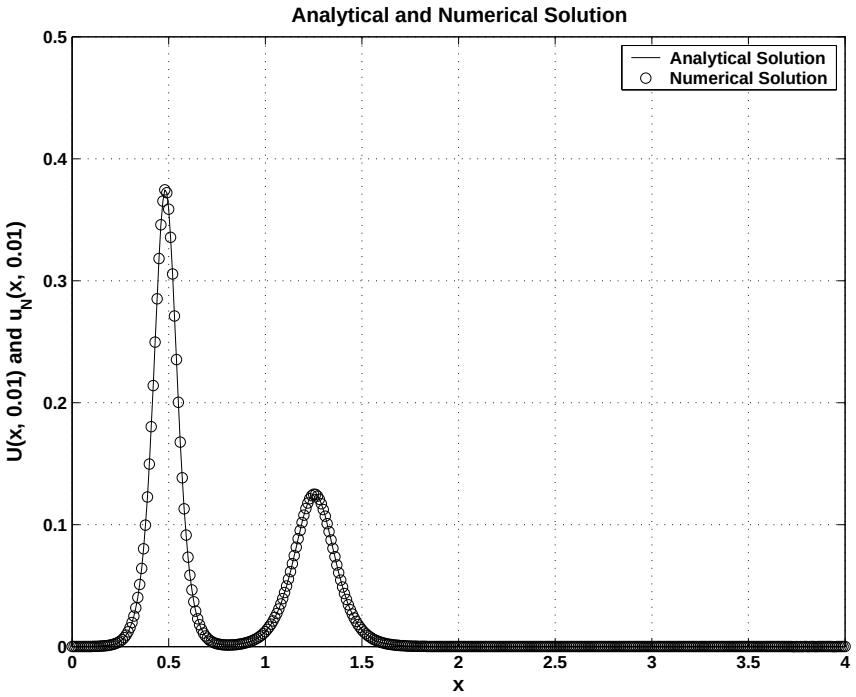

Figure 4: Plot of double soliton for problem (b) at t=0.01

Table 5: Comparison of numerical results of the problem (c) with the results obtained from [20]

t	Present Method		ANMC[20]	
	$\|E\|_\infty$	$\|E\|_2$	$\|E\|_\infty$	$\|E\|_2$
1	1.7485×10^{-6}	1.1245×10^{-5}	1.8048×10^{-5}	6.2366×10^{-5}
2	3.4562×10^{-6}	1.6758×10^{-5}	3.0373×10^{-5}	1.1264×10^{-4}
3	5.3945×10^{-6}	1.7459×10^{-5}	4.0088×10^{-5}	1.5537×10^{-4}
4	5.6421×10^{-6}	1.8147×10^{-5}	4.8347×10^{-5}	1.9400×10^{-4}
5	6.3214×10^{-6}	1.9171×10^{-5}	5.6090×10^{-5}	2.2943×10^{-4}

The exact solution [20] is

$$U(x,t) = 12\left\{\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{(3\cosh(x - 28t) + \cosh(3x - 36t))^2}\right\}.$$

The approximate solution obtained for $\tau = 0.001$, $\gamma = 5$ and $N = 8$, are compared with the results of [20]. We report maximum error and the discrete L^2-normed errors at $t = 0.1, 0.2$ and 0.3 in Table 6. As shown in Table 6, these solutions are in good agreement with each other. The results of the present method are much better than that of [20].

Table 6: Comparison of numerical results of the problem (d) with the results obtained from [20]

t	Present Method		ANMC[20]	
	$\|E\|_\infty$	$\|E\|_2$	$\|E\|_\infty$	$\|E\|_2$
0.1	2.3256×10^{-3}	2.4215×10^{-3}	5.6355×10^{-3}	1.5444×10^{-2}
0.2	4.5689×10^{-3}	3.2156×10^{-3}	2.3376×10^{-2}	6.2286×10^{-2}
0.3	5.7485×10^{-3}	4.6541×10^{-3}	5.9437×10^{-2}	1.6348×10^{-1}

4 Conclusion

In this paper, we discussed the well known Korteweg-de Vries equation. We proposed a numerical scheme to solve the third order nonlinear KdV equation using the pseudospectral method with artificial viscosity. The numerical results given in the previous section demonstrate the good accuracy of this scheme. For the test problems, the present method provide more accurate results than ANM[18], EFD[12] and ANMC[20]. So we have found advantages for the single soliton and double soliton solution by the Fourier pseudospectral method in getting the numerical solution of the KdV equation in terms of accuracy. From the errors plot, we observed the maximum error occurred just around peak position of wave amplitude in the case of single soliton and double soliton solution. The method is also capable of solving the KdV type equation with other types of boundary conditions and initial conditions.

Acknowledgements
The author is full of gratitude to the referees for their valuable suggestions and help in improving the presentation.

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