Inner invariant extensions of Dirac measures on compactly cancellative topological semigroups

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Abstract

Let $\mathcal{S}$ be a left compactly cancellative foundation semigroup with identity e and $M_a(\mathcal{S})$ be its semigroup algebra. In this paper, we give a characterization for the existence of an inner invariant extension of δ_e from $C_b(\mathcal{S})$ to a mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ in terms of asymptotically central bounded approximate identities in $M_a(\mathcal{S})$. We also consider topological inner invariant means on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ to study strict inner amenability of $M_a(\mathcal{S})$ and their relation with strict inner amenability of $\mathcal{S}$.

1 Introduction

Throughout this paper, $\mathcal{S}$ denotes a locally compact Hausdorff topological semigroup. The space of all bounded complex regular Borel measures on $\mathcal{S}$ is denoted by $M(\mathcal{S})$. This space with the convolution multiplication $*$ and the total variation norm defines a Banach algebra. The space of all measures $\mu \in M(\mathcal{S})$ for which the maps $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from $\mathcal{S}$ into $M(\mathcal{S})$ are weakly continuous is denoted by $M_a(\mathcal{S})$ (or $\widetilde{L}(\mathcal{S})$ as in [2]), where δ_x denotes the Dirac measure at x. $\mathcal{S}$ is called *foundation semigroup* if $\mathcal{S}$ coincides with the closure of the set $\bigcup\{\text{supp}(\mu) : \mu \in M_a(\mathcal{S})\}$. It is well-known that $M_a(\mathcal{S})$ is a closed two-sided L-ideal

*The research of the second author was in part supported by a grant from IPM(No. 85430035)
†The research of the third author was in part supported by a grant from IPM(No. 85430032)
Received by the editors November 2005.
Communicated by F. Bastin.
1991 *Mathematics Subject Classification :* Primary 43A07, Secondary 43A10, 43A15, 46H05.
Key words and phrases : Inner invariance, inner invariant extension, mixed identity, strict inner amenability, topological inner invariance.

of $M(\mathcal{S})$; see [2]. Let us point out that the second dual $M_a(\mathcal{S})^{**}$ of $M_a(\mathcal{S})$ is a Banach algebra with the first Arens product $\odot$ defined by the equations

$$(F \odot H)(f) = F(Hf),\ (Hf)(\mu) = H(f\mu),\ \text{ and }\ (f\mu)(\nu) = f(\mu * \nu)$$

for all $F, H \in M_a(\mathcal{S})^{**}$, $f \in M_a(\mathcal{S})^*$, and $\mu, \nu \in M_a(\mathcal{S})$.

Denote by $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ the set of all complex-valued bounded functions g on $\mathcal{S}$ that are μ-measurable for all $\mu \in M_a(\mathcal{S})$. We identify functions in $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ that agree μ-almost everywhere for all $\mu \in M_a(\mathcal{S})$. For every $g \in L^\infty(\mathcal{S}; M_a(\mathcal{S}))$, define

$$\|g\|_\infty = \sup\{\ \|g\|_{\infty,|\mu|} : \mu \in M_a(\mathcal{S})\ \},$$

where $\|.\|_{\infty,|\mu|}$ denotes the essential supremum norm with respect to $|\mu|$. Observe that $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with the complex conjugation as involution, the pointwise operations and the norm $\|.\|_\infty$ is a commutative C^*-algebra. The duality

$$\tau(g)(\mu) := \mu(g) = \int_{\mathcal{S}} g\ d\mu$$

defines a linear mapping τ from $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ into $M_a(\mathcal{S})^*$. It is well-known that if $\mathcal{S}$ is a foundation semigroup with identity, then τ is an isometric isomorphism of $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ onto $M_a(\mathcal{S})^*$; see Proposition 3.6 of Sleijpen [27]. Given any $\mu \in M_a(\mathcal{S})$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$, define the complex-valued functions $g \circ \mu$ and $\mu \circ g$ on $\mathcal{S}$ by

$$(g \circ \mu)(x) = \mu({}_x g)\ \text{ and }\ (\mu \circ g)(x) = \mu(g_x)$$

for all $x \in \mathcal{S}$, where $({}_x g)(y) = g(xy)$ and $(g_x)(y) = g(yx)$ for all $y \in \mathcal{S}$. It is clear that $g \circ \mu$ and $\mu \circ g$ are in $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with

$$\|g \circ \mu\|_\infty \le \|g\|_\infty\ \|\mu\|\ \text{ and }\ \|\mu \circ g\|_\infty \le \|g\|_\infty\ \|\mu\|.$$

Let $LUC(\mathcal{S})$ be the space of all *left uniformly continuous* on $\mathcal{S}$; recall that a function $g \in C_b(\mathcal{S})$ is called left uniformly continuous if the mapping $x \mapsto {}_x g$ from $\mathcal{S}$ into $C_b(\mathcal{S})$ is $\|.\|_\infty$-continuous, where $C_b(\mathcal{S})$ denotes the space of all bounded continuous complex-valued functions on $\mathcal{S}$; as usual $C_0(\mathcal{S})$ denotes the space of functions in $C_b(\mathcal{S})$ vanishing at infinity and $C_c(\mathcal{S})$ denotes its subspace of functions with compact support.

Let X be a closed subspace of $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ containing the constant functions on $\mathcal{S}$. A mean on X is a functional M with $\|M\| = M(1) = 1$. If, moreover, ${}_x g, g_x \in X$ for all $x \in \mathcal{S}$ and $g \in X$, we then say that m is *inner invariant* if

$$M({}_x g) = M(g_x) \qquad (x \in \mathcal{S}, g \in X).$$

The study of inner invariant means was initiated by Effros [8] and pursued by Akemann [1], H. Choda and M. Choda [4], M. Choda [5, 6] for discrete groups, Lau and Paterson [17], Losert and Rindler [20], Yuan [28] for topological groups, and by Ling [19] and the second and third authors [22] for discrete semigroups.

For a foundation semigroup $\mathcal{S}$ with identity e, the Dirac measure δ_e is always an inner invariant mean on $C_b(\mathcal{S})$. Several authors have been studied the possibility of inner invariant extension of δ_e to a mean on $L^\infty(\mathcal{S})$ in the case where $\mathcal{S}$ is a locally compact group; see [20] and [28]. Here, we consider the more general setting of left compactly cancellative topological semigroups; recall from [18] that $\mathcal{S}$ is said to be *left compactly cancellative* if $C^{-1}D$ is compact for all compact subsets C and D of $\mathcal{S}$. We give a characterization for the existence of an inner invariant extensions of δ_e to a mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ in terms of asymptotically central bounded approximate identities in $M_a(\mathcal{S})$. Motivated by an open problem arising from [24], we also consider topological inner invariant means on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ to study strict inner amenability of $M_a(\mathcal{S})$ and their relation with strict inner amenability of $\mathcal{S}$.

2 Inner invariant extensions of Dirac measures

Let us recall that an element E in $M_a(\mathcal{S})^{**}$ is called a *mixed identity* if

$$\mu \odot E = E \odot \mu = \mu \qquad (\mu \in M_a(\mathcal{S})).$$

It is easy to see that an element E of $M_a(\mathcal{S})^{**}$ is a mixed identity if and only if it is a weak* cluster point of a bounded approximate identity in $M_a(\mathcal{S})$; see [3], page 146. Furthermore, any mixed identity is a right identity of $M_a(\mathcal{S})^{**}$ but not a left identity in general.

Proposition 2.1. *Let $\mathcal{S}$ be a foundation semigroup with identity e. Then any extension of δ_e from $C_b(\mathcal{S})$ to $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with norm one is a mixed identity.*

Proof. Let E be an extension of δ_e from $C_b(\mathcal{S})$ to $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with norm one. For every $\mu \in M_a(\mathcal{S})$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ we have

$$(\mu \circ g)(x) = (\mu * \delta_x)(g)$$

for all $x \in \mathcal{S}$; it follows from the weak continuity of the map $x \mapsto \mu * \delta_x$ from $\mathcal{S}$ into $M_a(\mathcal{S})$ that $\mu \circ g \in C_b(\mathcal{S})$. Therefore

$$\begin{aligned}(\mu \odot E)(g) &= E(\mu \circ g)\\ &= (\mu \circ g)(e)\\ &= \mu(g);\end{aligned}$$

that is, $\mu \odot E = \mu$. Similarly $E \odot \mu = \mu$. Thus E is a mixed identity. ■

To prove the converse of Proposition 2.1, we need the following lemma.

Lemma 2.2. *Let $\mathcal{S}$ be a foundation semigroup with identity e. Suppose that M is a mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ with $M(h) = h(e)$ for all $h \in C_c(\mathcal{S})$. Then $M(g) = g(e)$ for all $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ continuous at e.*

Proof. Without loss of generality we may assume that g is non-negative and $g(e) = 0$. Given $\varepsilon > 0$, let

$$V_\varepsilon = \{x \in \mathcal{S} : \ g(x) < \varepsilon \}.$$

Then V_ε is a neighbourhood of e by continuity of g at e. Hence there exists $h \in C_c(\mathcal{S})$ with support in V_ε such that

$$0 \le h \le \| g\|_\infty \quad \text{and} \quad h(e) = \| g\|_\infty.$$

Thus

$$\begin{aligned} \| g\|_\infty + M(g) &= h(e) + M(g) \\ &= M(h) + M(g) \\ &= M(h+g) \\ &\leq \| h+g\|_\infty \\ &= \| g\|_\infty + \varepsilon. \end{aligned}$$

It follows that $M(g) \leq \varepsilon$ for all $\varepsilon > 0$; hence $M(g) = 0$ as required. ∎

The following theorem is indeed an improvement of Proposition 2.1 in [9].

Theorem 2.3. *Let $\mathcal{S}$ be a left compactly cancellative foundation semigroup with identity e, and let E be an element of $M_a(\mathcal{S})^{**}$ with norm one. Then the following assertions are equivalent.*

(a) *E is a mixed identity.*

(b) *E is a right identity.*

(c) *E is an extension of δ_e from $C_0(\mathcal{S})$ to $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$.*

(d) *E is an extension of δ_e from $C_b(\mathcal{S})$ to $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$.*

Proof. (a)$\Longrightarrow$(b). Let $F \in M_a(\mathcal{S})^{**}$ and (σ_α) be a net in $M_a(\mathcal{S})$ which converges to F in the weak* topology. Then

$$\sigma_\alpha \odot E \to F \odot E$$

in the weak* topology. So the result follows from that $\sigma_\alpha \odot E = \sigma_\alpha$ for all α.

(b)$\Longrightarrow$(c). Let (ν_γ) be a right approximate identity bounded by one converging to E in the weak* topology. Then for each $\mu \in M_a(\mathcal{S})$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ we have

$$\begin{aligned} E(\mu \circ g) &= \lim_\gamma \nu_\gamma(\mu \circ g) \\ &= \lim_\gamma (\mu * \nu_\gamma)(g) \\ &= \mu(g); \end{aligned}$$

that is, $E(\mu \circ g) = (\mu \circ g)(e)$. Now invoke Lemma 2.1 from [10], to conclude that

$$LUC(\mathcal{S}) = M_a(\mathcal{S}) \circ L^\infty(\mathcal{S}, M_a(\mathcal{S}))$$

and hence $E(f) = f(e)$ for all $f \in LUC(\mathcal{S})$. Since $\mathcal{S}$ is left compactly cancellative, from Lemma 1.2 of [11] it follows that

$$C_0(\mathcal{S}) \subseteq LUC(\mathcal{S}).$$

Thus $E(f) = f(e)$ for all $f \in C_0(\mathcal{S})$.

(c)$\Longrightarrow$(d). By Lemma 2.2, we only need to show that E is a mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. To that end, let m be the restriction of E to $LUC(\mathcal{S})$. Then by Theorem 2 of [11], there is $n \in C_0(\mathcal{S})^\perp$ such that

$$m = n + \mu \quad \text{and} \quad \|m\| = \|n\| + \|\mu\|,$$

where $\mu \in LUC(\mathcal{S})^*$ is defined by $\mu(f) = f(e)$ for all $f \in LUC(\mathcal{S})$. Since $\|E\| = 1$, we have $\|m\| \leq 1$, whence $n = 0$. Thus $E(1) = m(1) = 1$.

(d)$\Longrightarrow$(a). This follows from Proposition 2.1. ∎

Recall that a net (μ_γ) in $M_a(\mathcal{S})$ is called *asymptotically central* (resp. *weakly asymptotically central*) if

$$\delta_x * \mu_\gamma - \mu_\gamma * \delta_x \to 0$$

in the norm (resp. weak) topology for all $x \in \mathcal{S}$. In the following, let $P_1(M_a(\mathcal{S}))$ denote the set of all probability measures in $M_a(\mathcal{S})$.

Theorem 2.4. *Let $\mathcal{S}$ be a left compactly cancellative foundation semigroup with identity e. Then the following statements are equivalent.*

(a) *δ_e has an inner invariant extension to a mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$.*

(b) *There is a weakly asymptotically central approximate identity in $P_1(M_a(\mathcal{S}))$.*

(c) *There is an asymptotically central approximate identity in $P_1(M_a(\mathcal{S}))$.*

Proof. Suppose that (a) holds, and let E be an extension of δ_e from $C_b(\mathcal{S})$ to an inner invariant mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. Then E is a mixed identity by Proposition 2.1. Since $\mathcal{S}$ is a foundation semigroup with identity, it follows from [27] that $M_a(\mathcal{S})$ is the predual of the commutative C^*-algebra $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$; see also [12] and [23]. Thus $P_1(M_a(\mathcal{S}))$ is weak* dense in $P_1(M_a(\mathcal{S})^{**})$, the set of all means on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$; see Lemma 2.1 in [16]. So, there is a net (μ_γ) in $P_1(M_a(\mathcal{S}))$ which converges to E in the weak* topology. Thus, (μ_γ) is a weak approximate identity for $M_a(\mathcal{S})$, and therefore we may find an approximate identity (σ_α) in $P_1(M_a(\mathcal{S}))$ which converges to E in the weak* topology; see [3], page 146. Since E is inner invariant, it follows that

$$\delta_x * \sigma_\alpha - \sigma_\alpha * \delta_x \to 0 \quad (x \in \mathcal{S})$$

in the weak topology of $M_a(\mathcal{S})$. That is, (σ_α) a weakly asymptotically central approximate identity. A standard argument shows that (b) implies (c).

Finally, if there exists an asymptotically central approximate identity (ν_γ) in $P_1(M_a(\mathcal{S}))$, then any weak*-cluster point E of (ν_γ) in $M_a(\mathcal{S})^{**}$ is an inner invariant mean. Also, E is a mixed identity with norm one, and hence it follows from Theorem 2.3 that E is an extension of δ_e from $C_b(\mathcal{S})$ to $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. That is (a) holds. ∎

3 Strict inner amenability

A closed subspace X of $L^\infty(\mathcal{S}; M_a(\mathcal{S}))$ is called *topologically inner invariant* if

$$\mu \circ g,\; g \circ \mu \in X \quad (\mu \in M_a(\mathcal{S}), g \in X).$$

Let X be a topologically inner invariant closed subspace of $L^\infty(\mathcal{S}; M_a(\mathcal{S}))$ containing the constant functions. We say that a mean M is *topological inner invariant* on X whenever

$$M(\mu \circ g) = M(g \circ \mu) \quad (\mu \in M_a(\mathcal{S}), g \in X).$$

The notion of topological inner invariant means was introduced and studied by the third author [24] for a large class of Banach algebras known as Lau algebras. The subject of Lau algebras originated with the paper [15] published in 1983 by Lau in which he referred to them as F-algebras. Later on, in his useful monograph, Pier [26] introduced the name Lau algebra.

As pointed out in [23], $M_a(\mathcal{S})$ is a Lau algebra for all foundation semigroups $\mathcal{S}$ with identity; in this case, any mixed identity with norm one in $M_a(\mathcal{S})^{**}$ is a topological inner invariant mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. Following [24], $M_a(\mathcal{S})$ is called *strictly inner amenable* if there is a topological inner invariant mean m on $L^\infty(\mathcal{S}; M_a(\mathcal{S}))$ which is not a mixed identity.

Proposition 3.1. *Let $\mathcal{S}$ be a left compactly cancellative foundation semigroup with identity e, and suppose that there is a topological inner invariant mean on $LUC(\mathcal{S})$ not equal to δ_e. Then $M_a(\mathcal{S})$ is strictly inner amenable.*

Proof. Suppose that M is a topological inner invariant mean on $LUC(\mathcal{S})$ not equal to δ_e, and $\widetilde{M}$ is an extension of M from $LUC(\mathcal{S})$ to a mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. Then for $\mu \in M_a(\mathcal{S})$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ we have

$$\|\nu_\gamma \circ (\mu \circ g) - \mu \circ g\|_\infty = \|(\nu_\gamma * \mu - \mu) \circ g\|_\infty \to 0$$

and

$$\|\mu \circ (\nu_\gamma \circ g) - \mu \circ g\|_\infty = \|(\mu * \nu_\gamma - \mu) \circ g\|_\infty \to 0,$$

where (ν_γ) is an approximate identity of probability measures for $M_a(\mathcal{S})$; see [23]. It follows that

$$\lim_\gamma \widetilde{M}(\nu_\gamma \circ (\mu \circ g)) = \widetilde{M}(\mu \circ g) = \lim_\gamma \widetilde{M}(\mu \circ (\nu_\gamma \circ g)).$$

For every γ we have $\nu_\gamma \circ g \in LUC(\mathcal{S})$; see Lemma 2.1 from [10]. Therefore

$$M(\mu \circ (\nu_\gamma \circ g)) = M((\nu_\gamma \circ g) \circ \mu);$$

That is,

$$\widetilde{M}(\mu \circ (\nu_\gamma \circ g)) = \widetilde{M}((\nu_\gamma \circ g) \circ \mu;$$

moreover,

$$(\nu_\gamma \odot \widetilde{M})(\mu \circ g) = \widetilde{M}(\nu_\gamma \circ (\mu \circ g)),$$
$$\widetilde{M}((\nu_\gamma \circ g) \circ \mu) = (\nu_\gamma \odot \widetilde{M})(g \circ \mu).$$

Consequently,

$$\lim_\gamma (\nu_\gamma \odot \widetilde{M})(\mu \circ g) = \lim_\gamma (\nu_\gamma \odot \widetilde{M})(g \circ \mu).$$

Let E be a weak* cluster point of (ν_γ), Then E is a mixed identity and $\nu_\gamma \odot \widetilde{M}$ converges to $E \odot \widetilde{M}$ in the weak* topology of $M_a(\mathcal{S})^{**}$, and hence we get

$$(E \odot \widetilde{M})(\mu \circ g) = (E \odot \widetilde{M})(g \circ \mu).$$

This means that $E \odot \widetilde{M}$ is a topological inner invariant mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. Since $\widetilde{M}$ is not a mixed identity, $E \odot \widetilde{M}$ cannot be a mixed identity; that is $M_a(\mathcal{S})$ is strictly inner amenable. ∎

The following example shows that there is a locally compact non-discrete semigroup satisfying the hypothesis of Proposition 3.1 which is not a subset of any group.

Example 3.2. Let $T := \{0, 1, 2, ..., n\}$ be the discrete semigroup with the operation $xy = 0$ for all $x, y \in T \setminus \{1\}$ and $x1 = 1x = x$ for all $x \in T$. Then

$$\mathcal{S} := T \times SO(n, \mathbb{R})$$

is a compact foundation non-abelian semigroup with identity; see for example Palmer [25], page 80. It follows from the proof of Lemma 6.3 and Proposition 6.4 in [14] that $LUC(\mathcal{S})$ has an invariant mean M; that is, $M({}_xg) = M(g_x) = M(g)$ for all $x \in \mathcal{S}$ and $g \in LUC(\mathcal{S})$. In particular, M is not equal to the Dirac measure at the identity element of $\mathcal{S}$. Thus

$$M(\mu \circ g) = M(g \circ \mu) = M(g)$$

for all $\mu \in P_1(M_a(\mathcal{S}))$ and $g \in LUC(\mathcal{S})$; see [7], Lemma 2.3 and its proof, page 74. That is, M is also a topological inner invariant mean on $LUC(\mathcal{S})$.

Let $\mathcal{S}$ be a foundation semigroup with identity e. As pointed out δ_e is always an inner invariant mean on $C_b(\mathcal{S})$. We say that $\mathcal{S}$ is *strictly inner amenable* if there is an inner invariant mean m on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ which is not an extension of δ_e.

Proposition 3.3. *Let $\mathcal{S}$ be a non-discrete foundation semigroup with identity e. If there is a topological inner invariant mean in $M_a(\mathcal{S})$, then $\mathcal{S}$ is strictly inner amenable.*

Proof. Let $M \in M_a(\mathcal{S})$ be a topological inner invariant mean . Then M is in the center of $M_a(\mathcal{S})$, and so $M \circ g = g \circ M$ for all $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. This implies that for any $x \in \mathcal{S}$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$,

$$\begin{aligned} M({}_xg) &= \int_{\mathcal{S}} g(xy)\, dM(y) \\ &= (M \circ g)(x) \\ &= (g \circ M)(x) \\ &= \int_{\mathcal{S}} g(yx)\, dM(y) \\ &= M(g_x). \end{aligned}$$

So, M is an inner invariant mean. Finally, since $\mathcal{S}$ is not discrete, $M_a(\mathcal{S})$ does not have identity; see [2]. It follows that M is not a mixed identity. Now the proof is complete by Proposition 2.1. ∎

It is an open problem arising from [24] whether strict inner amenability of a locally compact group G is equivalent to strict inner amenability of $L^1(G)$?

In [21], Memarbashi and Riazi proved that strict inner amenability of G implies strict inner amenability of $L^1(G)$. The first and second authors [13] have recently shown that the converse is not true; however, they have proved that the converse is true if δ_e has an inner invariant extension to a mean on $L^\infty(G)$. Here, we show that the later result remains valid for certain topological semigroups.

Theorem 3.4. *Let $\mathcal{S}$ be a left compactly cancellative foundation semigroup with identity e. Suppose that δ_e has an inner invariant extension from $C_b(\mathcal{S})$ to a mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. Then strict inner amenability of $M_a(\mathcal{S})$ implies strict inner amenability of $\mathcal{S}$.*

Proof. Since δ_e has an inner invariant extension to from $C_b(\mathcal{S})$ to a mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$, it follows from Theorem 2.4 that $M_a(\mathcal{S})$ has an approximate identity (ν_γ) of probability measures such that

$$\|\delta_x * \nu_\gamma - \nu_\gamma * \delta_x\| \to 0 \qquad (x \in \mathcal{S}).$$

For each $x \in \mathcal{S}$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ we have

$$\nu_\gamma \circ ({}_x g) = (\delta_x * \nu_\gamma) \circ g \quad \text{and} \quad (g_x) \circ \nu_\gamma = g \circ (\nu_\gamma * \delta_x).$$

Now, let M be a topological inner invariant mean on $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. Then

$$\begin{aligned} (\nu_\gamma \odot M)({}_x g) &= M(\nu_\gamma \circ g_x) \\ &= M((\delta_x * \nu_\gamma) \circ g) \\ &= M(g \circ (\delta_x * \nu_\gamma)) \end{aligned}$$

and

$$\begin{aligned} (\nu_\gamma \odot M)(g_x) &= M(\nu_\gamma \circ g_x) \\ &= M(g_x \circ \nu_\gamma) \\ &= M(g \circ (\nu_\gamma * \delta_x)). \end{aligned}$$

Thus

$$\begin{aligned} |(\nu_\gamma \odot M)({}_x g - g_x)| &= |M(\nu_\gamma \circ {}_x g - \nu_\gamma \circ g_x)| \\ &= |M(g \circ (\delta_x * \nu_\gamma) - g \circ (\nu_\gamma * \delta_x))| \\ &\le \|M\| \, \|g\|_\infty \, \|\delta_x * \nu_\gamma - \nu_\gamma * \delta_x\| \to 0. \end{aligned}$$

Next, let E be a weak* cluster point of (ν_γ) in $M_a(\mathcal{S})^{**}$. Since $\nu_\gamma \odot M$ converging to $E \odot M$ in the weak* topology of $M_a(\mathcal{S})^{**}$, it follows that

$$(E \odot M)({}_x g) = (E \odot M)(g_x)$$

for all $x \in \mathcal{S}$ and $g \in L^\infty(\mathcal{S}, M_a(\mathcal{S}))$. That is $E \odot M$ is inner invariant. Since M is not a mixed identity, it follows that $E \odot M$ is not a mixed identity. Therefore, $E \odot M$ is not an extension of δ_e from $C_b(\mathcal{S})$ to $L^\infty(\mathcal{S}, M_a(\mathcal{S}))$ by Theorem 2.3 and the proof is complete. ∎

Acknowledgments. The authors would like to thank the referee of the paper for invaluable comments. The first author thanks the Center of Excellence for Mathematics at the University of Isfahan. The third author thanks the Center of Excellence for Mathematics at the Isfahan University of Technology.

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